Newton's Method and Linear Approximations

10/19/2011

Curves are tricky. Lines aren't.

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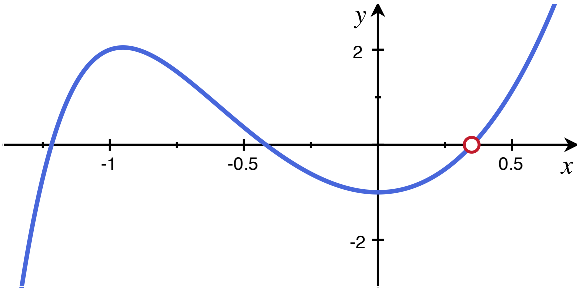
Newton's Method

Goal: Where is $f(x) = 0$?

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$



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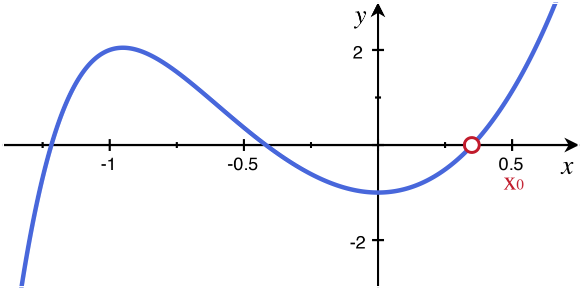

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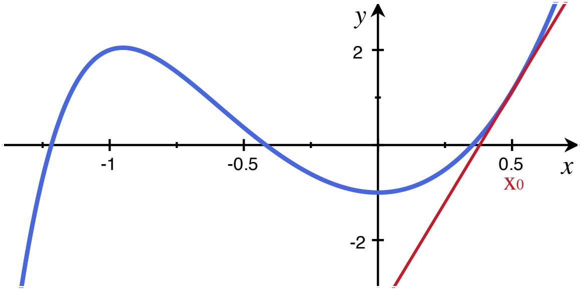

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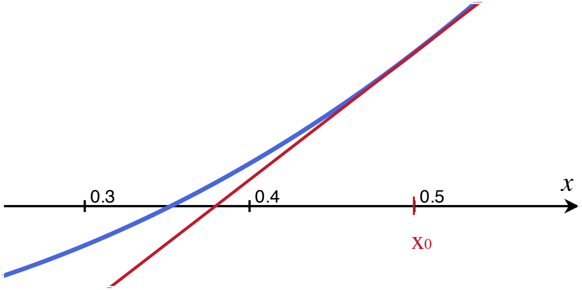

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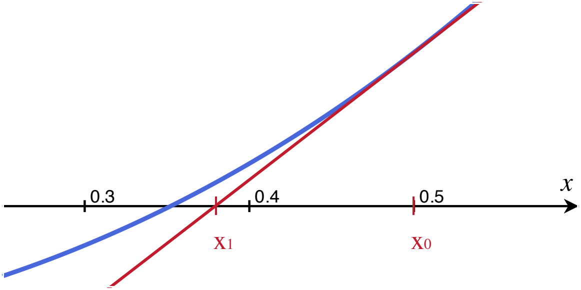

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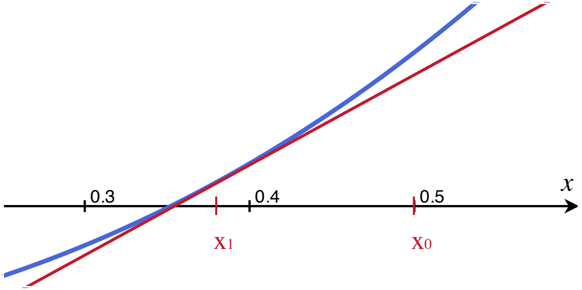

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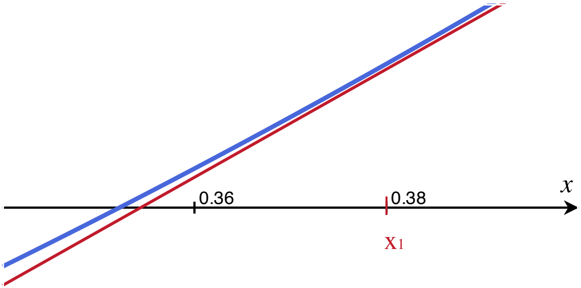

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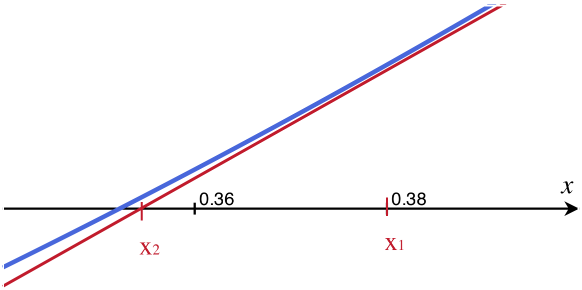

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$
$$f'(x) = 7x^6 + 9x^2 + 14x$$

i	x_i	$f(x_i)$	$f'(x_i)$	tangent line	x-intercept
0	0.5				
1					
2					
3					

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i	x_i	$f(x_i)$	$f'(x_i)$	tangent line	x-intercept
0	0.5	1.133	9.359	$y = 1.133 + 9.359(x - 0.5)$	0.379
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2					
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0	0.5	1.133	9.359	$y = 1.133 + 9.359(x - 0.5)$	0.379
1	0.379	0.170	6.619	$y = 0.170 + 6.619(x - 0.379)$	0.353
2					
3					

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i	x_i	$f(x_i)$	$f'(x_i)$	tangent line	x-intercept
0	0.5	1.133	9.359	$y = 1.133 + 9.359(x - 0.5)$	0.379
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3	0.352	0.00001	6.060	$y = 0.00001 + 6.060(x - 0.352)$	0.352

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Step 1: Pick a place to start. Call it x_0.

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$$0 = f(x_0) + f'(x_0) * (x - x_0) \quad \text{to get} \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This value is your x_1.

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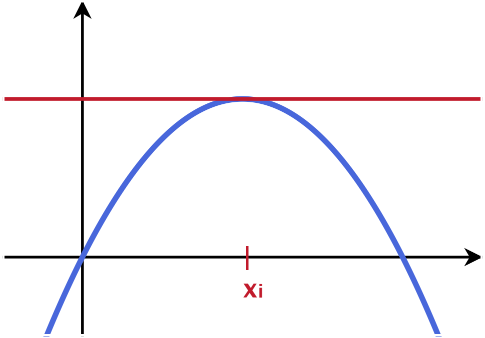
Step 4: Keep going until your x_i's stabilize.
What they stabilize to is an approximation of your root!

Caution!

Bad places to pick: Critical points! (where f'(x)=0)



Tangent line has no x-intercept!

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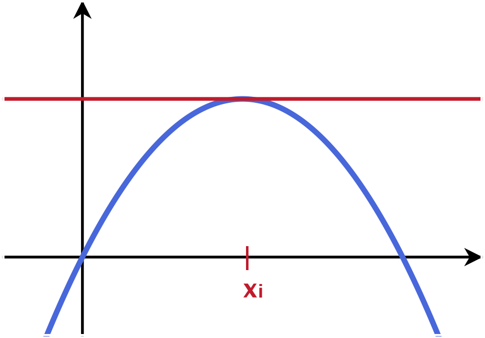
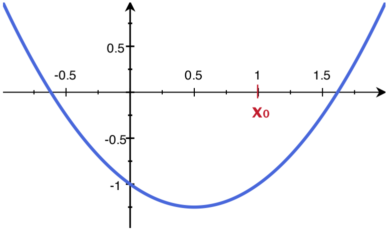


Tangent line has no x-intercept!

Even *near* critical points, the algorithm goes much slower.
Just stay away!

You try: Approximate a root of $f(x) = x^2 - x - 1$ near $x_0 = 1$.



$$f'(x) =$$

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	1			
1				
2				

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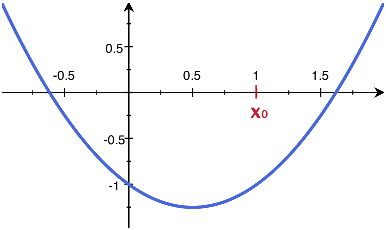

$$f'(x) = 2x - 1$$

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	1	-1	1	2
1	2	1	3	$5/3 \approx 1.667$
2	$5/3$	$1/9$	$7/3$	$34/21 \approx 1.619$

Back to the example:

$$f(x) = x^7 + 3x^3 + 7x^2 - 1$$

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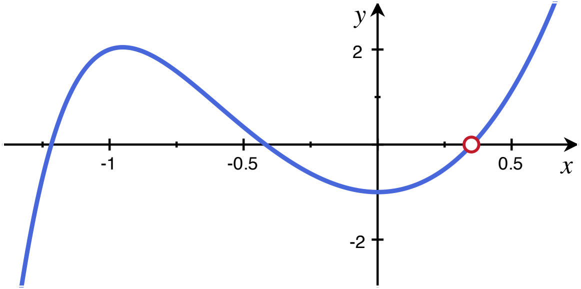

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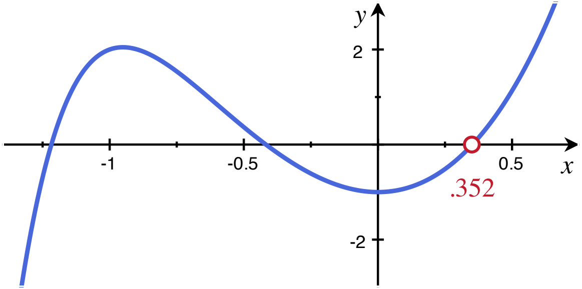

$r_3 \approx 0.352$

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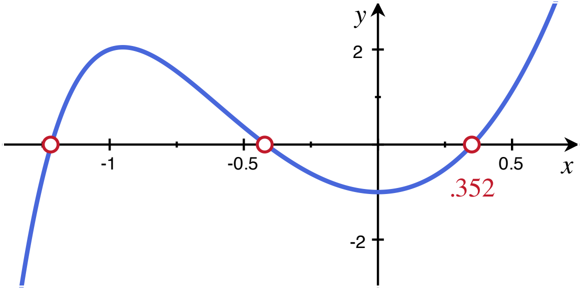

$r_1 \approx$ $\quad$ $r_2 \approx$ $\quad$ $r_3 \approx 0.352$

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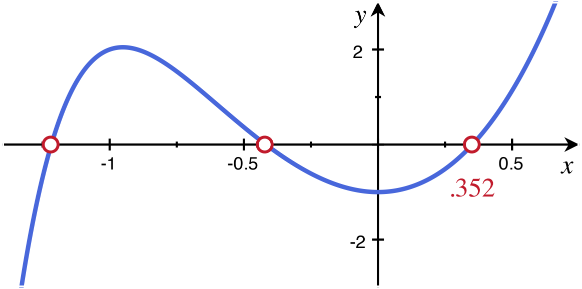

$r_1 \approx -1.217$ $\quad$ $r_2 \approx -0.418$ $\quad$ $r_3 \approx 0.352$

Linear approximations

Goal: approximate functions

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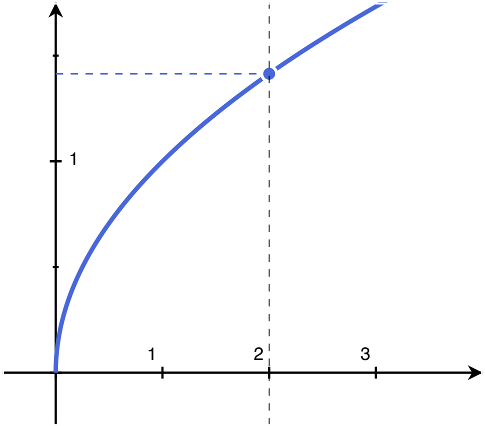
Example: approximate $\sqrt{2}$



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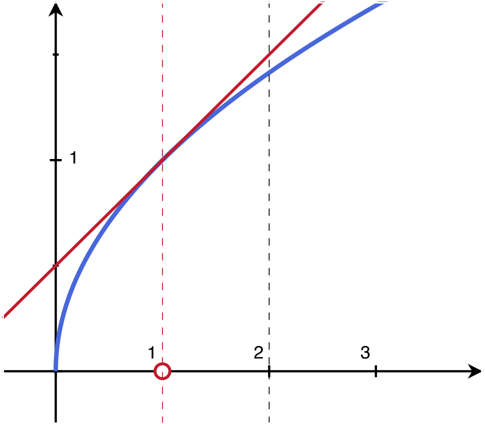

$$y = 1 + \frac{1}{2}(x-1)$$

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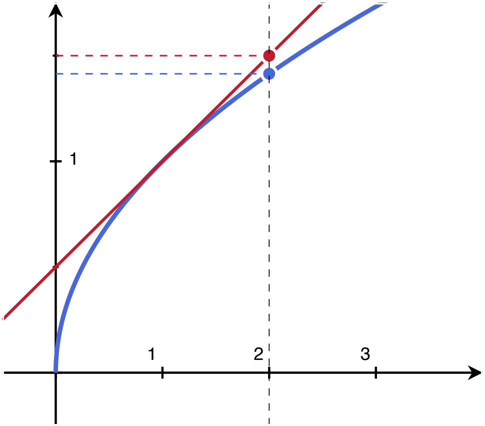

$$y = 1 + \frac{1}{2}(x - 1)$$

$$\sqrt{2} \approx 1 + \frac{1}{2}(2 - 1) = 1.5$$

Linear approximations

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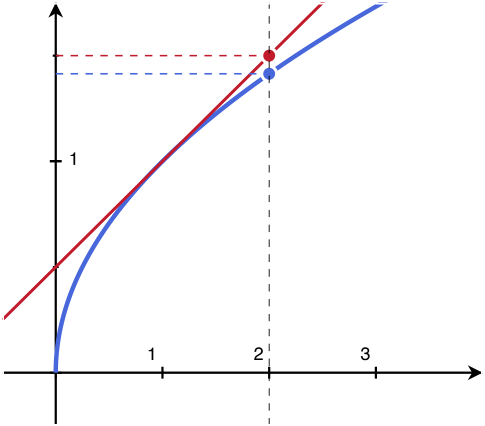

$$y = 1 + \frac{1}{2}(x - 1)$$

$$\sqrt{2} \approx 1 + \frac{1}{2}(2 - 1) = 1.5 \qquad (\sqrt{2} = 1.414...)$$

Linear approximations

If $f(x)$ is differentiable at a, then the tangent line to $f(x)$ at $x = a$ is

$$y = f(a) + f'(a) * (x - a).$$

For values of x *near* a, then

$$f(x) \approx f(a) + f'(a) * (x - a).$$

This is the *linear approximation* of f about $x = a$. We usually call the line $L(x)$.

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Our last approximation told us

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This isn't great... $(3^2 = 9)$

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The tangent line is

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

so

$$\sqrt{5} \approx L(5) = 2 + \frac{1}{4}(5 - 4) = \boxed{2.25}$$

Better! $(2.25^2 = 5.0625)$

Even better approximations...

The linear approximation is **the** line which satisfies

$$L(a) = f(a) + f'(a)(a - a) = \boxed{f(a)}$$

and

$$L'(a) = \frac{d}{dx}\big(f(a) + f'(a)(x - a)\big) = \boxed{f'(a)}$$

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A **better** approximation might be a quadratic polynomial $p_2(x)$ which **also** satisfies $p_2''(a) = f''(a)$:

$$\boxed{p_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2}$$

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or a cubic polynomial $p_3(x)$ which also satisfies $p_3^{(3)}(a) = f^{(3)}(a)$:

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and so on...

These approximations are called **Taylor polynomials** (read §2.14)