Modeling with Differential Equations: Introduction to the Issues

10/26/2011

Goal:
Given an equation relating a variable (e.g. x), a function (e.g. y), and its derivatives ($y', y'', \dots$), **what is y?**
i.e. How do I solve for y?

Goal:
Given an equation relating a variable (e.g. x), a function (e.g. y), and its derivatives $(y', y'', \dots)$, **what is y?**
i.e. How do I solve for y?

Why?
Many physical and biological systems can be modeled with differential equations. Also, it can be a lot harder to model a function long term than it is to measure how something changes as the system goes from one state to another.

Some examples

Obvervation: The rate of increase of a bacterial culture is proportional to the number of bacteria present at that time.

Obvervation: The motion of a mass on a spring is given by two opposing forces: (1) the force exerted by the mass in motion ($F = ma = m\frac{d^2}{dt^2}D$) and (2) the force exerted by the spring, proportional to the displacement from equilibrium ($F = kD$).

Some examples

Obvervation: The rate of increase of a bacterial culture is proportional to the number of bacteria present at that time.

Equation: $\dfrac{dP}{dt} = kP$

Obvervation: The motion of a mass on a spring is given by two opposing forces: (1) the force exerted by the mass in motion ($F = ma = m\frac{d^2}{dt^2}D$) and (2) the force exerted by the spring, proportional to the displacement from equilibrium ($F = kD$).

Equation: $m * \dfrac{d^2}{dt^2}D = -kD$

Some examples

Obvervation: The rate of increase of a bacterial culture is proportional to the number of bacteria present at that time.

Equation: $\dfrac{dP}{dt} = kP$

Solution: $P = Ae^{kt}$, where A is a constant.

(We did this in lecture 12)

Obvervation: The motion of a mass on a spring is given by two opposing forces: (1) the force exerted by the mass in motion ($F = ma = m\frac{d^2}{dt^2}D$) and (2) the force exerted by the spring, proportional to the displacement from equilibrium ($F = kD$).

Equation: $m * \dfrac{d^2}{dt^2}D = -kD$

Solution: $D = A\cos(t * \sqrt{k/m}) + B\sin(t * \sqrt{k/m})$, where A and B are constants.

(We did this in lecture 14, where $k/m = 1$)

Slope Fields

If you can write your differential equation like

$$\frac{dy}{dx} = F(x, y)$$

then you really have a way of saying

"If I'm standing at the point (a,b),
then I should move from here with slope $F(a, b)$."

Slope Fields

If you can write your differential equation like

$$\frac{dy}{dx} = F(x, y)$$

then you really have a way of saying

"If I'm standing at the point (a,b),
then I should move from here with slope $F(a, b)$."

Some examples:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dP}{dt} = kP$$

$$\frac{dx}{dt} = t^2 \sin(xt) + x^2$$

Slope Fields

If you can write your differential equation like

$$\frac{dy}{dx} = F(x, y)$$

then you really have a way of saying

"If I'm standing at the point (a,b),
then I should move from here with slope $F(a, b)$."

Some examples:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dP}{dt} = kP$$

$$\frac{dx}{dt} = t^2 \sin(xt) + x^2$$

Some non-examples:

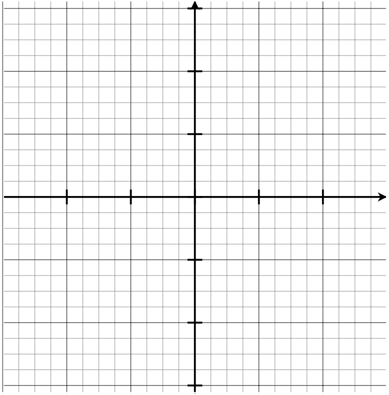
$$\frac{dy}{dx} = -\frac{x}{y} + \frac{d^2y}{dx^2}$$

$$\frac{dP}{dt} * \frac{d^2P}{dt^2} = kP$$

$$m * \frac{d^2D}{dt^2} = -kD$$

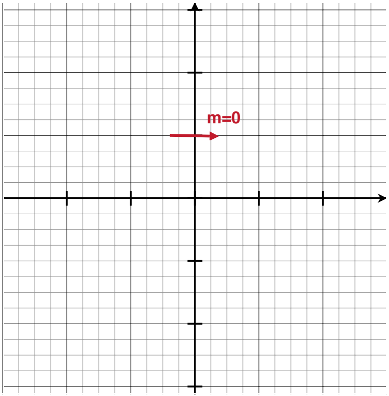
x	y	$\frac{dy}{dx} = -x/y$
0	1	
0	-1	
1	1	
1	-1	
-1	1	
-1	-1	
2	1	
1	2	
-2	0	

Slope field:



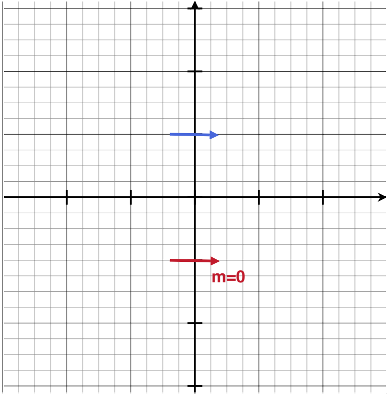
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	
1	1	
1	-1	
-1	1	
-1	-1	
2	1	
1	2	
-2	0	

Slope field:



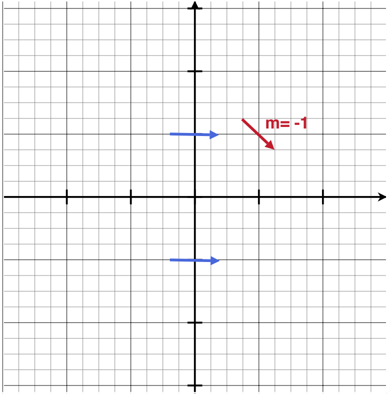
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	
1	-1	
-1	1	
-1	-1	
2	1	
1	2	
-2	0	

Slope field:



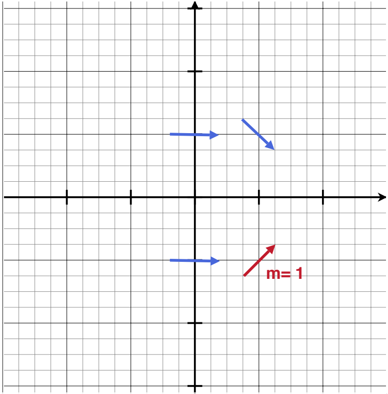
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	
-1	1	
-1	-1	
2	1	
1	2	
-2	0	

Slope field:



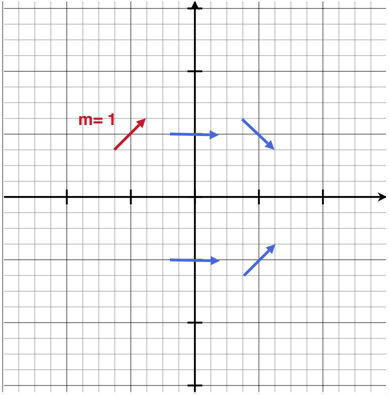
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	
-1	-1	
2	1	
1	2	
-2	0	

Slope field:



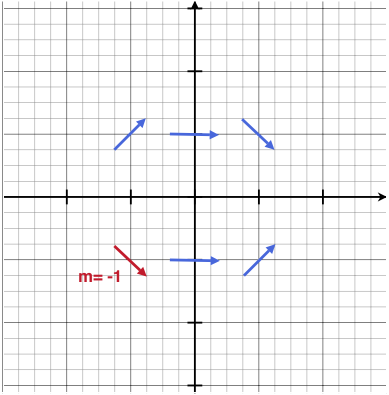
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	
2	1	
1	2	
-2	0	

Slope field:



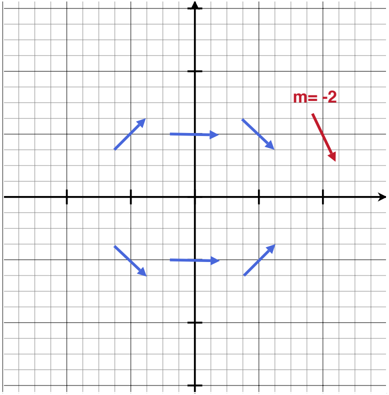
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	
1	2	
-2	0	

Slope field:



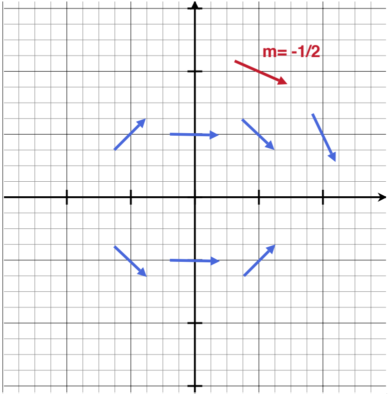
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	
-2	0	

Slope field:



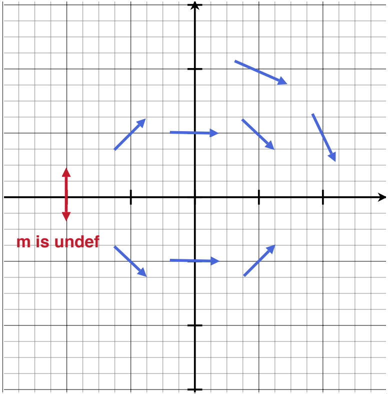
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	$-1/2$
-2	0	

Slope field:



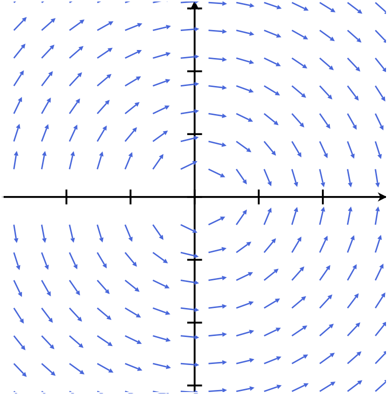
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	$-1/2$
-2	0	undef

Slope field:



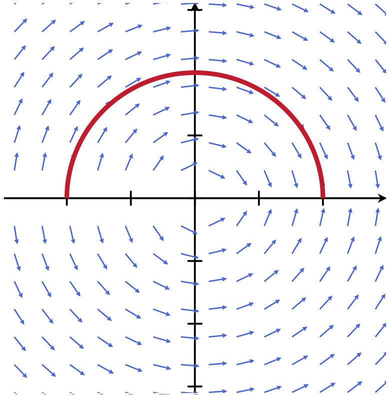
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	$-1/2$
-2	0	undef

Slope field:



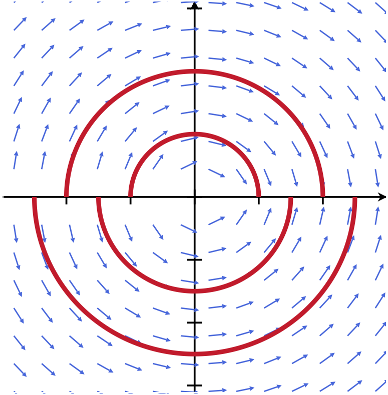
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	$-1/2$
-2	0	undef

Slope field:



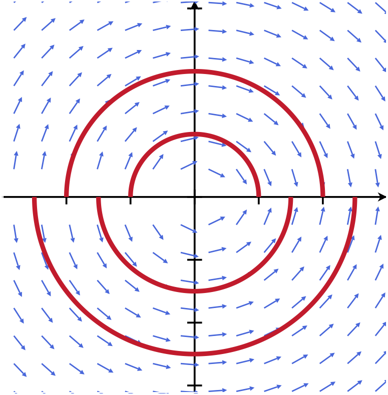
x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	$-1/2$
-2	0	undef

Slope field:



x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	$-1/2$
-2	0	undef

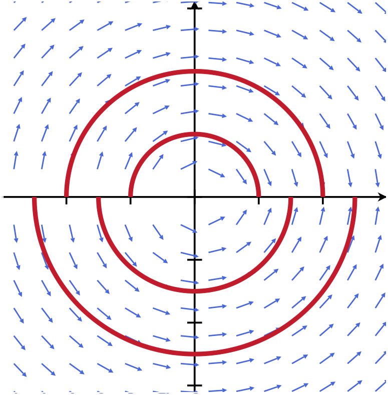
Slope field:



Arrows point in the direction of semicirles! $y = \pm\sqrt{r^2 - x^2}$?

x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	−1	0
1	1	− 1
1	−1	1
−1	1	1
−1	−1	− 1
2	1	− 2
1	2	− 1/2
−2	0	undef

Slope field:

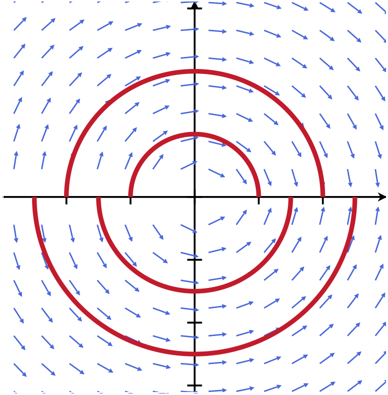


Arrows point in the direction of semicirles! $y = \pm\sqrt{r^2 - x^2}$?

Check: $$\frac{d}{dx} \pm \sqrt{r^2 - x^2} = \frac{-2x}{\pm 2\sqrt{r^2 - x^2}}$$

x	y	$\frac{dy}{dx} = -x/y$
0	1	0
0	-1	0
1	1	-1
1	-1	1
-1	1	1
-1	-1	-1
2	1	-2
1	2	$-1/2$
-2	0	undef

Slope field:



Arrows point in the direction of semicirles! $y = \pm\sqrt{r^2 - x^2}$?

Check: $$\frac{d}{dx} \pm \sqrt{r^2 - x^2} = \frac{-2x}{\pm 2\sqrt{r^2 - x^2}} = -\frac{x}{y} \quad ☺$$

Solving explicitly (get a formula!)

We've done. . .

1. Get lucky
   "what's a function you know whose derivative blah blah . . ."
2. Differential equations of the form

$$\frac{dy}{dx} = f(x)$$

   Find the antiderivative!

Today, we'll add

3. Differential equations of the form

$$\frac{dy}{dx} = f(x) * g(y)$$

   Use **"Separation of Variables"**

Separable Equations

A **separable differential equation** is one of the form

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

Separable Equations

A **separable differential equation** is one of the form

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

Some examples:

$$\frac{dy}{dx} = -\frac{x}{y} = (-x) * (\frac{1}{y})$$

$$\frac{dx}{dt} = t^2 \sec(x)$$

Separable Equations

A **separable differential equation** is one of the form

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

Some examples:

$$\frac{dy}{dx} = -\frac{x}{y} = (-x) * (\frac{1}{y})$$

$$\frac{dx}{dt} = t^2 \sec(x)$$

Some non-examples:

$$\frac{dy}{dx} = x + y$$

$$\frac{dx}{dt} = \frac{t + x}{xt^2}$$

Separable Equations

A **separable differential equation** is one of the form

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

Some examples:

$$\frac{dy}{dx} = -\frac{x}{y} = (-x) * (\frac{1}{y})$$

$$\frac{dx}{dt} = t^2 \sec(x)$$

Some non-examples:

$$\frac{dy}{dx} = x + y$$

$$\frac{dx}{dt} = \frac{t + x}{xt^2}$$

A separable equation is one in which we can put all of the y's and dy's (as products) on one side of the equation and all of the x's and dx's (as products) on the other...

Examples

(1) If $\quad \dfrac{dy}{dx} = -\dfrac{x}{y}, \qquad$ then $\quad y\ dy = -x\ dx.$

Examples

$$\text{(1) If} \quad \frac{dy}{dx} = -\frac{x}{y}, \qquad \text{then} \quad y\ dy = -x\ dx.$$

$$\text{(2) If} \quad \frac{dx}{dt} = t^2 \sec(x), \qquad \text{then} \quad \cos(x)\ dx = t^2\ dt.$$

Examples

$$(1) \text{ If } \quad \frac{dy}{dx} = -\frac{x}{y}, \qquad \text{then} \quad y\ dy = -x\ dx.$$

$$(2) \text{ If } \quad \frac{dx}{dt} = t^2 \sec(x), \qquad \text{then} \quad \cos(x)\ dx = t^2\ dt.$$

To solve (1), integrate both sides:

$$\int y\ dy = \int -x\ dx$$

Examples

$$(1)\ \text{If}\quad \frac{dy}{dx} = -\frac{x}{y}, \qquad \text{then}\quad y\ dy = -x\ dx.$$

$$(2)\ \text{If}\quad \frac{dx}{dt} = t^2 \sec(x), \qquad \text{then}\quad \cos(x)\ dx = t^2\ dt.$$

To solve (1), integrate both sides:

$$y^2/2 + c_2 = \int y\ dy = \int -x\ dx = -x^2/2 + c_1$$

Examples

$$(1)\ \text{If} \quad \frac{dy}{dx} = -\frac{x}{y}, \qquad \text{then} \quad y\ dy = -x\ dx.$$

$$(2)\ \text{If} \quad \frac{dx}{dt} = t^2 \sec(x), \qquad \text{then} \quad \cos(x)\ dx = t^2\ dt.$$

To solve (1), integrate both sides:

$$y^2/2 + c_2 = \int y\ dy = \int -x\ dx = -x^2/2 + c_1$$

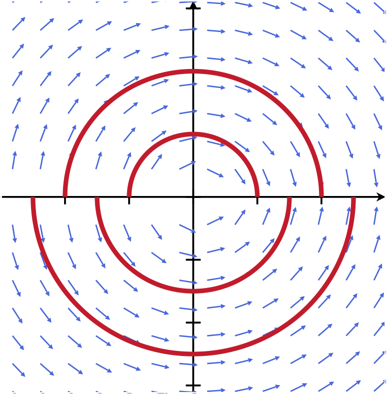
So

$$y = \pm\sqrt{2(-x^2/2 + c_1 - c_2)} = \pm\sqrt{a - x^2}$$

where $a = 2(c_1 - c_2)$.

Examples

Slope field for $\frac{dy}{dx} = -\frac{x}{y}$:



Suggested and checked $y = \pm\sqrt{r^2 - x^2}$

Examples

$$(1)\ \text{If}\quad \frac{dy}{dx} = -\frac{x}{y}, \qquad \text{then}\quad y\ dy = -x\ dx.$$

$$(2)\ \text{If}\quad \frac{dx}{dt} = t^2 \sec(x), \qquad \text{then}\quad \cos(x)\ dx = t^2\ dt.$$

To solve (1), integrate both sides:

$$y^2/2 + c_2 = \int y\ dy = \int -x\ dx = -x^2/2 + c_1$$

So

$$y = \pm\sqrt{2(-x^2/2 + c_1 - c_2)} = \pm\sqrt{a - x^2}$$

where $a = 2(c_1 - c_2)$.

Find an implicit formula for (2) (with no derivatives left in it)

How many solutions are there?

Existence?

How do I know I even get a solution?
An important result in the theory of differential equations is **Peano's Existence Theorem**, which states...

> If $\frac{dy}{dx} = F(x, y)$ and $y(a) = b$,
> where $F(x, y)$ is continuous in a domain D,
> then there is always **at least one solution** in the domain, and any such solution is differentiable.

Uniqueness?

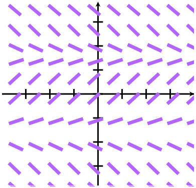
How do we know that there is not another solution?

> If, additionally, $F(x, y) = f(x)g(y)$, and if g' and f' are continuous, then solution is unique.

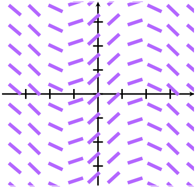
(1) Match the differential equations to the slope fields:

$$(A)\frac{dy}{dx} = \frac{1}{5}xy \quad (B)\frac{dy}{dx} = x+y \quad (C)\frac{dy}{dx} = \cos(x) \quad (D)\frac{dy}{dx} = \cos(y)$$

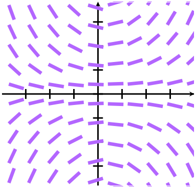
(a)



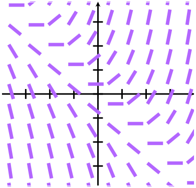
(b)



(c)



(d)



(2) Solve the initial value problems

$$(a) \ \frac{dy}{dx} = \frac{1}{5}xy, \qquad y(0) = 2;$$

$$(b) \ \frac{dy}{dx} = \sin(x)/y^2, \qquad y(0) = 3.$$

(1)

A: (c) B: (d) C: (b) D: (a)

(2)

(a) $\frac{dy}{dx} = \frac{1}{5} x y \rightarrow \frac{1}{y} dy = \frac{1}{5} x \, dx$

so

$$\int \frac{1}{y} dy = \int \frac{1}{5} x \, dx$$

thus

$$\ln(y) = \frac{1}{10} x^2 + c$$

$$\Rightarrow \boxed{y = e^{\frac{1}{10}x^2 + c} = e^{1/10 \, x^2} * e^c}$$ general

if $y(0) = 2$, then

$$2 = \underbrace{e^{\frac{1}{10} \cdot 0}}_{1} * e^c = e^c$$

so $$\boxed{y = e^{\frac{1}{10}x^2} * 2}$$ specific

(2) b

$$\frac{dy}{dx} = \sin(x)\, y^2 \quad \Rightarrow \quad y^2\, dy = \sin(x)\, dx$$

so

$$\int y^2\, dy = \int \sin(x)\, dx$$

so

$$\frac{1}{3} y^3 = -\cos(x) + c$$

$$\Rightarrow \quad \boxed{y = \sqrt[3]{3(-\cos(x) + c)}}$$

general

so since $y(0) = 3$,

$$3 = \sqrt[3]{3(-1 + c)}$$

$$9 = \frac{3^3}{3} = -1 + c$$

so $c = 10$

$$\boxed{y = \sqrt[3]{3(-\cos(x) + 10)}}$$

specific