Differentiation Rules

Oct 10 2011

Differentiability versus Continuity

Theorem

If $f'(a)$ exists, then f is continuous at a.

A function whose derivative exists at every point of an interval is

continuous and smooth,

i.e. it has no sharp corners.

Building the Toolbox

Theorem (Scalars)

If $y = f(x)$ is a differentiable function then,

$$(c \cdot f(x))' = c \cdot f'(x) \qquad \frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$

where c is a constant.

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Example

Since $\frac{d}{dx}x^2 = 2x$, we have $\frac{d}{dx}5x^2 = 5 \cdot 2x = 10x$.

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Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

In words,

the derivative of a sum is the sum of the derivatives.

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the derivative of a sum is the sum of the derivatives.

$$\frac{d}{dx}(3x^2+2x+7) = \frac{d}{dx}(3x^2)+\frac{d}{dx}(2x)+\frac{d}{dx}(7)$$

$$= 3\cdot 2x \quad + \quad 2\cdot 1 \quad + \quad 0 \qquad = \boxed{6x+2}$$

Calculate $\frac{d}{dx}\big((5x+3)(x+2)\big)$.

Caution! The answer is **not** *5. Expand first!*

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$$\begin{aligned}\frac{d}{dx}\big((5x+3)(x+2)\big) &= \frac{d}{dx}\big(5x^2+13x+6\big)\\ &= 5\cdot 2x+13\\ &= \boxed{\text{10x+13}}\end{aligned}$$

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Theorem (Products)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

In words, *the derivative of a product is*

the first times the derivative of the second,

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Try 2: Calculate $\frac{d}{dx}\big((5x+3)(x+2)\big)$:

$$\frac{d}{dx}\big((5x+3)(x+2)\big) = (5x+3)\cdot 1 + (x+2)\cdot 5 = \boxed{\text{10x+13}} \;☺$$

$$f\uparrow \qquad g\uparrow \qquad\qquad f \;\cdot\; g' \;+\; g \;\cdot\; f'$$

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Theorem (Reciprocals)

If $y = f(x)$ is a differentiable function, then wherever $f(x) \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}.$$

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Example:

Calculate $\frac{d}{dx}\left(\frac{1}{x^2}\right)$.

Here, $f(x) = x^2$, so

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2x}{(x^2)^2} = -\frac{2}{x^3} = -2x^{-3} \quad ☺$$

Some more examples of reciprocals

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Quotients:

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+3x+1}\right) = \frac{d}{dx}\left((x^2-1)\left(\frac{1}{x^2+3x+1}\right)\right)$$

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$$= (x^2-1)\frac{-(2x+3)}{(x^2+3x+1)^2} + \frac{1}{x^2+3x+1}(2x)$$

In general,

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Theorem (Quotient rule)

Wherever $g(x) \neq 0$,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

Building the Toolbox

Theorem

Let $(f \circ g)(x) = f(g(x))$.

Assume that g is differentiable at the point x and that f is differentiable at the point $g(x)$. Then the composite function $f \circ g$ is differentiable at the point x, and

$$\frac{d}{dx}((f \circ g)(x)) = \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Using Leibniz's notation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Chain rule: $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

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$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

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So

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 7x^6$$

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Back to reciprocals:

Chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$.

Calculate $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{d}{dx}\left((g(x))^{-1}\right)$

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Outside function: $f(x) = x^{-1}$
Inside function: $g(x)$

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So since

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we get

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{1}{(g(x))^2} \cdot g'(x) = -\frac{g'(x)}{(g(x))^2}$$

just like before!