# Warm-up:

Calculate

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if $f(x) = \sqrt{x}$.

Hint: remember $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$.

Multiply by

$$\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}.$$

$$f(x) = \sqrt{x}$$

so $f(x+h) = \sqrt{x+h}$

can't plug in
h=0 yet

so

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \begin{matrix} \to 0 \\ \\ \to 0 \end{matrix}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left( \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

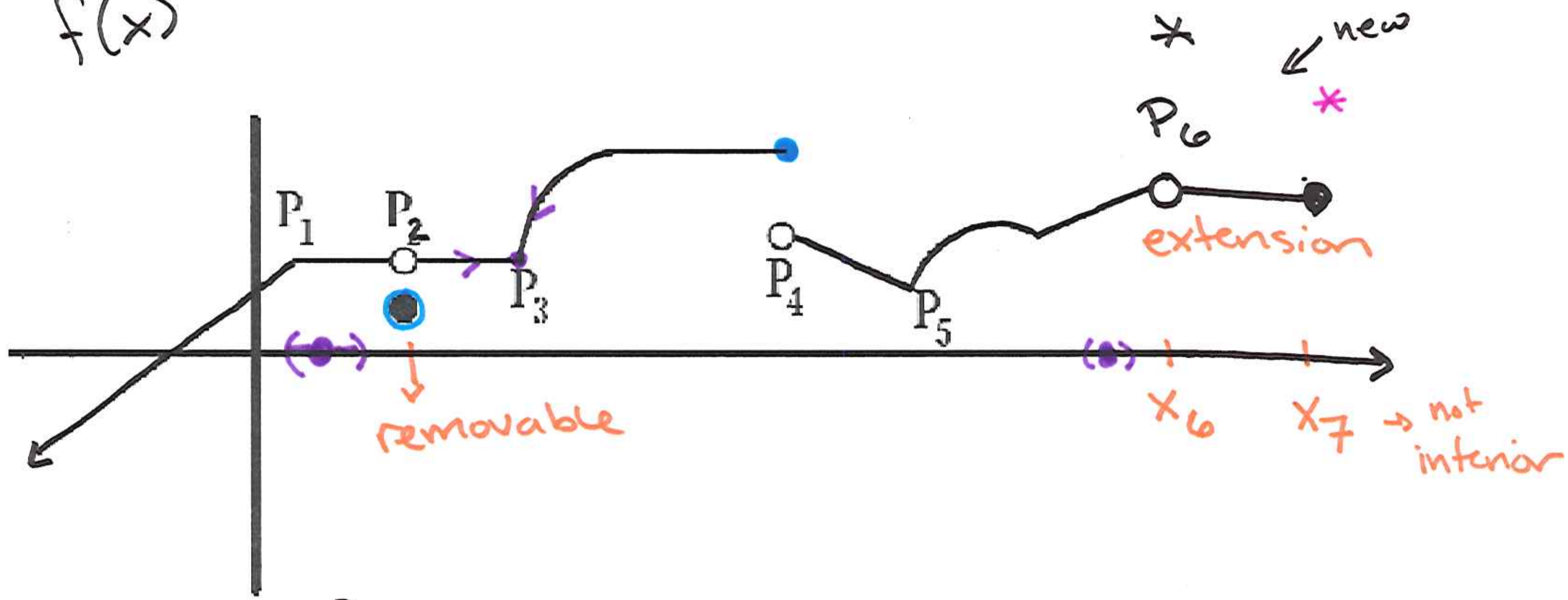
$$= \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

so

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$f(x)$




$f(x)$ is not continuous @ $P_2$, $P_4$

$f(x)$ is continuous "after $P_4$"

# Interior Point

An *interior point* of a set of real numbers is a point that can be enclosed in an open interval that is contained in the set.

↳ no endpts. $a < x < b$

sets we are interested in
are domains of functions.

From last example,
set is "all real #s
except $x_6$ • and $\leq x_7$"

# Definition

- A function is continuous at an interior point $c$ of its domain if $\lim_{x \to c} f(x) \underset{*}{=} f(c)$. $P_4$: no limit
  $P_2$: not equal

- If it is not continuous there, i.e. if either the limit does not exist or is not equal to $f(c)$ we will say that the function is discontinuous at $c$.

## Note:

To be continuous when $x = c$:

1. The function $f$ is defined at the point $x = c$,
2. The point $x = c$ is an interior point of the domain of $f$,
3. $\lim_{x \to c} f(x)$exists, call it $L$, and
4. $L = f(c)$.

$$\begin{cases} \lim_{x \to c^+} f(x) = L_+ \\ \lim_{x \to c^-} f(x) = L_- \\ \text{and} \\ L_- = L_+ \end{cases}$$

# Example

Is the function

$$f(x) = \begin{cases} x^2 & x < 1 \\ x^3 + 2 & 1 \le x \end{cases}$$

continuous at $x = 1$?

Checklist

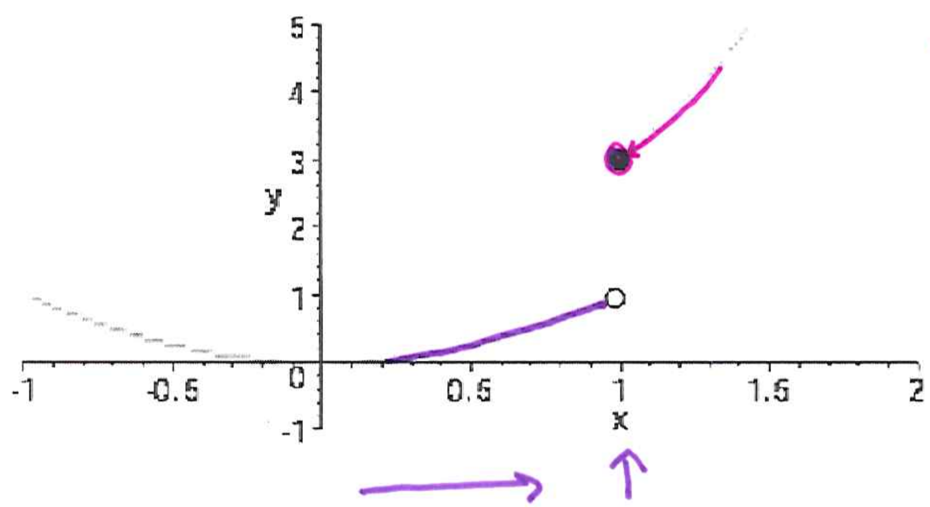
Q1 ✓

Q2 ✓

Q3 : no

no limit @ $x=1$.

discontinuous @ $x=1$




not left continuous, but is right continuous

## Right Continuity and Left Continuity

- A function $f$ is right continuous at a point $c$ if it is defined on an interval $[c, d]$ lying to the right of $c$ and if $\lim_{x \to c^+} f(x) = f(c)$.

- Similarly it is left continuous at $c$ if it is defined on an interval $[d, c]$ lying to the left of $c$ and if $\lim_{x \to c^-} f(x) = f(c)$.

From last page:

$$\lim_{x \to 1^-} f(x) = 1 \quad \text{but} \quad f(1) = 3$$

$$\neq$$

so not L-cont.

# Definition

A function $f$ is continuous at a point $x = c$ if $c$ is in the domain of $f$ and:

1. If $x = c$ is an interior point of the domain of $f$, then $\lim_{x \to c} f(x) = f(c)$.

2. If $x = c$ is not an interior point of the domain but is an endpoint of the domain, then $f$ must be right or left continuous at $x = c$, as appropriate.

$x_7$ is not interior, but $f(x)$ is L. cont @ $x_7$.

- A function $f$ is said to be a continuous function if it is continuous at every point of its domain. (not just interior)

- A point of discontinuity of a function $f$ is a point in the domain of $f$ at which the function is not continuous.

① if $x = c$ is interior,

and $\lim_{x \to c} f(x) = f(c)$

then $f(x)$ is cont

@ $x = c$

② if $x = c$ is not interior,

and $\lim_{x \to c^{\pm}} f(x) = f(c)$

then $f(x)$ is cont

@ $x = c$

③ if $f(x)$ is cont

@ all pts in domain,

then $f(x)$ is continuous.

# Facts

- All polynomials,
- Rational functions, → b/c jumps only happen @ domain gaps
- Trigonometric functions,
- The absolute value function, and
- The exponential and logarithm functions

are continuous.

# Example

- The rational function $f(x) = \frac{x^2-4}{x-2}$ is a continuous function.
- The domain is all real numbers except 2.
- $\lim_{x\to 2} f(x) = 4$ exists.

$x^2-4=(x+2)(x-2)$

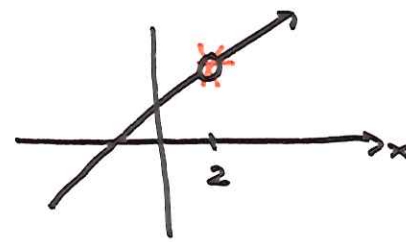
so $f(x)=\begin{cases} \text{undf} & x=2 \\ x+2 & \text{else} \end{cases}$

It has a *continuous extension*

add a pt to domain and stay continuous.

$$F(x) = \begin{cases} f(x) & \textit{if } x \textit{ is in the domain of } f \\ 4 & \textit{if } x = 2. \end{cases}$$

$= x+2$



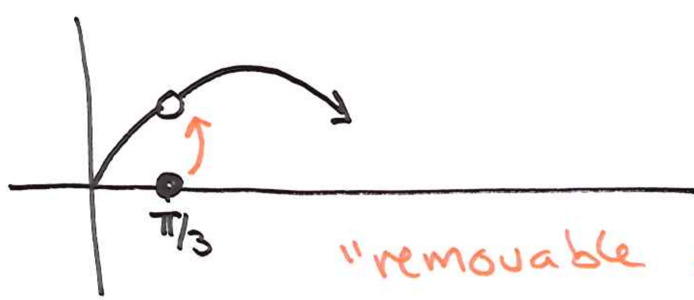
# Example

The function

$$f(x) = \begin{cases} \sin x & x \neq \pi/3 \\ 0 & x = \pi/3 \end{cases}$$

is discontinuous at $\pi/3$.

We can "remove" the discontinuity by redefining the value of $f$ at $\pi/3$.




"removable discontinuity"

# Definition

- If $c$ is a discontinuity of a function $f$, and if $\lim_{x\to c} f(x) = L$ exists, then $c$ is called a removable discontinuity. The discontinuity is removed by defining $f(c) = L$.

- If $f$ is not defined at $c$ but $\lim_{x\to c} f(x) = L$ exists, then $f$ has a continuous extension to $x = c$ by defining $f(c) = L$.

# Example

Suppose that $f(x)$ is defined piecewise as

$$f(x) = \begin{cases} -x^2 + 1 & x < 2 \\ x + k & x > 2 \end{cases}$$

Let us find a value of the constant $k$ such that $f$ has a continuous extension to $x = 2$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} -x^2 + 1 = -3$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x + K = 2 + K$$

$\lim_{x \to 2} f(x)$ only exists if $-3 = 2 + K \to \boxed{K = -5}$

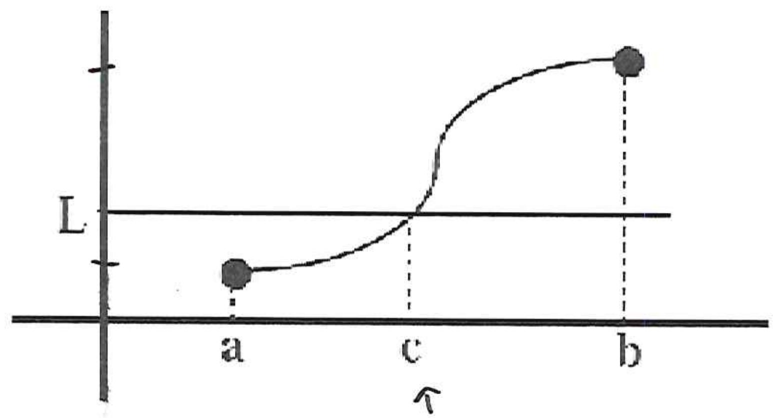


$$F(x) = \begin{cases} f(x) & \text{if } x \neq 2 \\ -3 & \text{if } x = 2 \end{cases}$$

# The Intermediate Value Theorem

all in
domain

If a function $f$ is continuous on a closed interval $[a, b]$, and if $f(a) < L < f(b)$ (or $f(a) > L > f(b)$), then there exists a point $c$ in the interval $[a, b]$ such that $f(c) = L$.



# Example

Show that the equation $x^5 - 3x + 1 = 0$ has a solution in the interval $[0, 1]$.

* pols are continuous

* every closed interval is in my domain

if $f(x) = x^5 - 3x + 1$

$f(0) = 1, \quad f(1) = -1$

so there's some pt btwn 0 & 1 for which $f(c) = 0$ b/c

$-1 < 0 < 1.$

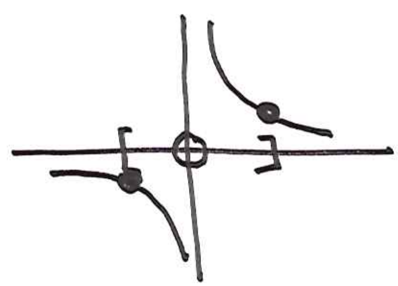
# Example

Does the equation $1/x = 0$ have a solution?

$\frac{1}{x}$ is continuous.

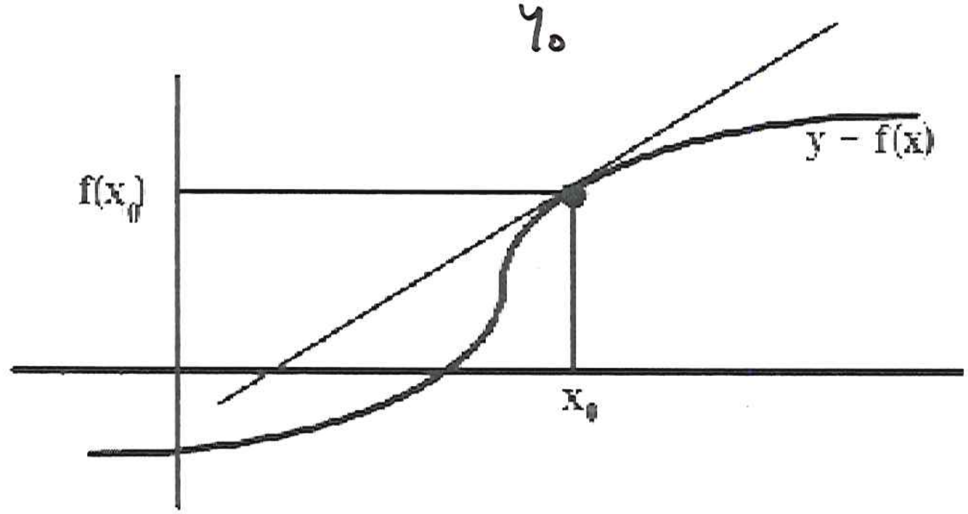
if I pick a closed interval in domain, it will have to be to one side of $x=0$ or the other

IVT does not apply.



# The Tangent Line and Their Slope

- **The Tangent Line Problem** Given a function $y = f(x)$ defined in an open interval and a point $x_0$ in the interval, define the tangent line at the point $(\underline{x_0}, \underbrace{f(x_0)}_{y_0})$ on the graph of $f$.




pt- slope:

$y - y_0 = m(x - x_0)$

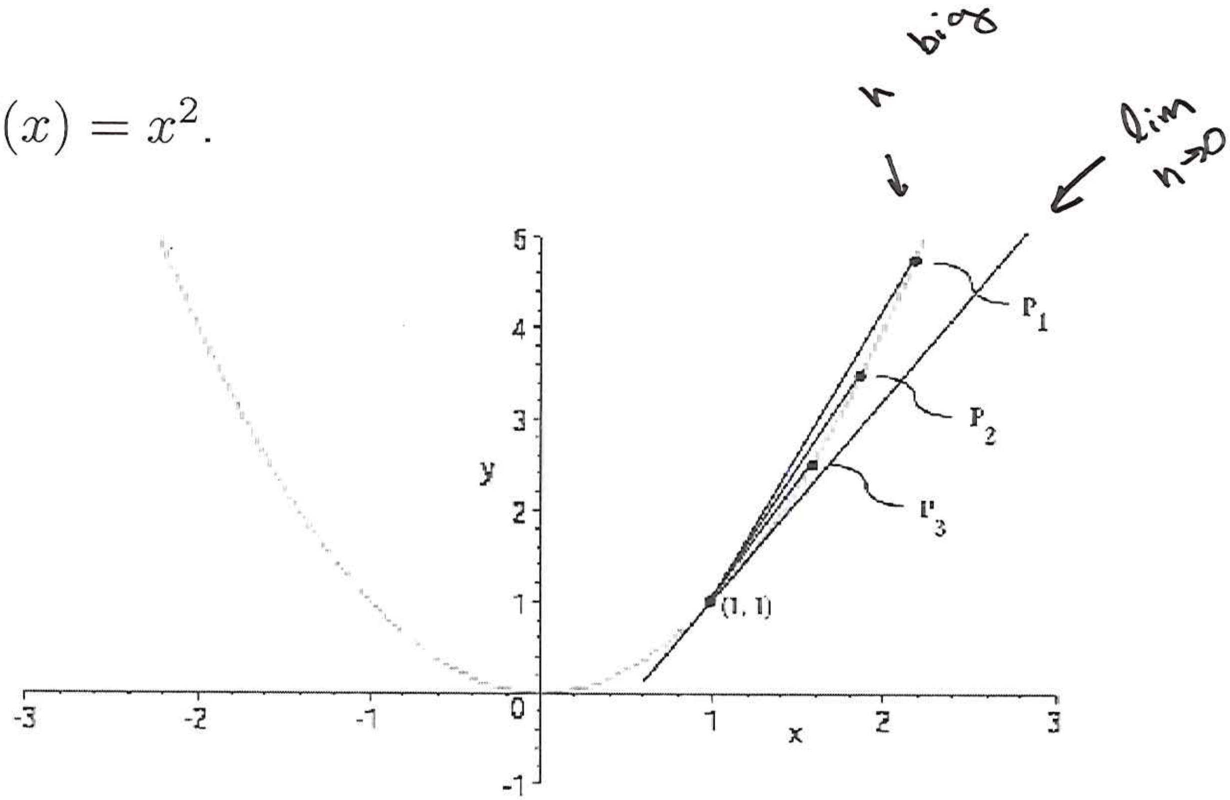
$y_0 = f(x_0)$

$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

@ $x = x_0$.

# Example

Let $f(x) = x^2$.



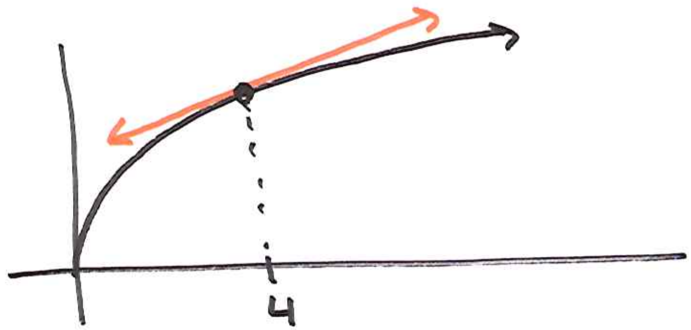
# Definition

Given a function $f$ and a point $x_0$ in its domain, the slope of the tangent line at the point $(x_0, f(x_0))$ on the graph of $f$ is

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

# Example

Given $f(x) = \sqrt{x}$, find the equation of the tangent line at $x = 4$.




$x_0 = 4$

$y_0 = \sqrt{4} = 2$

evaluated at

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Bigg|_{x=4}$$

from warm up

$$= \frac{1}{2\sqrt{x}} \Bigg|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\boxed{y - 2 = \frac{1}{4}(x-4)}$$