Tangent and normal lines.

Recall: in order to write down the equation for a line, it's usually easiest to start with point-slope form:

$$y = m(x - x_0) + y_0, \qquad \text{where } m = \text{slope, and } (x_0, y_0) \text{ is a point on the line.}$$

For a line tangent to a curve $y = f(x)$ at $x = a$,
$m = f'(a)$ and $(x_0, y_0) = (a, f(a))$.

For a line normal to a curve $y = f(x)$ at $x = a$,
$m = -1/f'(a)$ and $(x_0, y_0) = (a, f(a))$.

A. The basics

Find the equation of the tangent line to...

(1) $f(x) = x^2$ at $x = 3$.

(2) $f(x) = x^3$ at $x = 2$.

(3) $f(x) = x^2 + 2x + 3$ at $x = 1$.

(4) $f(x) = x^2 - 4$ at $x = 5$.

(5) $f(x) = x^3 - 1$ at $x = -1$.

(6) $f(x) = \frac{x+1}{x-3}$ at $x = 2$.

(7) $f(x) = \sqrt{x}$ at $x = 9$.

(8) $f(x) = \frac{1}{x}$ at $x = 2$.

(9) $f(x) = \frac{1}{x^2}$ at $x = -8$.

(10) $f(x) = e^x$ at $x = 0$.

(11) $f(x) = e^x$ at $x = 1$.

(12) $f(x) = \ln(x)$ at $x = e$.

(13) $f(x) = \ln(x)$ at $x = 17$.

(14) $f(x) = \sin(x)$ at $x = \pi/3$.

(15) $f(x) = \cos(x)$ at $x = \pi/4$.

(16) $f(x) = \tan(x)$ at $x = -\pi/4$.

Answers

(1) $y = 6x - 9$ (2) $y = 12x - 16$ (3) $y = 4x + 2$ (4) $y = 10x - 29$

(5) $y = 3x + 1$ (6) $y = 5x - 4$ (7) $y = \frac{1}{6}x + \frac{3}{2}$ (8) $y = -\frac{1}{4}x + 1$

(9) $y = \frac{1}{2^8}x + \frac{3}{2^6}$ (10) $y = x + 1$ (11) $y = ex$ (12) $y = \frac{1}{e}x$

(13) $y = \frac{1}{17}x - 1 + \ln(17)$ (14) $y = \frac{1}{2}x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$ (15) $y = -\frac{1}{\sqrt{2}}x + \frac{\pi+4}{4\sqrt{2}}$ (16) $y = 2x + \frac{\pi}{2} - 1$

B. A little trickier

(1) Find the equations of the tangent and normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$.

(2) Find the equations of the tangent and normal to the curve $y = \cot^2 x - 2\cot x + 2$ at $x = \pi/4$.

(3) [*parametric curves*] Find the equation of the tangent to the curve given by the equations $x = \theta + \sin(\theta)$ and $y = 1 + \cos(\theta)$ at $\theta = \pi/4$.

(4) [*parametric curves*] Find the equation of the tangent to the curve given by the equations $x = a\cos\theta$ and $y = b\sin\theta$ at $\theta = \pi/4$.

(5) [*parametric curves*] For a general t find the equation of the tangent and normal to the curve given by the equations $x = a\cos t$ and $y = b\sin t$.

(6) [*parametric curves*] For a general t find the equation of the tangent and normal to the curve $x = a\sec t$, $y = b\tan t$.

(7) [*parametric curves*] For a general t find the equation of the tangent and normal to the curve given by the equations $x = a(t + \sin t)$ and $y = b(1 - \cos t)$.

(8) [*implicit curves*] Find the equations of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$.

(9) Find the equation of the tangent to the curve $y = \sec^4 x - \tan^4 x$ at $x = \pi/3$.

(10) Find the equation of the normal to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$.

(11) Find the equation of the normal to the curve $y = \dfrac{1 + \sin x}{\cos x}$ at $x = \pi/4$.

(12) Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.

(13) Show that the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ are at right angles to eachother.

(14) Find the points on the curve $2a^2y = x^3 - 3ax^2$ where the tangent is parallel to the x-axis.

(15) For the curve $y(x-2)(x-3) = x-7$ show that the tangent is parallel to the x-axis at the points for which $x = 7 \pm 2\sqrt{5}$.

(16) Find the points on the curve $y = 4x^3 - 2x^5$ at which the tangent passes through the origin.

(17) [*implicit curves*] Find the points on the circle $x^2 + y^2 = 13$ where the tangent is parallel to the line $2x + 3y = 7$.

(18) Find the point on the curve $y = 3x^2 + 4$ at which the tangent is perpendicular to a line whose slope is $-1/6$.

(19) [*implicit curves*] Find the equations of the normals to the curve $2x^2 - y^2 = 14$ parallel to the line $x + 3y = 4$.

(20) Find the equation of the tangent to the curve $x^2 + 2y = 8$ which is perpendicular to the line $x - 2y + 1 = 0$.

(21) [*implicit curves*] If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)} = 1$ show that $(a\cos\alpha)^n + (b\sin\alpha)^n = p^n$.

Answers

(1) $y = 2(x-1) + 3$, $y = -\frac{1}{2}(x-1) + 3$

(2) $y = 1$, $x = \pi/4$

(3) $y = \frac{1+\sqrt{2}/2}{-\sqrt{2}/2}(x - \pi/4 - \sqrt{2}/2) + 1 + \sqrt{2}/2$

(4) $y = -\frac{a}{b}(x - a\sqrt{2}/2) + b\sqrt{2}/2$

(5) $y = -\frac{a\sin(t)}{b\cos(t)}(x - a\cos(t)) + b\sin(t)$, $y = \frac{b\cos(t)}{a\sin(t)}(x - a\cos(t)) + b\sin(t)$

(6) $y = \frac{a\tan(t)}{b\sec(t)}(x - a\sec(t)) + b\tan(t)$, $y = -\frac{b\sec(t)}{a\tan(t)}(x - a\sec(t)) + b\tan(t)$

(7) $y = \frac{a(1+\cos(t))}{b(\sin(t))}(x - a(t+\sin t)) + b(1-\cos t)$, $y = -\frac{b(\sin(t))}{a(1+\cos(t))}(x - a(t+\sin t)) + b(1-\cos t)$

(8) $y = -\frac{8}{3\sqrt{5}}x + \frac{12}{\sqrt{5}}$, $y = \frac{3\sqrt{5}}{8}x + \frac{7\sqrt{5}}{12}$

(9) $y = 16\sqrt{3}x - 16\pi/\sqrt{3} + 7$

(10) $y = \frac{1}{12}x - \frac{\pi}{24} + 4$

(11) $y = (\frac{1}{\sqrt{2}} - 1)x + \frac{16+12\sqrt{2}+\pi}{8+4\sqrt{2}}$

(12) (show they have the same m)

(13) (show their slopes satisfy $m_1 = -1/m_2$)

(14) $x = 0, 2a$

(15) (show $m = 0$)

(16) $x = 0, \pm\sqrt{2/3}$

(17) $(2,3), (-2,-3)$

(18) $y = 6x + 1$

(19) $y = -\frac{1}{3}(x-3) + 2, y = -\frac{1}{3}(x+3) - 2$

(20) $y = -2x - 2$