Differentiation Rules

Oct 10 2011

Differentiability versus Continuity

Theorem

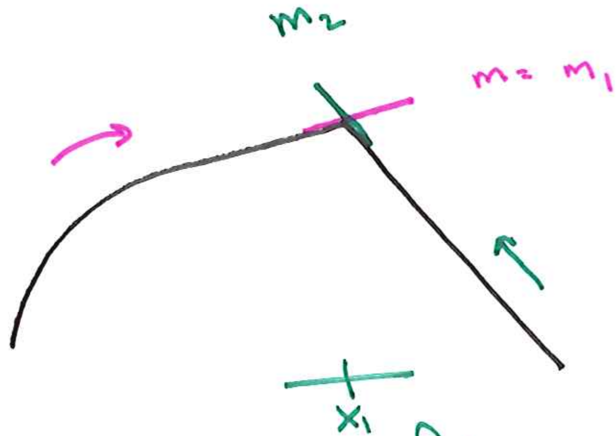
If $f'(a)$ exists, then f is continuous at a.

A function whose derivative exists at every point of an interval is

continuous and smooth,

i.e. it has no sharp corners.

ex of cont but not diff'ble :



* $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ DNE

$h \to 0^-$: m_1

$h \to 0^+$: m_2

* $\lim\limits_{x \to x_1} f(x)$ exists.

Building the Toolbox

Theorem (Scalars)

If $y = f(x)$ is a differentiable function then,

$$(c \cdot f(x))' = c \cdot f'(x) \qquad \frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$

where c is a constant.

pf

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

ex: $\frac{d}{dx} x^2 = 2x$

what is $\frac{d}{dx} 3x^2$.

$$\frac{d}{dx} cf(x)$$

$$= \lim_{h \to 0} \frac{\left(cf(x+h) - cf(x)\right)}{h}$$

$$= \lim_{h \to 0} \frac{c\left(f(x+h) - f(x)\right)}{h}$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= c\, f'(x).$$

▨

Building the Toolbox

Theorem (Scalars)

If $y = f(x)$ is a differentiable function then,

$$(c \cdot f(x))' = c \cdot f'(x) \qquad \frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$

where c is a constant.

Example

Since $\frac{d}{dx}x^2 = 2x$, we have $\frac{d}{dx}5x^2 = 5 \cdot 2x = 10x$.

Building the Toolbox

Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

In words,

the derivative of a sum is the sum of the derivatives.

$$\frac{d}{dx}\left(f(x)+g(x)\right) =$$

$$\lim_{h\to 0} \frac{f(x+h)+g(x+h) - \left(f(x)+g(x)\right)}{h}$$

$$\vdots$$

$$= \lim_{h\to 0} \frac{f(x+h) \cdot f(x)}{h}$$

$$+ \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Building the Toolbox

Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

In words,

the derivative of a sum is the sum of the derivatives.

$$\frac{d}{dx}(3x^2+2x+7) =$$

Building the Toolbox

Theorem (Sums)

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

In words,

the derivative of a sum is the sum of the derivatives.

$$\begin{aligned}\frac{d}{dx}(3x^2+2x+7) &= \frac{d}{dx}(3x^2)+\frac{d}{dx}(2x)+\frac{d}{dx}(7)\\ &= 3\cdot 2x \quad + \quad 2\cdot 1 \quad + \quad 0 \quad = \boxed{6x+2}\end{aligned}$$

Calculate $\frac{d}{dx}\big((5x+3)(x+2)\big)$.

Caution! The answer is **not** *5. Expand first!*

Calculate $\frac{d}{dx}\big((5x+3)(x+2)\big)$.

Caution! The answer is **not** *5. Expand first!*

$$\begin{aligned}\frac{d}{dx}\big((5x+3)(x+2)\big) &= \frac{d}{dx}\big(5x^2+13x+6\big)\\ &= 5\cdot 2x+13\\ &= \boxed{\text{10x+13}}\end{aligned}$$

Building the Toolbox

Theorem (Products)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

In words, *the derivative of a product is*

the first times the derivative of the second,

plus the second times the derivative of the first.

Building the Toolbox

Theorem (Products)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x).$$

In words, *the derivative of a product is*

the first times the derivative of the second,

plus the second times the derivative of the first.

Try 2: Calculate $\frac{d}{dx}\big((5x+3)(x+2)\big)$:

$$\frac{d}{dx}\big(\underbrace{(5x+3)}_{f\,\uparrow}\underbrace{(x+2)}_{g\,\uparrow}\big) = \underbrace{(5x+3)}_{f}\cdot\underbrace{1}_{g'} + \underbrace{(x+2)}_{g}\cdot\underbrace{5}_{f'} = \boxed{\text{10x+13}}\ ☺$$

product of 3 fns:

$$\cancel{3x}\ \overbrace{(5x+3)}^{f}\overbrace{(x+2)(2x-1)}^{g}$$ → same answer.

$$\underbrace{\big((5x+3)(x+2)\big)}_{f}\underbrace{(2x-1)}_{g}$$

$$\frac{d}{dx}(5x+3)(x+2)(2x-1)$$

$$= (5x+3)(x+2)\cdot\left(\frac{d}{dx}(2x-1)\right)$$

$$+ (2x-1)\cdot\Big(\underbrace{\frac{d}{dx}(5x+3)(x+2)}_{\text{another product rule}}\Big)$$

$$\vdots$$

$$= (5x+3)(x+2)\cdot 2$$

$$+ (2x-1)\left[10x+13\right]$$

Building the Toolbox

Theorem (Reciprocals)

If $y = f(x)$ is a differentiable function, then wherever $f(x) \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}.$$

$$\frac{d}{dx}\frac{1}{f(x)} = \lim_{h\to 0}\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)-f(x+h)}{f(x+h)\,f(x)}\right)$$

$$= \lim_{h\to 0} - \underbrace{\frac{(f(x+h)-f(x))}{h}} \cdot \underbrace{\frac{1}{f(x+h)\,f(x)}}$$

$$= (-1) \quad \cdot \quad f'(x) \quad \cdot \quad {}^{1}/f^{2}(x)$$

$$\frac{d}{dx}x^{-2} = -2\,x^{-3}$$

Building the Toolbox

Theorem (Reciprocals)

If $y = f(x)$ is a differentiable function, then wherever $f(x) \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}.$$

Example:

Calculate $\frac{d}{dx}\left(\frac{1}{x^2}\right)$.

Building the Toolbox

Theorem (Reciprocals)

If $y = f(x)$ is a differentiable function, then wherever $f(x) \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}.$$

Example:

Calculate $\frac{d}{dx}\left(\frac{1}{x^2}\right)$.

Here, $f(x) = x^2$, so

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2x}{(x^2)^2} = -\frac{2}{x^3} = -2x^{-3} \quad ☺$$

Some more examples of reciprocals

$$\text{Rule: } \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}.$$

Some more examples of reciprocals

$$\text{Rule: } \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}.$$

$$\frac{d}{dx}\left(\frac{1}{x^2+3x+1}\right) = -\frac{2x+3}{(x^2+3x+1)^2}$$

Some more examples of reciprocals

$$\text{Rule: } \tfrac{d}{dx}\left(\tfrac{1}{f(x)}\right) = -\tfrac{f'(x)}{f(x)^2}.$$

$$\frac{d}{dx}\left(\frac{1}{x^2+3x+1}\right) = -\frac{2x+3}{(x^2+3x+1)^2}$$

Quotients:

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+3x+1}\right) = \frac{d}{dx}\left((x^2-1)\left(\frac{1}{x^2+3x+1}\right)\right)$$

Some more examples of reciprocals

Rule: $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f(x)^2}$.

$$\frac{d}{dx}\left(\frac{1}{x^2+3x+1}\right) = -\frac{2x+3}{(x^2+3x+1)^2}$$

Quotients:

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+3x+1}\right) = \frac{d}{dx}\left((x^2-1)\left(\frac{1}{x^2+3x+1}\right)\right)$$

$$= (x^2-1)\frac{-(2x+3)}{(x^2+3x+1)^2} + \frac{1}{x^2+3x+1}(2x)$$

In general,

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(\left(\frac{1}{g}\right)\cdot f\right)$$

$$=$$

In general,

$$\begin{aligned}\frac{d}{dx}\left(\frac{f}{g}\right) &= \frac{d}{dx}\left(\left(\frac{1}{g}\right)\cdot f\right)\\ &= \left(\frac{1}{g}\right)\cdot\frac{d}{dx}(f) + f\cdot\frac{d}{dx}\left(\frac{1}{g}\right) && \text{(product rule)}\end{aligned}$$

In general,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f}{g}\right) &= \frac{d}{dx}\left(\left(\frac{1}{g}\right)\cdot f\right) \\
&= \left(\frac{1}{g}\right)\cdot\frac{d}{dx}(f) + f\cdot\frac{d}{dx}\left(\frac{1}{g}\right) && \text{(product rule)} \\
&= \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right) && \text{(reciprocal rule)}
\end{aligned}$$

In general,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f}{g}\right) &= \frac{d}{dx}\left(\left(\frac{1}{g}\right)\cdot f\right) \\
&= \left(\frac{1}{g}\right)\cdot\frac{d}{dx}(f) + f\cdot\frac{d}{dx}\left(\frac{1}{g}\right) && \text{(product rule)} \\
&= \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right) && \text{(reciprocal rule)} \\
&= \frac{f'}{g} - \frac{f\cdot g'}{g^2}
\end{aligned}$$

In general,

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{f}{g}\right) &= \frac{d}{dx}\left(\left(\frac{1}{g}\right)\cdot f\right) \\
&= \left(\frac{1}{g}\right)\cdot\frac{d}{dx}(f) + f\cdot\frac{d}{dx}\left(\frac{1}{g}\right) && \text{(product rule)} \\
&= \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right) && \text{(reciprocal rule)} \\
&= \frac{f'}{g} - \frac{f\cdot g'}{g^2} \\
&= \frac{g\cdot f' - f\cdot g'}{g^2} && \text{(common denominator)}
\end{aligned}
$$

In general,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f}{g}\right) &= \frac{d}{dx}\left(\left(\frac{1}{g}\right)\cdot f\right) \\
&= \left(\frac{1}{g}\right)\cdot\frac{d}{dx}(f) + f\cdot\frac{d}{dx}\left(\frac{1}{g}\right) && \text{(product rule)} \\
&= \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right) && \text{(reciprocal rule)} \\
&= \frac{f'}{g} - \frac{f\cdot g'}{g^2} \\
&= \frac{g\cdot f' - f\cdot g'}{g^2} && \text{(common denominator)}
\end{aligned}$$

Theorem (Quotient rule)

Wherever $g(x) \neq 0$,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

Building the Toolbox

Theorem

Let $(f \circ g)(x) = f(g(x))$.

Assume that g *is differentiable at the point* x *and that* f *is differentiable at the point* $g(x)$. *Then the composite function* $f \circ g$ *is differentiable at the point* x, *and*

$$\frac{d}{dx}((f \circ g)(x)) = \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Using Leibniz's notation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

$$\text{Chain rule:}\quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7 + 5}\right)$.

Here,

$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

$$\text{Chain rule:}\quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

Here,

$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

So

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 7x^6$$

$$\text{Chain rule:} \quad \frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x).$$

Example

Calculate $\frac{d}{dx}\left(\sqrt{x^7+5}\right)$.

Here,

$$f(x) = \sqrt{x} = x^{1/2} \quad \text{and} \quad g(x) = x^7 + 5.$$

So

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 7x^6$$

and so

$$\frac{d}{dx}\left(\sqrt{x^7+5}\right) = \frac{1}{2\sqrt{x^7+5}} \cdot 7x^6.$$

Back to reciprocals:

Chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$

$$\text{Calculate } \frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{d}{dx}\left((g(x))^{-1}\right)$$

Back to reciprocals:

Chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$

$$\text{Calculate } \frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{d}{dx}\left((g(x))^{-1}\right)$$

Outside function: $f(x) = x^{-1}$
Inside function: $g(x)$

Back to reciprocals:

Chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$.

Calculate $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{d}{dx}\left((g(x))^{-1}\right)$

Outside function: $f(x) = x^{-1}$
Inside function: $g(x)$

So since

$$\frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2} \qquad \text{and} \qquad \frac{d}{dx}(g(x)) = g'(x),$$

Back to reciprocals:

Chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$.

$$\text{Calculate } \frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{d}{dx}\left((g(x))^{-1}\right)$$

Outside function: $f(x) = x^{-1}$
Inside function: $g(x)$

So since

$$\frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2} \qquad \text{and} \qquad \frac{d}{dx}(g(x)) = g'(x),$$

we get

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{1}{(g(x))^2} \cdot g'(x) = -\frac{g'(x)}{(g(x))^2}$$

just like before!

Calculate $\frac{d}{dx}$ of ...

① $(3x^2+x+1)(5x+1)$

Two ways:

ⓐ expand first.

$$(3x^2+x+1)(5x+1) = 15x^3+8x^2+6x+1$$

$$\Big\downarrow \frac{d}{dx}$$

$$45x^2+16x+6$$

ⓑ product rule:

$$\frac{d}{dx}(3x^2+x+1)(5x+1)$$

$$= (3x^2+x+1)\cdot 5 + (5x+1)(6x+1)$$

"done" here.
but to compare to ⓐ ...

$$= 15x^2+5x+5 + 30x^2+11x+1$$

$$= 45x^2+16x+6$$

:)

(2) $y = (3x^2 + x + 1)(5x+1)^2$

Lots of ways, including:

(a) Expand all the way:

$$(3x^2+x+1)(5x+1)^2 = 75x^4 + 55x^3 + 38x^2 + 11x + 1$$

so $$\frac{dy}{dx} = 4\cdot 75x^3 + 3\cdot 55x^2 + 2\cdot 38x + 11$$
$$= \boxed{300x^3 + 165x^2 + 76x + 11}$$

(b) Expand the $(5x+1)^2$:

$$(3x^2+x+1)(5x+1)^2 = (3x^2+x+1)(25x^2+10x+1)$$

$\frac{d}{dx}$

product rule:

$$(3x^2+x+1)(50x+10) + (25x^2+10x+1)(6x+1)$$

"done"

$$= 150x^3 + 80x^2 + 60x + 10$$
$$+ 150x^3 + 85x^2 + 16x + 1$$
$$= 300x^3 + 165x^2 + 76x + 11$$

(c) two product rules:

$$\frac{d}{dx}\Big(\big((3x^2+x+1)(5x+1)\big)(5x+1)\Big)$$
$$= \underbrace{(3x^2+x+1)(5x+1)\cdot 5} + (5x+1)\cdot\underbrace{\frac{d}{dx}(3x^2+x+1)(5x+1)}$$
$$= 75x^3 + 40x^2 + 30x + 5 + (5x+1)\left[(3x^2+x+1)\cdot 5 + (5x+1)(6x+1)\right]$$
$$= \cdots = 300x^3 + 165x^2 + 76x + 1.$$

(d) My favorite:

product and chain rule:

$$\frac{d}{dx}(3x^2+x+1)(5x+1)^2$$

$$\left.\begin{aligned} &= (3x^2+x+1)\cdot 2(5x+1)^1\cdot 5 \\ &\quad + (5x+1)^2\cdot(6x+1) \end{aligned}\right\} \text{ done.}$$

$$\begin{aligned} &= 150x^3 + 80x^2 + 60x + 10 \\ &\quad + 150x^3 + 85x^2 + 16x + 1 \\ &= 300x^3 + 165x^2 + 76x + 11 \end{aligned}$$

③ $y = (5x+1)^{10}$

Lots of ways, for example...

ⓐ Be obnoxious and expand first:

$$(5x+1)^{10} = 9{,}765{,}625\,x^{10} + 19{,}531{,}250\,x^{9} + 17{,}578{,}125\,x^{8} + 9{,}375{,}000\,x^{7} + 3{,}281{,}250\,x^{6} + 787{,}500\,x^{5} + 131{,}250\,x^{4} + 15{,}000\,x^{3} + 1125\,x^{2} + 50x + 1$$

so

$$\frac{dy}{dx} = 97{,}656{,}250\,x^{9} + 175{,}781{,}250\,x^{8} + 140{,}625{,}000\,x^{7} + 65{,}625{,}000\,x^{6} + 19{,}687{,}500\,x^{5} + 3{,}937{,}500\,x^{4} + 525{,}000\,x^{3} + 45{,}000\,x^{2} + 2250x + 50.$$

ⓑ Chain rule:

$$\frac{d}{dx}(5x+1)^{10} = 10\,(5x+1)^{9} * 5$$

(which happens to be if you expand)

④ $(3x^2+x+1)(5x+1)^{10}$

Product, then chain:

$$(3x^2+x+1)\cdot\frac{d}{dx}(5x+1)^{10} + (5x+1)^{10}\frac{d}{dx}(3x^2+x+1)$$

$$= (3x^2+x+1)\cdot 10(5x+1)^9\cdot 5$$

$$+ (5x+1)^{10}(6x+1).$$

↖ "done",

but notice there's a quick rout to factorization:

pull out $(5x+1)^9$ that the two terms have in common:

$$= (5x+1)^9\left(50(3x^2+x+1)+(5x+1)(6x+1)\right)$$

$$= (5x+1)^9\left(150x^2+50x+50+30x^2+11x+1\right)$$

$$= (5x+1)^9\left(\underbrace{180x^2+66x+51}_{\text{no real roots!}}\right)$$

⑤ $\dfrac{\sqrt{x^2-x}}{x+x^{-1}}$

Many ways, including...

ⓐ Quotient rule: $y=\dfrac{f}{g}$

where

$f=(x^2-x)^{1/2}$; $g=x+x^{-1}$

so

$f'=\frac{1}{2}(x^2-x)^{-1/2}$; $g'=1-x^{-2}$

so $\dfrac{dy}{dx}=\dfrac{f'g-g'f}{g^2}=\dfrac{\overset{(2x-1)*}{\vee}\frac{1}{2}(x^2-x)^{-1/2}\cdot(x+x^{-1})-(1-x^{-2})(x^2-x)^{1/2}}{(x+x^{-1})^2}$

"done"

ⓑ product rule: $f\cdot(g)^{-1}$

where $f=(x^2-x)^{1/2}$; $g=x+x^{-1}$

$$\frac{d}{dx}f\cdot(g)^{-1}=f\cdot\frac{d}{dx}(g)^{-1}+(g)^{-1}\frac{d}{dx}f$$

$$=(x^2-x)^{1/2}\cdot\left(-(x+x^{-1})^{-2}\cdot(1-x^{-2})\right)$$

$$+(x+x^{-1})^{-1}\cdot\frac{1}{2}(x^2-x)^{-1/2}$$ ← "done"

$$=-\frac{(x^2-x)^{1/2}(1-x^2)}{(x+x^{-1})^2}+\frac{(x+x^{-1})\cdot\frac{1}{2}(x^2-x)^{-1/2}}{(x+x^{-1})^2}$$ ← same as ⓐ ✓

⑥ $\dfrac{1}{\sqrt[3]{x^2+7x^{1/2}}}$

Many ways, including...

ⓐ reciprocal rule: $\dfrac{1}{f}$

where $f = (x^2+7x^{\frac{1}{2}})^{1/3}$

since $f' = \frac{1}{3}(x^2+7x^{1/2})^{-2/3} \cdot (2x + 7/2\, x^{-1/2})$

$$\frac{dy}{dx} = -\frac{\frac{1}{3}(x^2+7x^{1/2})^{-2/3}(2x+\frac{7}{2}x^{-1/2})}{(\sqrt[3]{x^2+7x^{1/2}})^2}$$

"done"

$$= -\frac{2x+\frac{7}{2}x^{-1/2}}{3(x^2+7x^{1/2})^{4/3}}.$$

← nicer.

ⓑ Rewrite first, then chain rule:

$$y = (x^2+7x^{1/2})^{-1/3}$$

so $\dfrac{dy}{dx} = -\frac{1}{3}(x^2+7x^{1/2})^{-4/3}(2x+\frac{7}{2}x^{-1/2})$

$$= -\frac{2x+7/2\,x^{-1/2}}{3(x^2+7x^{1/2})^{4/3}},$$

just as before!