Derivatives

Oct 7. 2011

Warm-up:

Calculate as many

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

of the following as you can in the next few minutes:

$f(x) = x$,

x^2, x^3, ← expand the numerators

$\frac{1}{x}$, $\frac{1}{x^2}$, ← common denominators

$\sqrt{x}$ ← mult by $\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$

$\sqrt[3]{x}$

↖ mult by (top & bot):

$$\left(\sqrt[3]{x+h}\right)^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2$$

$$= (x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}$$

$f(x) = x$

$$\lim_{h \to 0} \frac{(x+h) - x}{h} \overset{*}{=} \lim_{h \to 0} \frac{h}{h} = \boxed{1}$$

$f(x) = x^2$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$\overset{*}{=} \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h$$

$$= \boxed{2x}$$

$f(x) = x^3$

recall: $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

so $(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$.

So, if $h \neq 0$

$$* \quad \frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2$$

So

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^3\right)$$

$$= \boxed{3x^2} + 0 + 0$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$\lim_{h \to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} = \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{x-(x+h)}{(x+h)(x)}\right)$$

$$\overset{*}{=} \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{-h}{(x+h)x}\right)$$

$$= \lim_{h \to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2} = \boxed{-x^{-2}}$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$\lim_{h \to 0} \frac{\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)}{h} = \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{x^2-(x+h)^2}{(x+h)^2 x^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{x^2-(x^2+2xh+h^2)}{(x+h)^2 \; x^2}\right)$$

$$\overset{*}{=} \lim_{h \to 0} \frac{1}{h}\left(\frac{-2xh+h^2}{(x+h)^2 x^2}\right) = \lim_{h \to 0} \frac{-2x+h}{(x+h)^2 x^2}$$

$$= -\frac{2x}{x^4} = -\frac{2}{x^3} = \boxed{-2x^{-3}}$$

From last class:

$$f(x) = \sqrt{x}$$

so $f(x+h) = \sqrt{x+h}$

so

can't plug in $h=0$ yet

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \begin{matrix}\to 0 \\ \to 0\end{matrix}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

so

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2x^{1/2}}$$

$$f(x) = \sqrt[3]{x} = x^{1/3} :$$

Since

$$(a-b)(a^2 + ab + b^2)$$

$$= a^3 + a^2 b + a b^2$$

$$\quad - (a^2 b + a b^2 + b^3)$$

$$= a^3 - b^3 ,$$ let $a = \sqrt[3]{x+h}$, $b = \sqrt[3]{x}$

we have

$$\left((x+h)^{1/3} - x^{1/3}\right)\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)$$

$$= (x+h)^{3/3} - x^{3/3} = x + h - x = h .$$

so

$$\lim_{h \to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} \cdot \frac{\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)}{\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)}$$

$$\overset{*}{=} \lim_{h \to 0} \frac{h}{h\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)}$$

$$= \frac{1}{x^{2/3} + x^{1/3} x^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \boxed{\frac{1}{3} x^{-2/3}}$$

$f(x)$	$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
$x^{-1} = \frac{1}{x}$	$-x^{-2}$
x^{-2}	$-2x^{-3}$
$x^{1/2}$	$+\frac{1}{2}x^{-1/2}$
$x^{1/3}$	$\frac{1}{3}x^{-2/3}$

Definition

The derivative of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will say that a function f is differentiable at a point $x = a$ if the derivative function f' exists at a.

Suppose we consider the piecewise defined function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$

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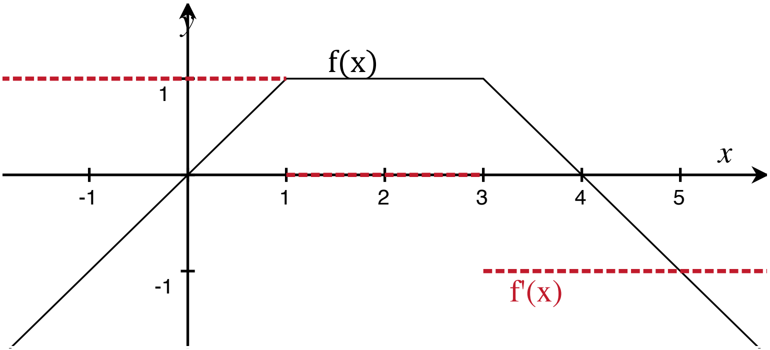
It's derivative is:

$$f(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

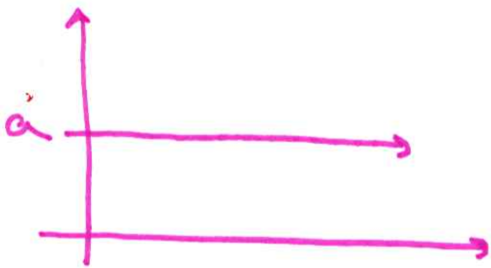


y
f(x)
1
x
-1
1
2
3
4
5
-1
f'(x)

More examples

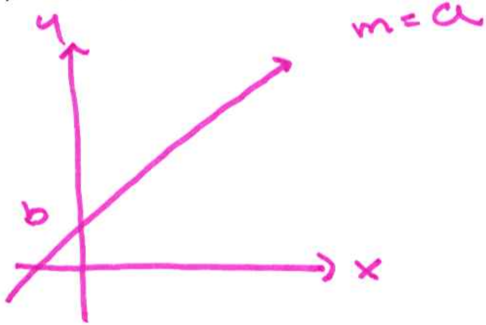
If a and b are constants:

$$f(x) = a$$



$$f'(x) = 0$$

$$f(x) = ax + b$$



$$f'(x) = a$$

Derivative "rule"

$f(x)$	$f'(x)$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
x^{-1}	$-x^{-2}$
x^{-2}	$-2x^{-3}$
$x^{1/2}$	$(1/2)x^{-1/2}$
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If $f(x) = x^a$ then $f'(x) = ax^{a-1}$

Find an equation of the tangent line to the graph of $f(x) = x^{4/3}$ at the point where $x = 1$.

$$y = f(1) + f'(1)(x - 1).$$

$$f(x) = x^{4/3} \longrightarrow f(a) = 1$$

$$f'(x) = \frac{4}{3} x^{\frac{1}{3}}$$

$$a = 1$$

so $f'(a) = \frac{4}{3}$

so $y - 1 = \frac{4}{3}(x-1)$

$$y = 1 + \frac{4}{3}(x-1)$$

$\uparrow$ $f(1)$ $\quad$ $\uparrow$ $f'(1)$

$$= \frac{4}{3}x - \frac{1}{3}$$

$f(x) = x^{4/3}$

$a = 2$

$f(a) = 2^{4/3} = 2.52$

$y - 2.52 = m(x-2)$

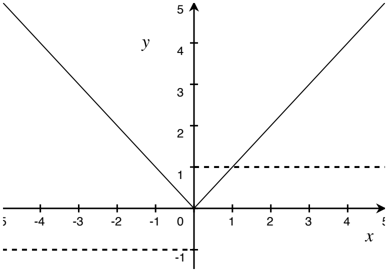
$m = f'(2) = \frac{4}{3}(2)^{1/3} = 1.68$

aprox:

$y - \underline{2.52} = \underline{1.68}(x-2)$

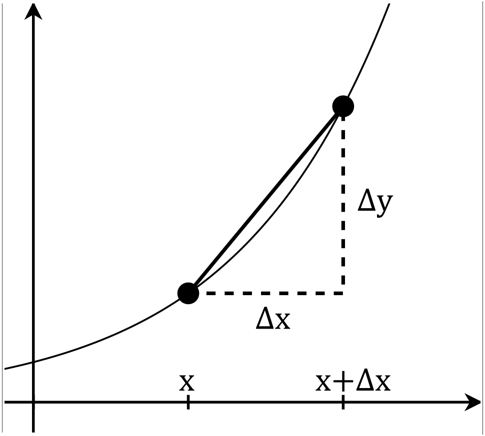
Find the derivative of $f(x) = |x|$.

$$\lim_{h \to 0^+} \frac{|0+h| - 0}{h} = 1$$
$$\lim_{h \to 0^-} \frac{|0+h| - 0}{h} = -1$$



Notation for the Derivative

$$y' = D_x y = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x).$$



Example

For the function $y = f(x) = 1/x$, find the slope of its tangent line at $x = 2$. Compare it with the average rate of change over the interval $[2, 3]$.

Higher Order Derivatives

When we differentiate a function $f(x)$ we obtain a new function $f'(x)$.
The derivative is again a candidate for differentiation, and we call its derivative *the second derivative* of $f(x)$.
So long as the derivatives exist we can continue this process to obtain a succession of higher derivatives.

Higher Order Derivatives ...

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = D_x^2 y = D_x^2 f(x).$$

The nth derivative, where n is a positive integer

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x) = D_x^n y = D_x^n f(x).$$