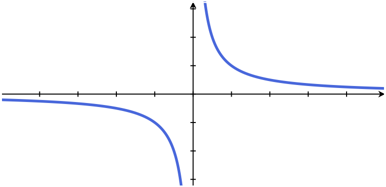
,

Exponential Growth and Decay

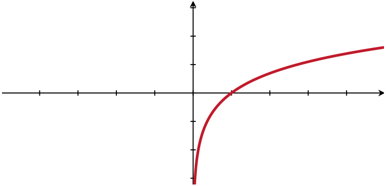
10/28/2011

Antiderivative of $1/x$

$1/x$:



$\ln(x)$:



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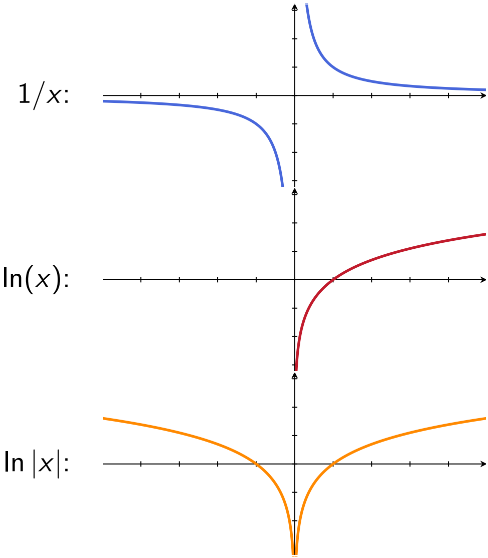

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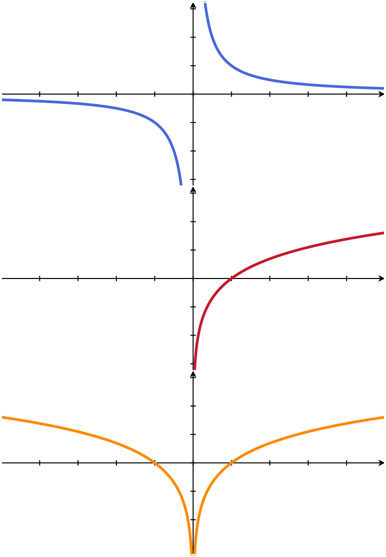
$\ln|x|$:



So

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

Warm up: Decide whether each statement is true or false by taking a derivative of the RHS and seeing if it's the function inside the integral. **If false**, calculate the real antiderivative.

1. $\displaystyle\int \frac{1}{3x+5}\ dx = \frac{1}{3}\ln|3x+5| + C$

2. $\displaystyle\int e^{4x}\ dx = e^{4x} + C$

3. $\displaystyle\int \frac{1}{e^x}\ dx = \ln|e^x| + C$
   [hints: $e^x > 0$, so $\ln|e^x| = \ln(e^x)$. Also, $1/e^x = e^{-x}$]

4. $\displaystyle\int \cos(-14x+32)\ dx = -\frac{1}{14}\sin(-14x+32) + C$

5. $\displaystyle\int \frac{1}{2x}\ dx = \frac{1}{2}\ln(2x) + C$

Review of antiderivatives we know so far

$\int x^a \; dx = \frac{1}{a+1} x^{a+1} + C$	$\int \sec^2(x) \; dx = \tan(x) + C$
$\int \frac{1}{x} \; dx = \ln\|x\| + C$	$\int \csc^2(x) \; dx = -\cot(x) + C$
$\int e^x \; dx = e^x + C$	$\int \sec(x)\tan(x) \; dx = \sec(x) + C$
$\int \sin(x) \; dx = -\cos(x) + C$	$\int \csc(x)\cot(x) \; dx = -\csc(x) + C$
$\int \cos(x) \; dx = \sin(x) + C$	

If $F'(x) = f(x)$ and $G'(x) = g(x)$, and a and b are constants, then

$$\int \Big(a * f(x) + b * g(x)\Big) \; dx = a * F(x) + b * G(x) + C$$

and $$\int f(a * x + b) dx = \frac{1}{a} f(a * x + b) + C$$

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The plan:

1. Put it into math, i.e. Write down an initial value problem.

1'. Look at slope fields to make sure the IVP makes sense.

2. Find the general solution to the IVP.
3. Plug in the points and find the particular solution.
4. Calculate the value of the solution when $t = 24$.

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Step 1: Put into math. Initial value problem:

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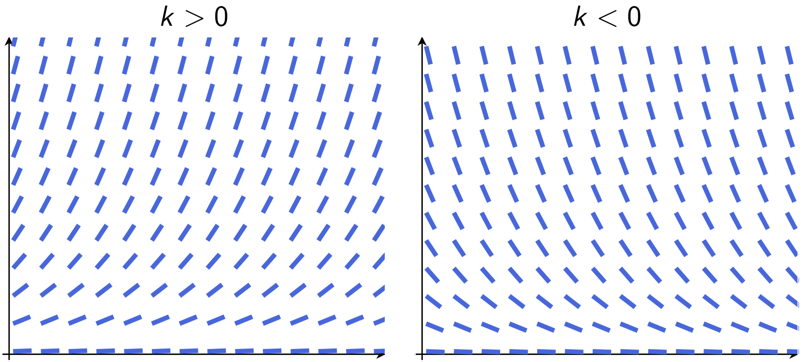

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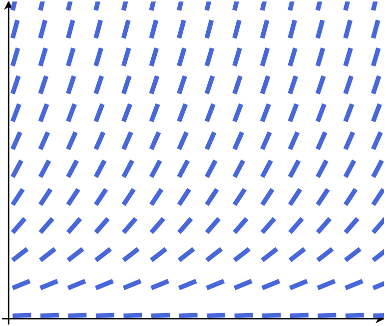
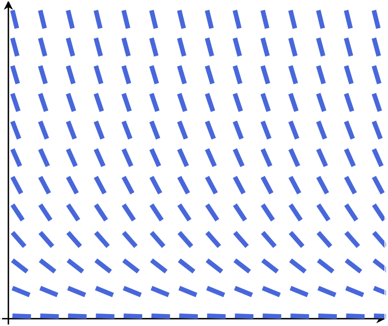
$k > 0$



$k < 0$



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Step 2: Find the general solution. To solve: Separate!

$$\int \frac{1}{y} dy = \int k \ dt$$

LHS: $\int \frac{1}{y} \ dy = \ln|y| + c_1$

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$$\int \frac{1}{y} dy = \int k \; dt$$

LHS: $\int \frac{1}{y} \; dy = \ln|y| + c_1$
RHS: $\int k \; dt = kt + c_2$

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Putting it together:

$$\ln|y| = kt + C$$

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$$\ln |y| = kt + C \implies |y| = e^{kt+C} = e^C * e^k t$$

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$$\ln|y| = kt + C \implies |y| = e^{kt+C} = e^C * e^k t$$
$$\implies y = \pm e^C * e^k t = Ae^{kt}.$$

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$$\begin{aligned} \ln|y| = kt + C &\implies |y| = e^{kt+C} = e^{C} * e^{k} t \\ &\implies y = \pm e^{C} * e^{k} t = Ae^{kt}. \end{aligned}$$

General solution: $\boxed{y = Ae^{kt}}$

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$$700 = y(0) = Ae^0 = A, \quad \text{so } y = 700e^{kt}$$

$$900 = 700e^{12k} \implies 12k = \ln(900/700) = ln(9/7)$$
$$\implies k = \frac{1}{12}\ln(9/7) \approx \boxed{0.021}$$

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Note: another way to write this is

$$y = 700e^{t*\frac{1}{12}\ln(9/7)} = 700\left(e^{ln(9/7)}\right)^{t/12} = 700\left(\frac{9}{7}\right)^{t/12}$$

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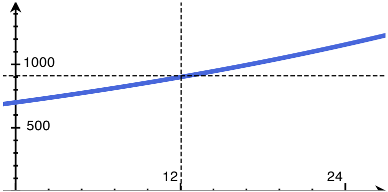

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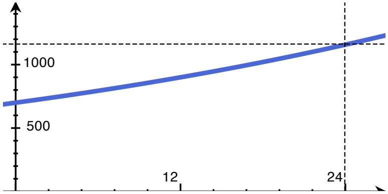

$$y(24) = 700\left(\frac{9}{7}\right)^{24/12} = 700\left(\frac{9}{7}\right)^{2} = 8100/7 \approx 1157.14$$

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

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Step 1: Put into math. Initial value problem:

$$\frac{dy}{dt} = k(y - 70), \qquad y(0) = 370, \quad y(10) = 340$$

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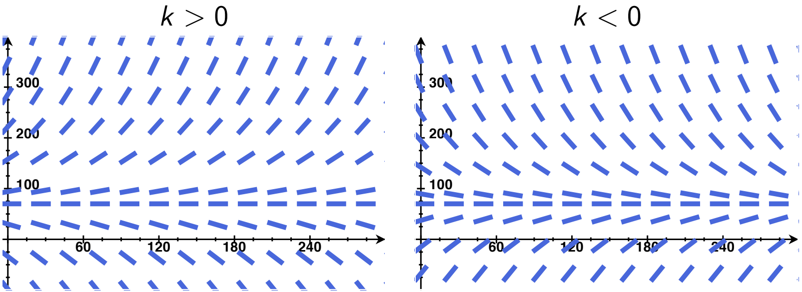

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Step 2: Find the general solution. To solve: Separate!

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LHS: $\int \frac{1}{y-70} dy = \ln|y-70| + c_1$,

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$$y - 70 = \pm e^{kt+c} = \pm e^c * e^{kt} = Ae^{kt} \qquad \text{where } A = \pm e^c,$$

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and so

$$\boxed{y = Ae^{kt} + 70}$$

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General solution: $\boxed{y = Ae^{kt} + 70}$

What do we expect from k and A?

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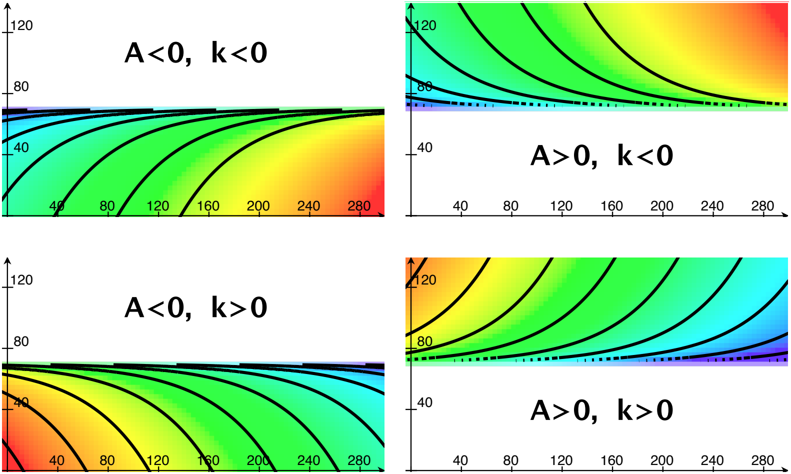

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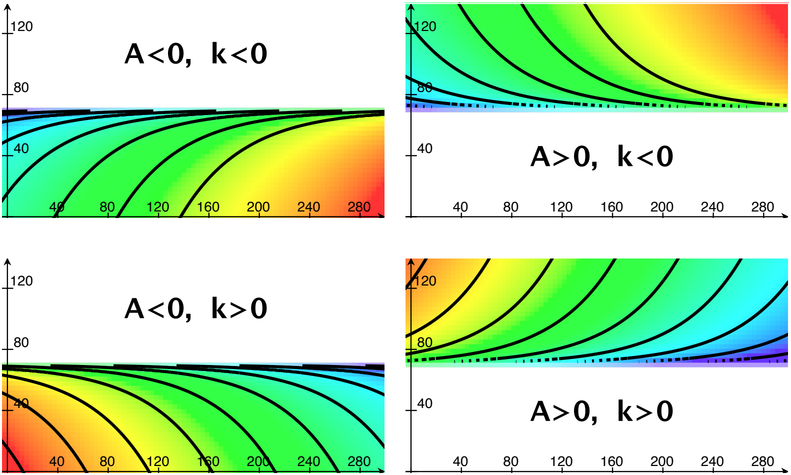
What do we expect from k and A? $A > 0,\ k < 0$



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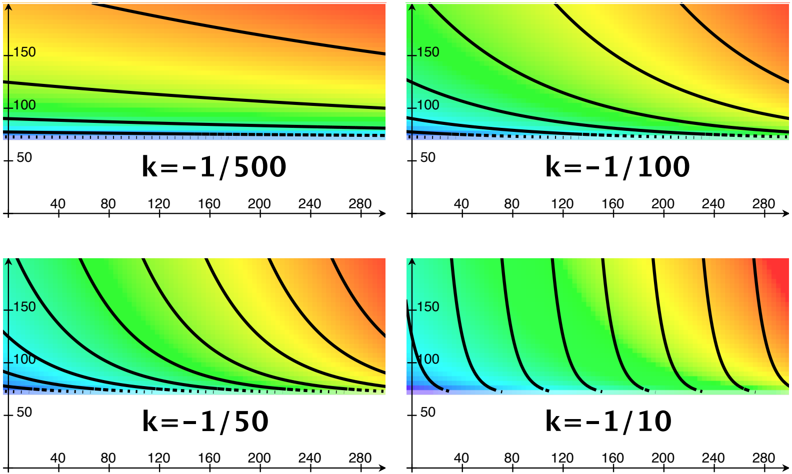

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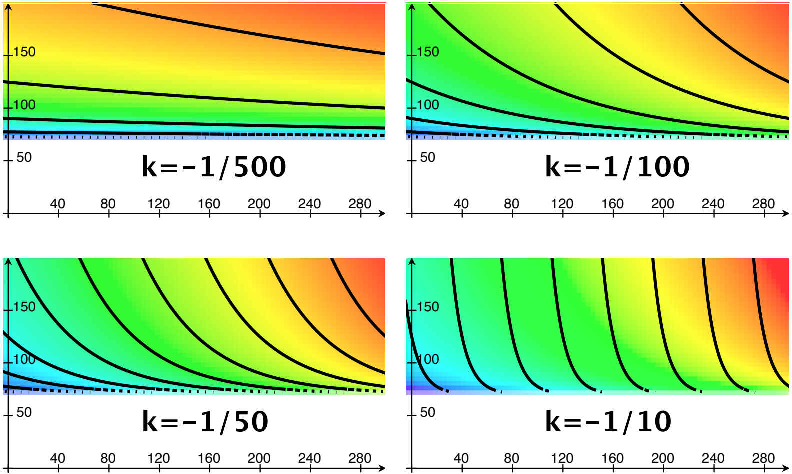

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Step 3: Plug in points and find particular solution

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Step 3: Plug in points and find particular solution

$$370 = y(0) = Ae^0 + 70, \qquad \text{so } \boxed{A = 300}$$

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Step 3: Plug in points and find particular solution

$$370 = y(0) = Ae^0 + 70, \qquad \text{so } \boxed{A = 300}$$

$$340 = y(10) = 300e^{k*10} + 70$$

,

$$\text{so } k = \frac{1}{10} \ln\left(\frac{350 - 70}{300}\right) = \boxed{\ln(.9)/10 \approx -0.0105}.$$

$$\boxed{\text{IVP: } \frac{dy}{dt} = k(y-70), \qquad y(0) = 370, \quad y(10) = 340}$$

General solution: $\boxed{y = Ae^{kt} + 70}$

What do we expect from k and A? $A > 0$, $k < 0$, and k small

Step 3: Plug in points and find particular solution

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,

$$\text{so } k = \frac{1}{10} \ln\left(\frac{350-70}{300}\right) = \boxed{\ln(.9)/10 \approx -0.0105}.$$

So the particular solution is

$$\boxed{y = 300e^{t*\ln(.9)/10} + 70}$$

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

(a) How hot is the pie after 20 minutes?

(b) How long will it take for the center of the pie to cool to 100°F?

Particular solution: $\boxed{y = 300e^{t*\ln(.9)/10} + 70}$

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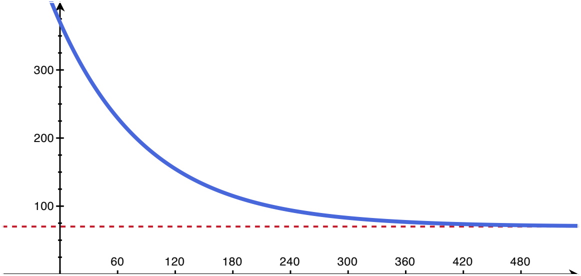

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Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

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Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

(b) $100 = 300e^{t*\ln(.9)/10} + 70$

So $e^{t*\ln(.9)/10} = 30/300 = 1/10$,

Example 2: Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F.

(a) How hot is the pie after 20 minutes?

(b) How long will it take for the center of the pie to cool to 100°F?

Particular solution: $\boxed{y = 300e^{t*\ln(.9)/10} + 70}$

Answers:

(a) $y(20) = 300e^{20*\ln(.9)/10} + 70 = \boxed{313}$

(b) $100 = 300e^{t*\ln(.9)/10} + 70$

So $e^{t*\ln(.9)/10} = 30/300 = 1/10$, and so

$$t = \frac{10}{\ln(.9)} \ln(.1) \approx \boxed{218.543}$$

Example 3: The isotope thorium-239 decays at a rate proportional to the amount present, and has a half-life of 24.1 days. How long does 10 grams of thorium-234 take to decay to 1 gram?

"Half-life": The time it takes for an amount of stuff to halve in size.

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To do:

Separate to get general solution;
Plug in points to get specific solution;
Solve $y(t) = 1$ for t