Antiderivatives and Initial Value Problems

10/19/2011

Warm up

If $\dfrac{d}{dx}f(x) = 2x$, what is $f(x)$?

Can you think of another function that $f(x)$ could be?

If $\dfrac{d}{dx}f(x) = 3x^2 + 1$, what is $f(x)$?

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Some other candidates:

$$f(x) = x^2 + 1, \quad x^2 - 2, \quad x^2 + 13\pi, \quad x^2 - 143.7$$

If $\frac{d}{dx}f(x) = 3x^2 + 1$, what is $f(x)$?

$$f(x) = x^3 + x$$

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Definition

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Examples:

1. An antiderivative of $f(x) = 2x$ is $F(x) = x^2$.
2. Another antiderivative of $f(x) = 2x$ is $F(x) = x^2 + 1$.
3. There are *lots* of antiderivatives of $f(x) = 2x$ which look like $F(x) = x^2 + c$.

Suppose that h is differentiable in an interval I,
and $h'(x) = 0$ for all x in I.

Then h is a constant function!
i.e. $h(x) = C$ for all $x \in I$, where c is a constant.

So, if $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative must be of the form $F(x) + c$.

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So, if $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative must be of the form $F(x) + c$.

Example: **All** of the antiderivatives of $f(x) = 2x$ look like

$$F(x) = x^2 + c$$

for some constant c.

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Example:

$$\int 2x \; dx = x^2 + c.$$

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$$\int x^2 \; dx = \frac{1}{3}x^3 + c, \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = \frac{1}{3} * 3x^2 = x^2$$

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$$\int x^{-3} \; dx =$$

$$\int x^k \; dx =$$

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$$\int x^5 \, dx = \frac{1}{6}x^6 + c, \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{1}{6}x^6 + c\right) = \frac{1}{6} * 6x^5 = x^5$$

$$\int x^{-3} \, dx = \frac{1}{-2}x^{-2} + c, \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{1}{-2}x^{-2} + c\right) = x^{-3}$$

$$\int x^k \, dx = \frac{1}{k+1}x^{k+1} + c, \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{1}{k+1}x^{k+1} + c\right) = x^k$$

Except!! What if $k = -1$?

$$\int x^k \, dx = \frac{1}{k+1} x^{k+1} + c \qquad \text{if } k \neq -1$$

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$$\int \sin(x) \, dx = -\cos(x) + c$$

$$\int \cos(x) \, dx = \sin(x) + c$$

$$\int e^x \, dx = e^x + c$$

$$\int \sec^2(x) \, dx = \tan(x) + c$$

Theorem (Opposite of sum and constant rules)

Suppose the functions f and g both have antiderivatives on the interval I. Then for any constants a and b, the function $af + bg$ has an antiderivative on I and

$$\int \big(a * f(x) + b * g(x)\big) dx = a \int f(x) dx + b \int g(x) dx$$

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To solve: find a function $f(x)$ that satisfies the equation identically when substituted for the unknown function y, i.e. let $y = f(x)$.

Examples

Solve the differential equation $y' = 2x + sinx.$

Solve the differential equation $\dfrac{d^2y}{dx^2} + y = 0.$

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$$y = x^2 - \cos(x) + c$$

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$$y = \sin(x),\ y = \cos(x),\ y = a\cos(x) + b\sin(x)$$

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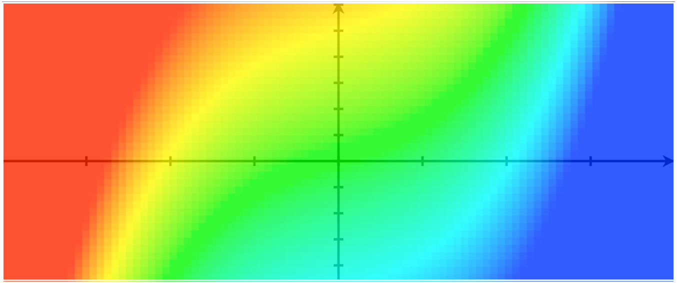

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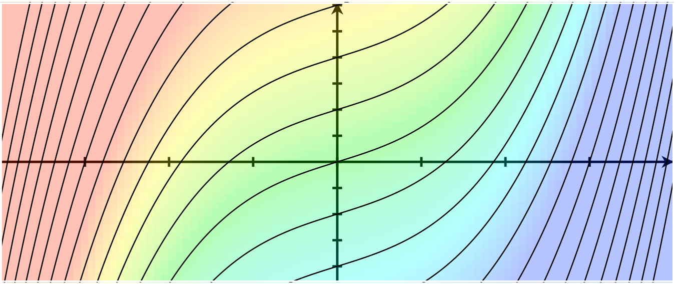

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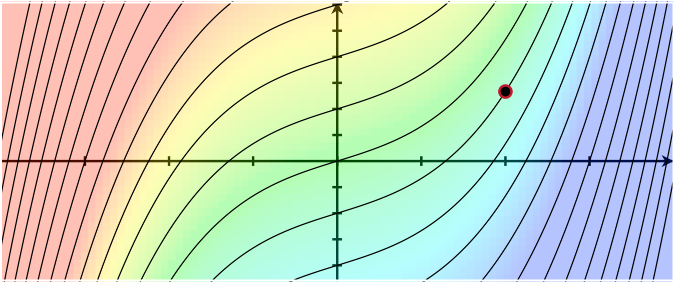

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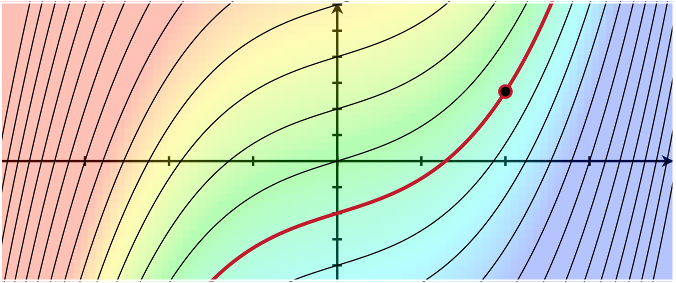

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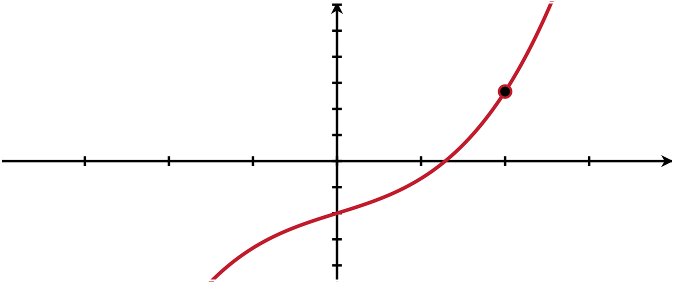
general solution:	$y = \frac{1}{3}x^3 + x + c$
particular solution:	$y = \frac{1}{3}x^3 + x - 2$



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Definition

An **initial-value problem** is a differential equation together with enough additional conditions to specify the constants of integration that appear in the general solution.

The **particular solution of the problem** is then a function that satisfies both the differential equation and also the additional conditions.

Solve the initial value problem

$$\frac{dy}{dx} = 2x + \sin(x)$$

subject to $y(0) = 0$.

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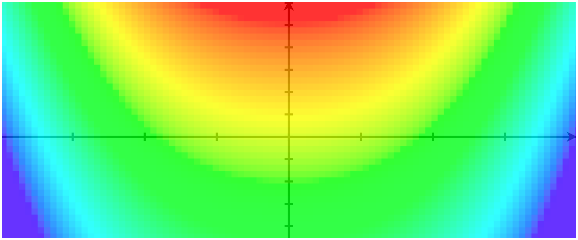

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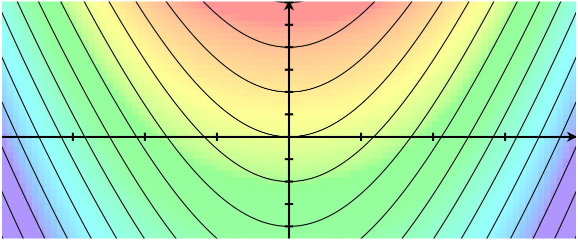

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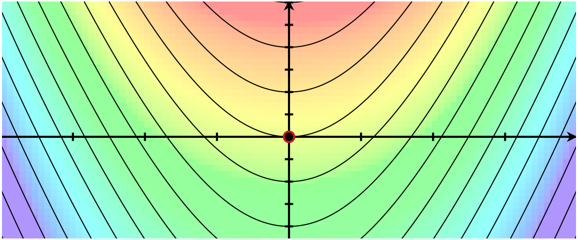

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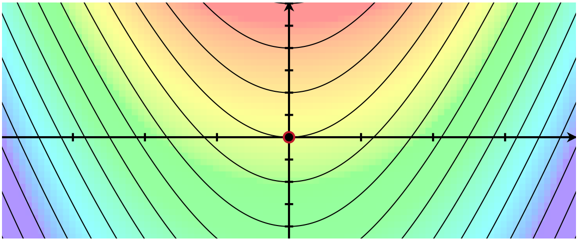

Algebraically: get a particular solution by solving

$$\mathbf{0} = \mathbf{y}(\mathbf{0}) = (0)^2 - \cos(0) + c = -1 + c \quad \text{(for } c\text{)}$$

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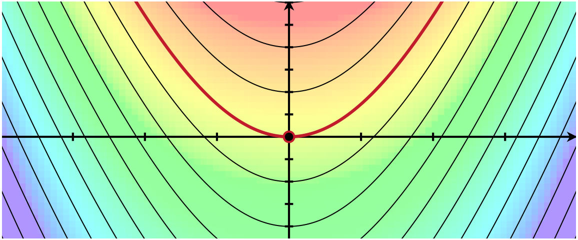

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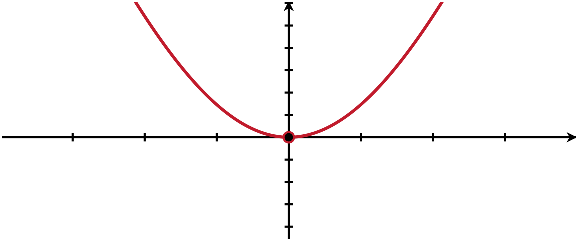

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Step 1: Calculate the antiderivative of $\cos(x)$ to find the general solution for y'.

Step 2: Plug in the values $y'(\frac{\pi}{2}) = 2$ to calculate c.

Step 3: Write down the *particular* solution for y'.

Step 4: Calculate the antiderivative of your particular solution in Step 3 to find the *general solution for y*.

Step 5: Plug in the values $y(\frac{\pi}{2}) = 3\pi$ to solve for the new constant.

Step 6: Write down the *particular* solution for y.

Solve the initial-value problem $y'' = \cos x,\ y'(\frac{\pi}{2}) = 2,\ y(\frac{\pi}{2}) = 3\pi.$

Step 1: Calculate the antiderivative of $\cos(x)$ to find the general solution for y'. Ans: $\boxed{y' = \sin(x) + c}$

Step 2: Plug in the values $y'(\frac{\pi}{2}) = 2$ to calculate c.
Ans: $2 = \sin(\pi/2) + c = 1 + c$, so $\boxed{c = 1}$

Step 3: Write down the *particular* solution for y'. Ans: $\boxed{y' = \sin(x) + 1}$

Step 4: Calculate the antiderivative of your particular solution in Step 3 to find the *general solution for* y.
Ans: $\boxed{y = -\cos(x) + x + d}$

Step 5: Plug in the values $y(\frac{\pi}{2}) = 3\pi$ to solve for the new constant.
Ans: $3\pi = -\cos(\pi/2) + \pi/2 + d = \pi/2 + d$ so $\boxed{d = 5\pi/2}$

Step 6: Write down the *particular* solution for y.
Ans: $\boxed{y' = -\cos(x) + x + 5\pi/2}$

Word problem:
An object dropped from a cliff has acceleration $a = -9.8\ m/sec^2$ under the influence of gravity. What is the function $s(t)$ that models its height at time t?

Initial value problem:
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Suppose that a baseball is thrown upward from the roof of a 100 meter high building. It hits the street below eight seconds later. What was the initial velocity of the baseball, and how high did it rise above the street before beginning its descent?

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$$\frac{d^2s}{dt^2} = -9.8, \qquad s(0) = 100, \; s(8) = 0.$$

Use your solution to
(1) calculate $s'(0)$, and
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