The Mean Value Theorem

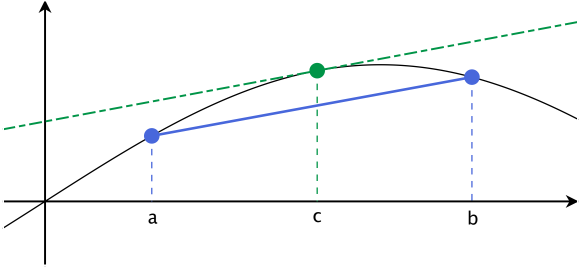
Oct 14 2011

The Mean Value Theorem

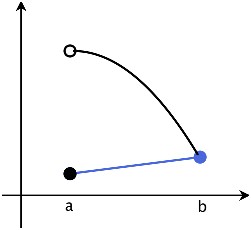
Theorem

Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that f' exists on the open interval (a, b). Then there exists a point c in (a, b) such that

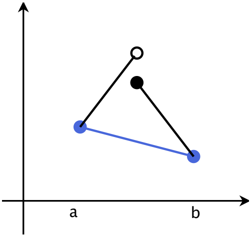
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



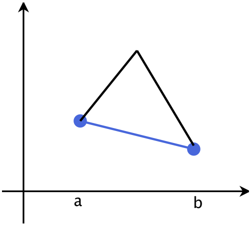
Bad examples



Discontinuity
at an endpoint



Discontinuity
at an interior point



No derivative
at an interior point

Examples

Does the mean value theorem apply to $f(x) = |x|$ on $[-1, 1]$?

How about to $f(x) = |x|$ on $[1, 5]$?

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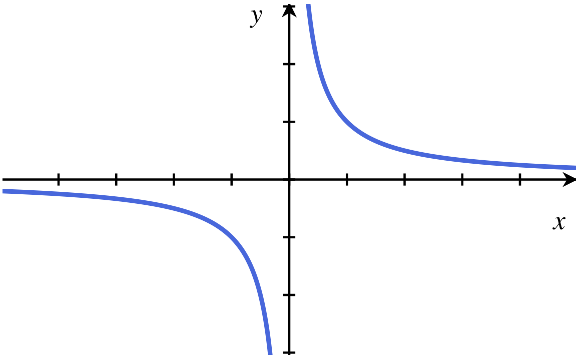
(No! Because $f(x)$ is not differentiable at $x = 0$.)

How about to $f(x) = |x|$ on $[1, 5]$?

(Yes! Because $f(x) = x$ on this domain, which is differentiable.)

Example

Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$?



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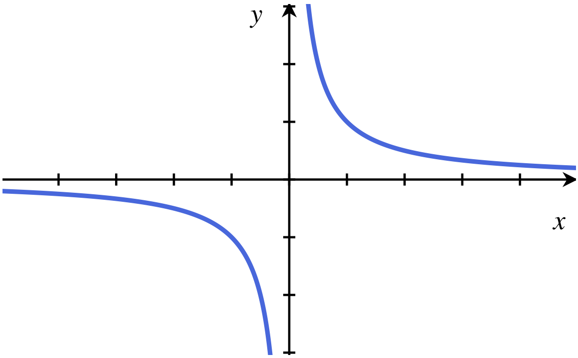

ANY closed interval on the domain!

Example

Let's verify the conclusion of the Mean Value Theorem for the function $f(x) = (x+1)^3 - 1$ on the interval $[-3, 1]$...

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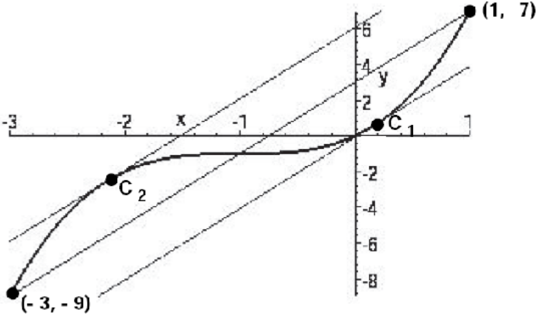

Optimizing functions!

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Some definitions:

If f is defined on an interval (a, b), then

1. f is *increasing* on (a, b) if, for any two points x_1 and x_2 in (a, b), we get

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2.$$

2. f is *nondecreasing* on (a, b) if, for any two points x_1 and x_2 in (a, b), we get

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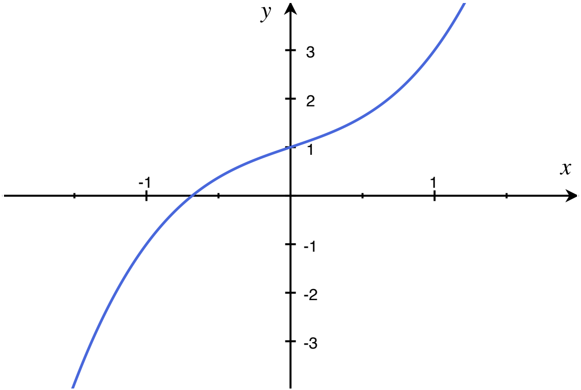
$$\text{Think about } \lim_{h \to 0} \frac{f(x_1 + h) - f(x)}{h}.$$

It's positive!

f is **increasing** on (a, b) $\implies$ $f'(x)$ is **positive**
f is **nondecreasing** on (a, b) $\implies$ $f'(x)$ is **nonnegative**
f is **decreasing** on (a, b) $\implies$ $f'(x)$ is **negative**
f is **nonincreasing** on (a, b) $\implies$ $f'(x)$ is **nonpositive**

Examples

On what interval is the function $f(x) = x^3 + x + 1$ increasing (decreasing)?



$$f'(x) = 3x^2 + 1$$

Find the intervals on which the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing.

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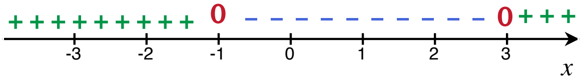

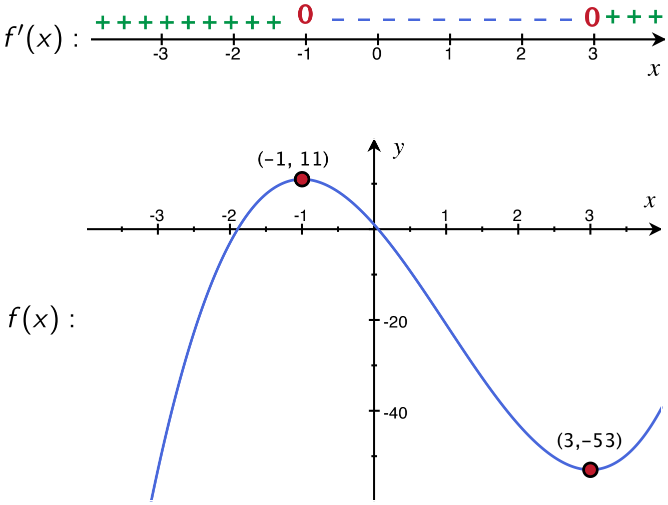

f′(x) :
+ + + + + + + + +
0
- - - - - - - - - - -
0
+ + +
-3
-2
-1
0
1
2
3
x
f(x) :
(−1, 11)
(3,−53)
y
x
-3
-2
-1
1
2
3
-20
-40

If f is continuous on a closed interval $[a, b]$, then there is a point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).

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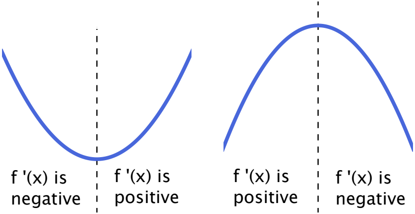
what's going on right before c?
what's going on right after c?



Example

For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

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3	53
-4	-151
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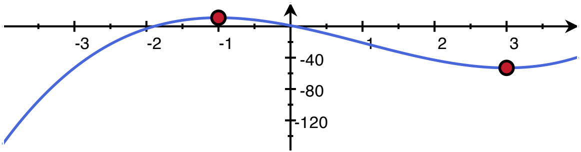
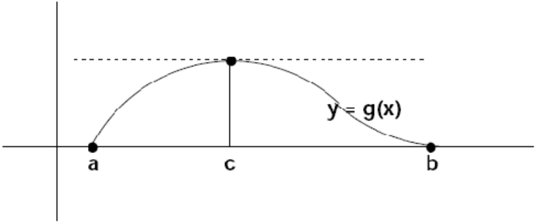


Rolle's Theorem

Theorem

Suppose that the function g is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $g(a) = 0$ and $g(b) = 0$ then there exists a point c in the open interval (a, b) where $g'(c) = 0$.



Let's use Rolle's Theorem to show that the equation $x^5 - 3x + 1 = 0$ has exactly three real roots!

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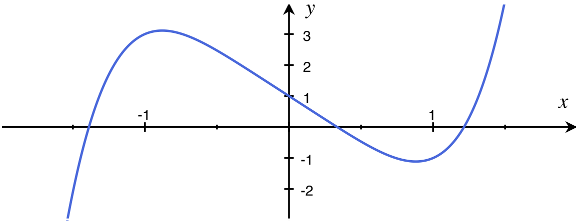


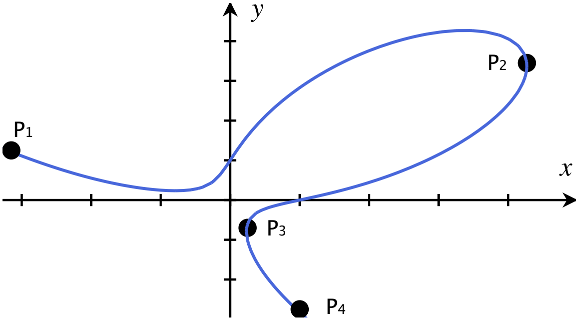
Implicit Differentiation

Many curves are not the graphs of functions.

For example, circle of radius 1 does not pass the “vertical line test” and hence is not the graph of a function.

It is, however, the graph of the equation $x^2 + y^2 = 1$!

Another example: the equation $x^3 - 8xy + y^3 = 1$ cannot be explicitly solved for y as a function of x.



GOAL: Find the derivative of $f(x)$ without explicitly solving the equation.

Implicit Differentiation

Think of y as a mystery function.
Whenever you have to take it's derivative, just call it y'!

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$$\text{LHS: } \frac{d}{dx}(1) = 0, \text{ and}$$

$$\text{RHS: } \frac{d}{dx}(x^2 + y^2) = 2x + 2y * y' \text{ (chain rule!)}$$

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$$\text{RHS: } \frac{d}{dx}(x^2 + y^2) = 2x + 2y * y' \text{ (chain rule!)}$$

So $2x + 2y * y' = 0$... solve for y'!

$$y' = -\frac{x}{y}$$

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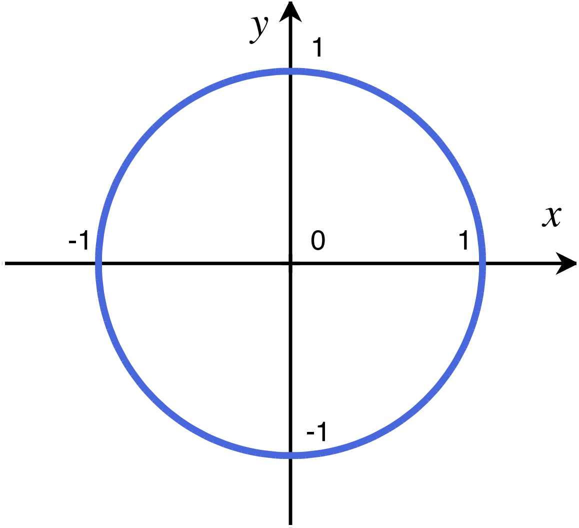
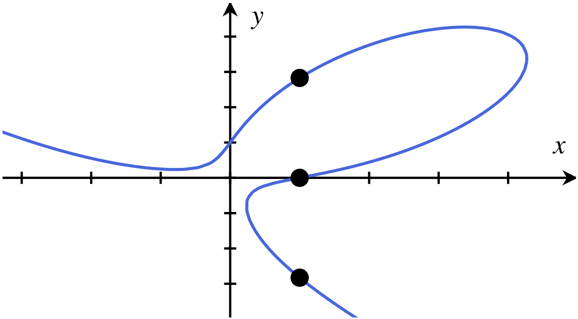


1. Use implicit differentiation to find y' when $xy^2 + x^2y - 6 = 0$.

2. Find the equation of the tangent line to the graph of $xy^2 + x^2y - 6 = 0$ at the point $(1, 2)$.

Return to the equation $x^3 - 8xy + y^3 = 1$ with which we begin this section. Find the slope at the points on the curve for which $x = 1$.



Suppose a differentiable function f has an inverse f^{-1}. Find the derivative of f^{-1}.