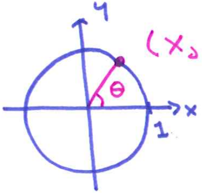
Derivatives of the Trigonometric Functions

Oct 12 2011

Trig Identities:



$(x, y) = (\cos\theta, \sin\theta)$

Resulting identities:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\cos^2(\theta) + \sin^2\theta = 1$$

Other useful identities:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

The derivative of sine

$$\frac{d}{dx}\sin x =$$

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$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

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$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

The derivative of sine

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h}$$

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$$= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

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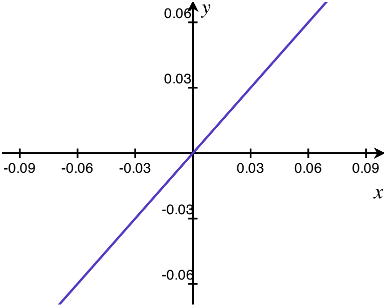
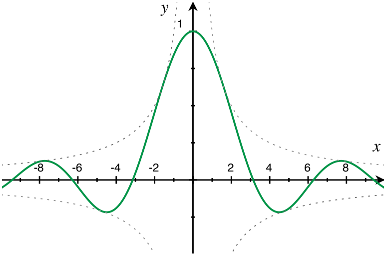
$$= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

Recall: $\cos(0)=1$ and $\sin(0)=0$

Near $x = 0$, $\sin(x) \approx x$:

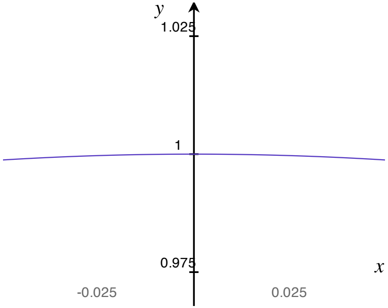


Graph of $\frac{\sin(x)}{x}$:

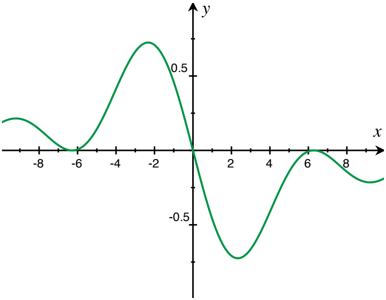


$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Near $x = 0$, $\cos(x) \approx 1$:



Graph of $\frac{\cos(x)-1}{x}$:



$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

The derivative of sine

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}
\end{aligned}
$$

The derivative of sine

$$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \sin(x) * 0 + \cos(x) * 1$$

The derivative of sine

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}\\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \sin(x)*0+\cos(x)*1\\
&= \boxed{\cos(x)}
\end{aligned}$$

The derivative of cosine

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$$= \lim_{h\to 0}\frac{\cos(x)(\cos(h)-1) - \sin(x)\sin(h)}{h}$$

The derivative of cosine

$$
\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h} \\
&= \cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}
\end{aligned}
$$

The derivative of cosine

$$\frac{d}{dx}\cos x = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h}$$

$$= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$

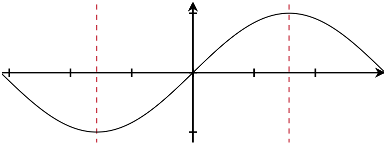
$$= \cos(x) * 0 - \sin(x) * 1$$

The derivative of cosine

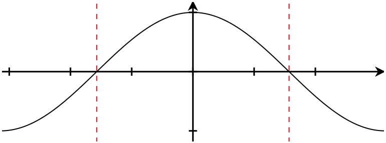
$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h} \\
&= \cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \cos(x)*0-\sin(x)*1 \\
&= \boxed{-\sin(x)}
\end{aligned}$$

Does it make sense?

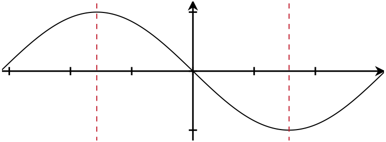
$y = \sin(x)$:



$y = \cos(x)$:



$y = -\sin(x)$:



Examples

On your own, calculate:

1. $\frac{d}{dx}\sin(2x)$
2. $\frac{d}{dx}\sin\left(x^2+\frac{1}{x}\right)$
3. $\frac{d}{dx}\cos(3x+\sqrt{x})$
4. $\frac{d}{dx}\sin(x)\cos(x)$
5. $\frac{d}{dx}\sin(\cos(x^2+2))$

Examples

On your own, calculate:

1. $\frac{d}{dx}\sin(2x) = \boxed{2 * \sin(2x)}$
2. $\frac{d}{dx}\sin\left(x^2 + \frac{1}{x}\right)$
   $= \frac{d}{dx}\sin\left(x^2 + x^{-1}\right) = \boxed{(2x - x^{-2})\cos(x^2 + x^{-1})}$
3. $\frac{d}{dx}\cos(3x + \sqrt{x})$
   $= \frac{d}{dx}\cos(3x + x^{1/2}) = \boxed{(3 + \frac{1}{2}x^{-\frac{1}{2}})(-\sin(3x + x^{1/2}))}$
4. $\frac{d}{dx}\sin(x)\cos(x) = \sin(x)(-\sin(x)) + \cos(x)\cos(x)$
   $= \boxed{\cos^2(x) - \sin^2(x)} = \cos(2x)$
5. $\frac{d}{dx}\sin(\cos(x^2 + 2)) = \cos(\cos(x^2 + 2)) * \frac{d}{dx}\left(\cos(x^2 + 2)\right)$
   $$= \cos(\cos(x^2 + 2)) * (-\sin(x^2 + 2)) * \frac{d}{dx}\left(x^2 + 2\right)$$
   $$= \boxed{\cos(\cos(x^2 + 2)) * (-\sin(x^2 + 2)) * (2x)}$$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x)$

2. $\frac{d}{dx}\cot(x)$

3. $\frac{d}{dx}\sec(x)$

4. $\frac{d}{dx}\csc(x)$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)}$

2. $\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)}$

3. $\frac{d}{dx}\sec(x) = \frac{d}{dx}(\cos(x))^{-1}$

4. $\frac{d}{dx}\csc(x) = \frac{d}{dx}(\sin(x))^{-1}$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{\cos(x)\cdot(\cos(x)) - \sin(x)\,(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= \boxed{\sec^2(x)}$$

$$\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)} = \cdots$$

$$= \frac{d}{dx}\left(\tan(x)\right)^{-1} = -\left(\tan(x)\right)^{-2}\cdot\sec^2(x)$$

$$= -\frac{\cos^2(x)}{\sin^2(x)}\cdot\frac{1}{\cos^2(x)} = -\csc^2(x$$

$$\frac{d}{dx}\left(\cos(x)\right)^{-1} = \frac{\ }{(\sin(x))^2}$$

$$= -\frac{1}{(\cos(x))^2} \cdot -\sin(x)$$

$$= \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x)$$

$$= \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)}$$

guess

$$\frac{d}{dx}\csc(x) = -\csc(x) \cdot \cot(x)$$

On your own, fill in the rest of the trig functions:

1. $\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \boxed{\sec^2(x)}$

2. $\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)} = \boxed{-\csc^2(x)}$

3. $\frac{d}{dx}\sec(x) = \frac{d}{dx}(\cos(x))^{-1} = \boxed{\sec(x)\tan(x)}$

4. $\frac{d}{dx}\csc(x) = \frac{d}{dx}(\sin(x))^{-1} = \boxed{-\csc(x)\cot(x)}$

Example

Compute the derivative of

$$y = \left(x + \tan^3\left(\csc^2(17x)\right)\right)^4.$$

$$\frac{d}{dx}\left(x + \tan^3(\csc^2(17x))\right)^4$$

$$= 4\left(x + \tan^3(\csc^2(17x))\right)^3$$

$$* \Big[1 + 3\tan^2(\csc^2(17x))$$

$$* \sec^2(\csc^2(17x))$$

$$* 2\csc(17x)$$

$$* (-\csc(17x)\cdot\cot(17x))$$

$$* 17 \Big]$$

this is all

$$\frac{d}{dx}\tan^3(\csc^2(17x))$$