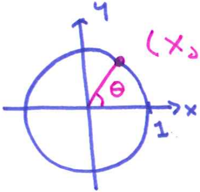
Derivatives of the Trigonometric Functions

Oct 12 2011

Trig Identities:



$(x, y) = (\cos\theta, \sin\theta)$

Resulting identities:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\cos^2(\theta) + \sin^2\theta = 1$$

Other useful identities:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

The derivative of sine

$$\frac{d}{dx}\sin x =$$

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$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

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$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

The derivative of sine

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}
\end{aligned}$$

The derivative of sine

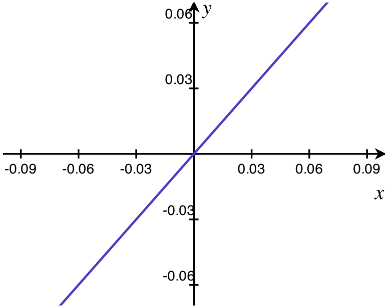
$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}
$$

The derivative of sine

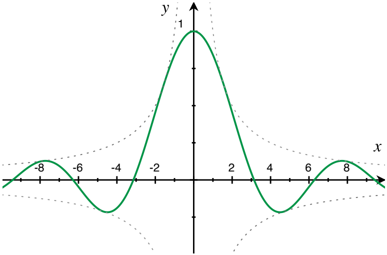
$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}
\end{aligned}$$

Recall: $\cos(0) = 1$ and $\sin(0) = 0$

Near $x = 0$, $\sin(x) \approx x$:

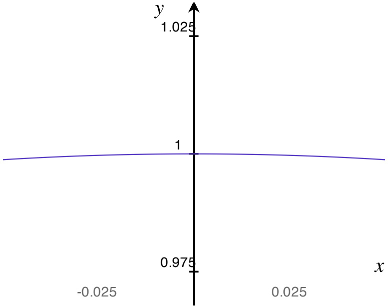


Graph of $\frac{\sin(x)}{x}$:

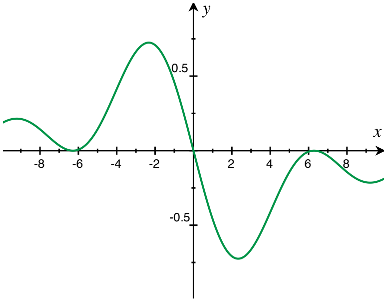


$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Near $x = 0$, $\cos(x) \approx 1$:



Graph of $\frac{\cos(x)-1}{x}$:



$$\boxed{\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0}$$

The derivative of sine

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}
\end{aligned}
$$

The derivative of sine

$$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \sin(x) * 0 + \cos(x) * 1$$

The derivative of sine

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}$$

$$= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)*0+\cos(x)*1$$

$$= \boxed{\cos(x)}$$

The derivative of cosine

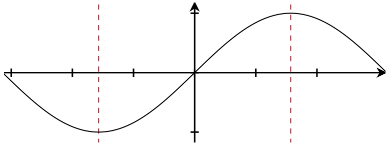
$$\frac{d}{dx}\cos x =$$

The derivative of cosine

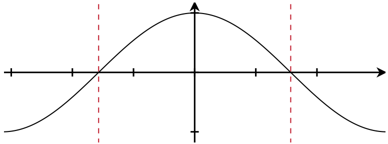
$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

Does it make sense?

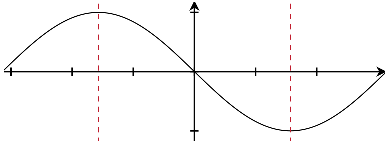
$y = \sin(x)$:



$y = \cos(x)$:



$y = -\sin(x)$:



Examples

On your own, calculate:

1. $\frac{d}{dx}\sin(2x)$
2. $\frac{d}{dx}\sin\left(x^2+\frac{1}{x}\right)$
3. $\frac{d}{dx}\cos(3x+\sqrt{x})$
4. $\frac{d}{dx}\sin(x)\cos(x)$
5. $\frac{d}{dx}\sin(\cos(x^2+2))$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x)$

2. $\frac{d}{dx}\cot(x)$

3. $\frac{d}{dx}\sec(x)$

4. $\frac{d}{dx}\csc(x)$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)}$

2. $\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)}$

3. $\frac{d}{dx}\sec(x) = \frac{d}{dx}(\cos(x))^{-1}$

4. $\frac{d}{dx}\csc(x) = \frac{d}{dx}(\sin(x))^{-1}$

Example

Compute the derivative of

$$y = \left(x + \tan^3\left(\csc^2(17x)\right)\right)^4.$$