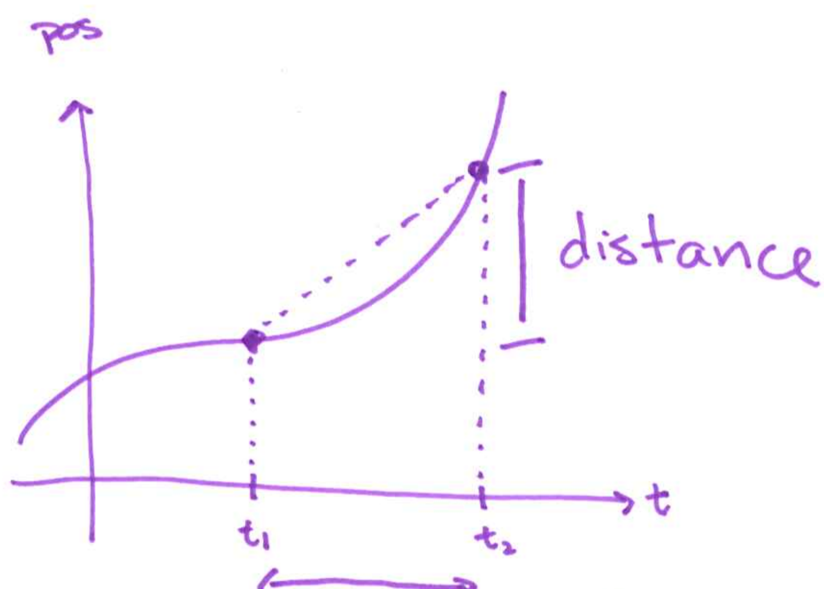
The average velocity during a time interval is the distance traveled divided by the elapsed time, i.e.

$$\text{Average velocity over}[t_1, t_2] = \frac{\text{distance traveled}}{t_2 - t_1}.$$



Δ time $= h$

$t_2 = t_1 + h$

Definition

Let $x(t)$ be a function that gives the position at time t of an object moving on the x-axis. Then

$$\text{Ave vel}[t_1, t_2] = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

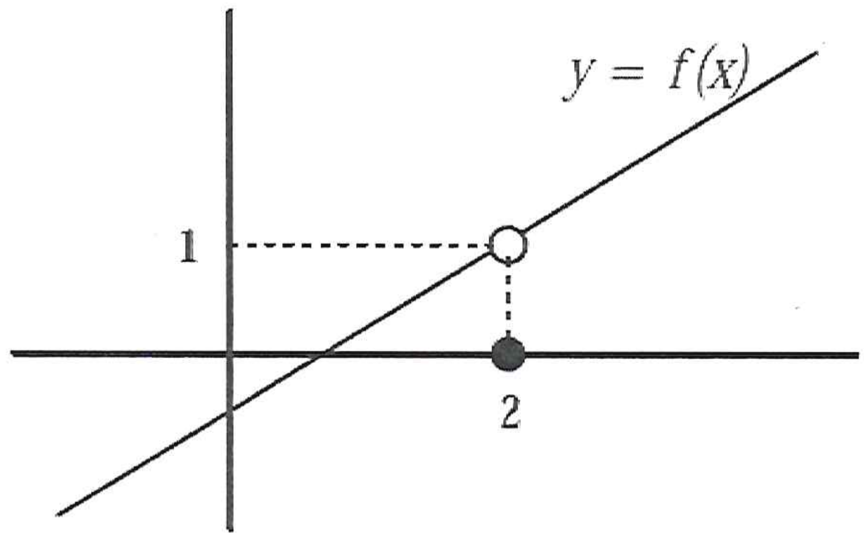
$$\text{Velocity}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}.$$

let the time gap get small

Limit of a Function – Definition

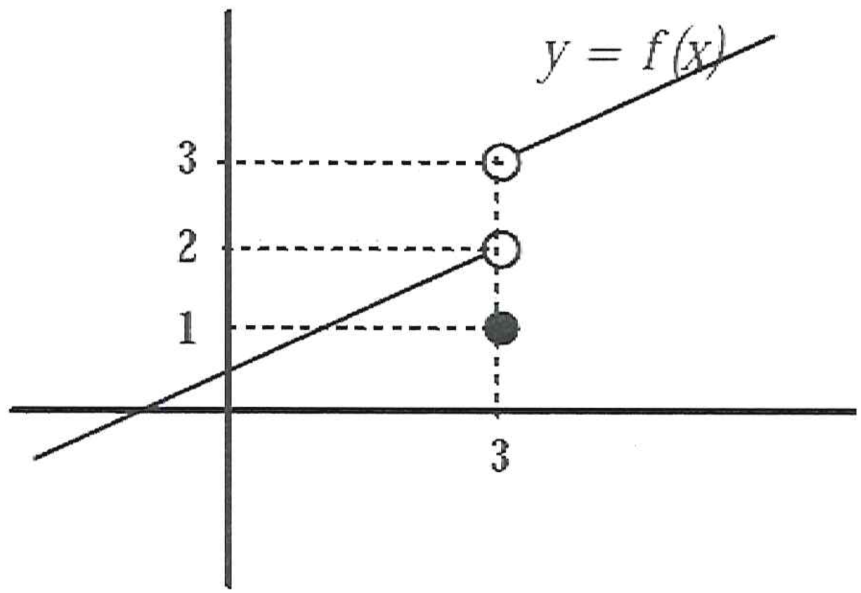
We say that a function f approaches the limit L as x approaches a, written $\lim_{x \to a} f(x) = L$, if we can make $f(x)$ as close to L as we please by taking x sufficiently close to a.

Example



$$\lim_{x \to 2} f(x) = 1$$

$f(2)$ is ~~undefined~~ 0.



$f(3)$ is ~~undef~~. 1

$\lim_{x \to 3^-} f(x) \rightarrow$: the limit from the left is 2

$\lim_{x \to 3^+} f(x) \rightarrow$: ——— " ——— right is 3

Theorem. *The limit of f as $x \longrightarrow a$ exists if and only if both the right-hand and left-hand limits exist and have the same value. I.e.*

$$\lim_{x \longrightarrow a} f(x) = L \Leftrightarrow \lim_{x \longrightarrow a^-} f(x) = L \textit{ and } \lim_{x \longrightarrow a^+} f(x) = L.$$

Examples

Compute the limits:

- $\lim_{x\to 2} \frac{x-2}{x+3}$ $= \frac{2-2}{2+3} = \frac{0}{5} = 0$
- $\lim_{x\to +1} \frac{x^2-1}{x-1}$ $= \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$ if $x \neq 1$ $= \lim_{x\to 1} \underbrace{x+1}_{\substack{f(x)=x \\ g(x)=1}} = 2$
- $\lim_{x\to 0} \frac{1}{x}$ DNE

$\lim_{x\to 0^-} \frac{1}{x}$ is another story.

Theorem. *If* $\underbrace{\lim_{x\to a} f(x) = A}_{\text{exists!}}$ *and* $\underbrace{\lim_{x\to a} g(x) = B}_{\text{exists!}}$ *both exist, then*

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A + B$

2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A - B$

3. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$

* 4. $\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x) = A/B$ $\boxed{(B \neq 0)}$.

↑ except if $\lim_{x\to a} g(x) = 0$

If $\lim_{x\to a} g(x) = 0$, lots can happen.

Examples

1. $\lim_{x \to 1} \frac{x^2 - 2x + 3}{x^3 + 3x - 1}$ $= 2/3$

2. $\lim_{x \to 0} \frac{|x|}{x}$ DNE

3. Let $f(x) = 1/x$. Compute $\lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}$

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

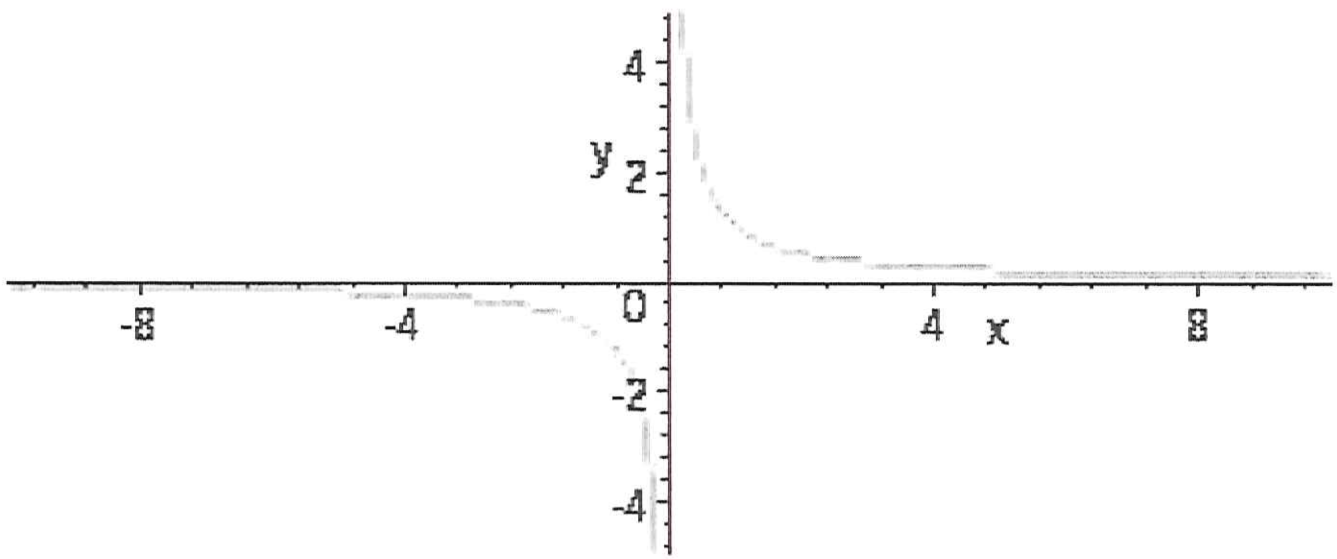
$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\left(\frac{x - (x+h)}{x(x+h)}\right)}{h} = \frac{\left(-h / x(x+h)\right)}{h}$$

Limits at Infinity

$\lim_{x\to\infty} f(x) = L$ means that the value of $f(x)$ approaches L as the value of x approaches $+\infty$. This means that $f(x)$ can be made as close to L as we please by taking the value of x sufficiently large. Similarly, $\lim_{x\to\infty} f(x) = L$ means that $f(x)$ can be made as close to L as we please by taking the value of x sufficiently small (in the negative direction).

Example

$\lim_{x\to\infty} 1/x = 0.$



$\lim_{x \to -\infty} \frac{1}{x} = 0$

$\lim_{x \to 0} \frac{1}{x}$ DNE

More Examples

Evaluate the limits:

1. $\lim_{x\to\infty} \frac{x-1}{x^3+2}$ $= 0$

2. $\lim_{x\to\infty} \frac{3x^2-2x+1}{4x^2-1}$ $= 3/4$?

3. $\lim_{x\to\infty} \frac{x^4-x^2+2}{x^3+3}$ $\to \infty$
(DNE)

Dominant Term Rue

(rational functions)

For the limit $\lim_{x\to\infty} P(x)/Q(x)$, where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m,

1. If $n < m$, the limit is 0,
2. If $n > m$, the limit is $\pm\infty$,
3. If $n = m$, the limit is the quotient of the coefficients of the highest powers.

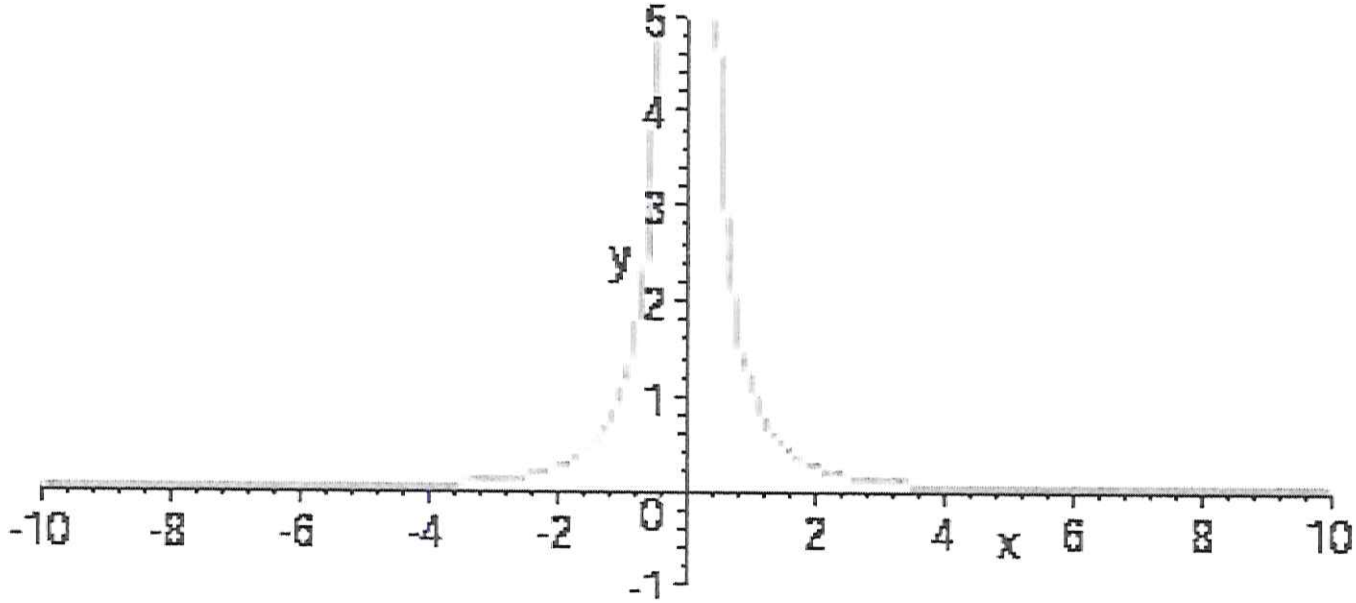
Example

Evaluate the limit:

$$\lim_{x \to \infty} \frac{x}{\sqrt{3x^2 + 2}}$$

Infinite Limits

Compute the limit $\lim_{x\to 0} 1/x^2$. DNE



$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Evaluate $\lim_{x \to \pi/2} \tan x$.   DNE