Calibrated representations of 2-boundary Temperley-Lieb algebras

(arXiv:2009.02812)

Zajj Daugherty

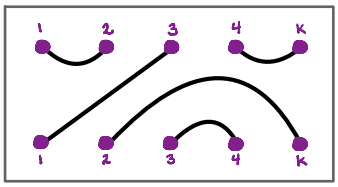
The City College of NY
& The CUNY Graduate Center
(Soon: Reed College)

Arun Ram

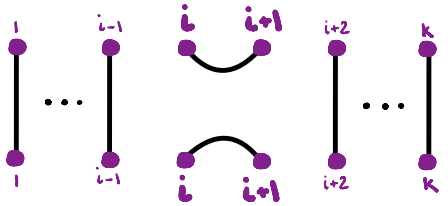
University of Melbourne

Classical Temperley-Lieb

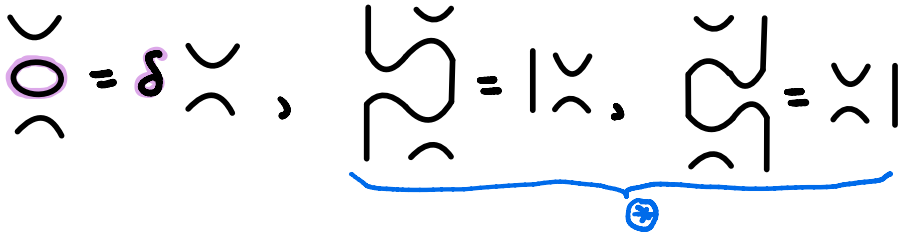
$$T_k(\delta) = \mathbb{C}\{\text{non-crossing matching diagrams on } 2k \text{ vertices}\} / \langle \bigcirc = \delta \rangle$$

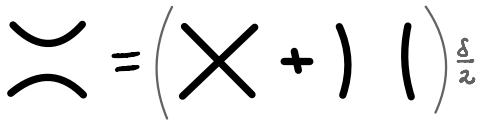


generators: $e_i =$



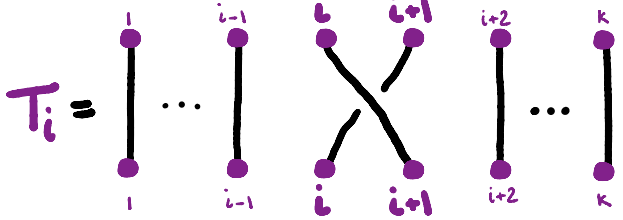
(local) relations: $e_i^2 = \delta e_i$, $e_i e_{i\pm1} e_i = e_i$, $e_i e_{i-1} e_i = e_i$

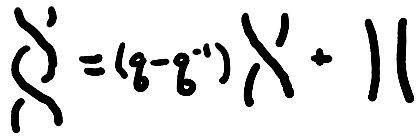


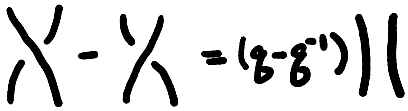
Thm: $T_k(\delta) = \mathbb{C}S_k / \circledast$ by setting 

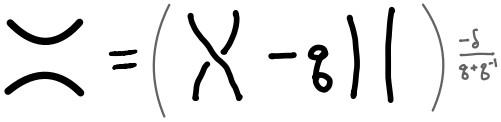
S_k: permutation diagrams

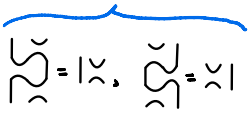
The (finite, type A) Hecke algebra H_k is the quotient of $\mathbb{C}\{\text{Braids on } k \text{ strands}\}$ by $(T_i - q)(T_i + q^{-1}) = 0$

$T_i =$ 

ie. 

ie. 

Thm: $T_k(\delta) = H_k / \circledast$ by setting 

$\circledast$: 

Idea: From the perspective of the Hecke algebra H_n, the relations $\circledast$ amount to Killing certain H_k-modules.

Precisely the projections that correspond to

Schur-Weyl Duality

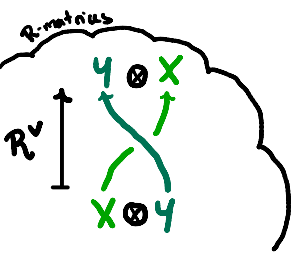
$$H_k \curvearrowright \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{k} \curvearrowleft U_q\mathfrak{sl}_2(\mathbb{C})$$

H_k acts via R-matrices; $U_q\mathfrak{sl}_2(\mathbb{C})$ acts via Hopf-alg action

$\downarrow \circledast$

$T_k(\delta)$ $\curvearrowleft$

$\delta = q + q^{-1} = [2]_q$



★ Produces a duality between

$U_q\mathfrak{sl}_2(\mathbb{C})$-modules ↪ integer partitions w/ ≤ 2 parts.

$\vdots$

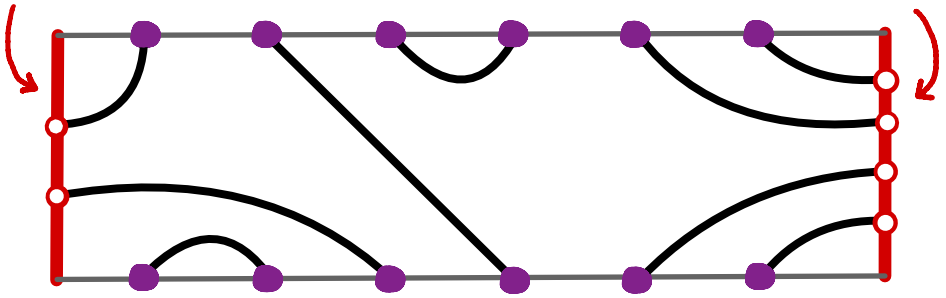
$T_k(q+q^{-1})$-modules

Two-boundary Temperley-Lieb algebra

↳ Mitra, Nienhuis, De Gier '04
↳ De Gier, Nichols '09

Generalize the tensor space:

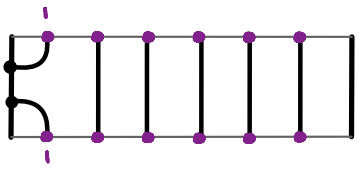
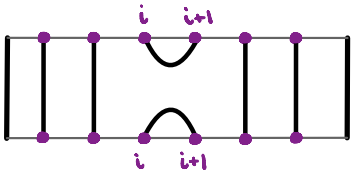
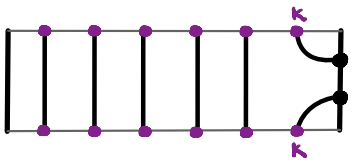
$$M \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes N$$



$U_q \mathfrak{sl}_2(\mathbb{C})$ — Hopf-alg action

{ Non-crossing diagrams w/an even number of connections on each side, but no "bubbles" on either side.

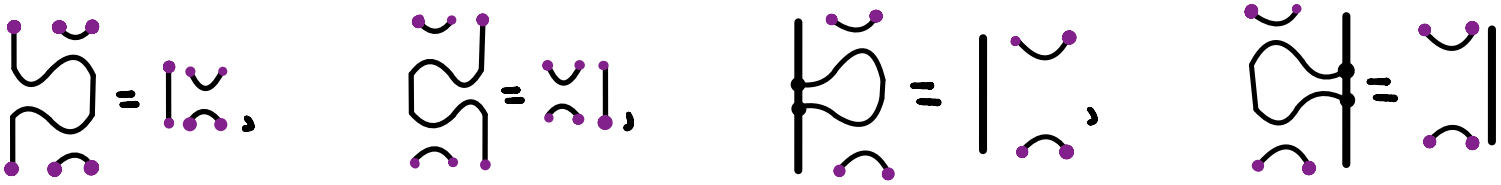
$TL_K = TL_K(\delta, \delta_0, \delta_K)$ are generated over $\mathbb{C}$ by

$e_0 =$ , $e_i =$ , $e_K =$ .

with local relations...

$$e_i e_{i\pm1} e_i = e_i, \quad 1 \le i \le K-1$$

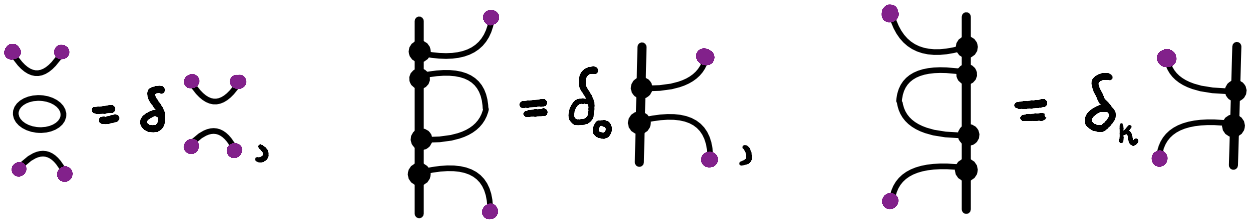
(Killing certain projections)



$$e_i^2 = \delta_i e_i, \quad 0 \le i \le K$$

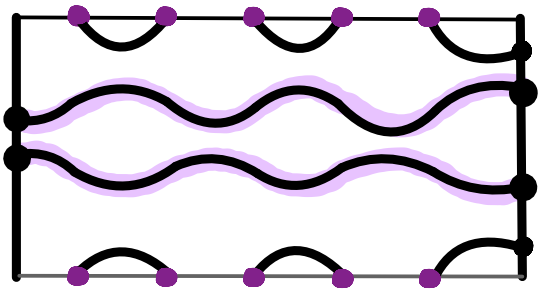
$\delta_i = \cdots = \delta_{K-1} = \delta$

Note: loops on boundaries get resolved differently!



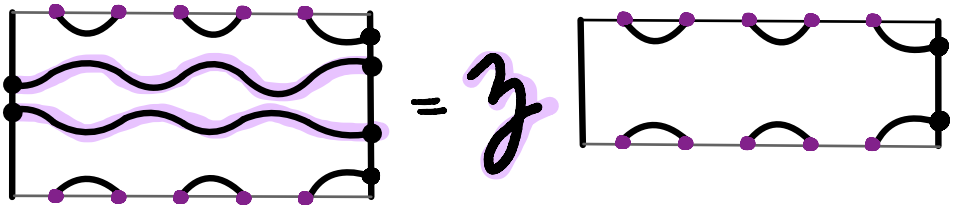
(Relations coming from projection behavior.)

Given our relations above, there still isn't a way to eliminate horizontal strands like these



$\Rightarrow TL_K$ is infinite-dimensional

But



for some $z \in Z(TL_K)$.

So the action of these elements are proportional on any simple module, and

$$Bl_K = TL_K \Big/ \langle \text{⟨image⟩} = z \rangle$$

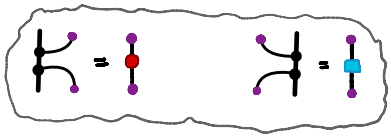
is finite dim'l.

fix another parameter in $\mathbb{C}$

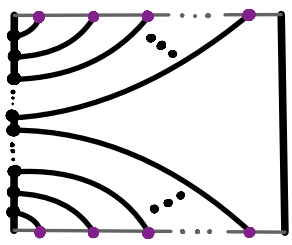
"Simplectic blob algebra" (DeGier - Nichols) Green, Martin, Parker, King...

Good for exploring representations of TL_K : Bl_K diagrammatically..

"Cellularity" "Quasihereditiy" special parameters...



Example Consider the left ideal of Bl_n generated by $d =$



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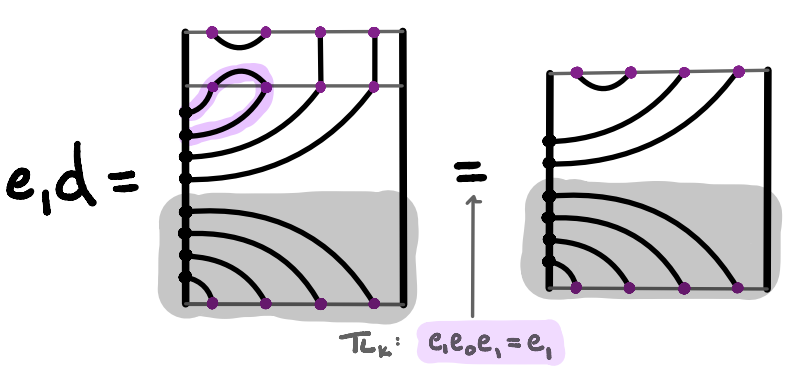
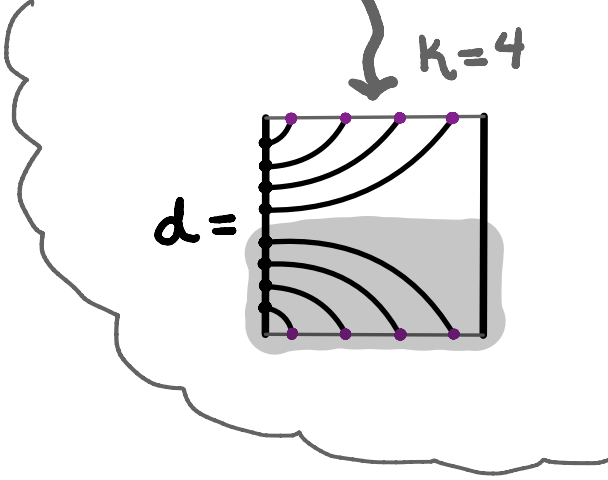
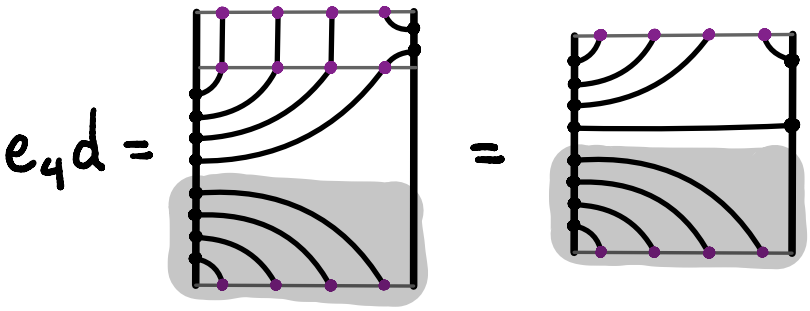
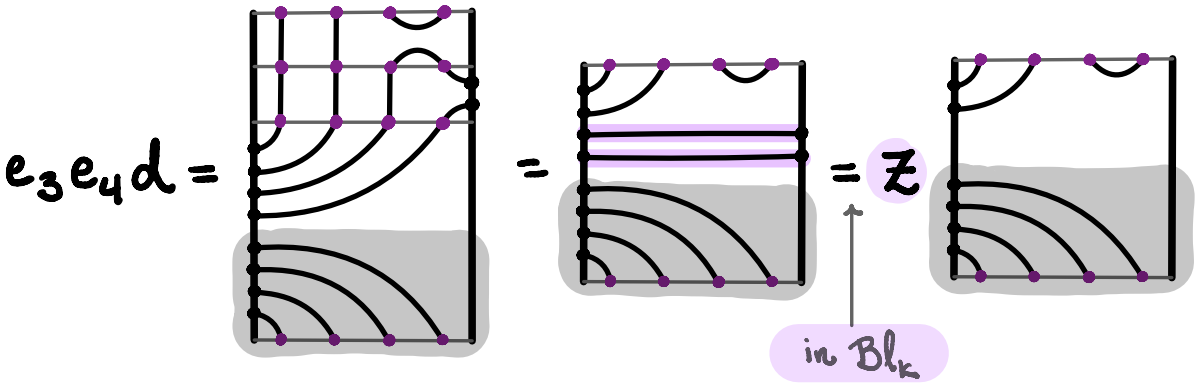




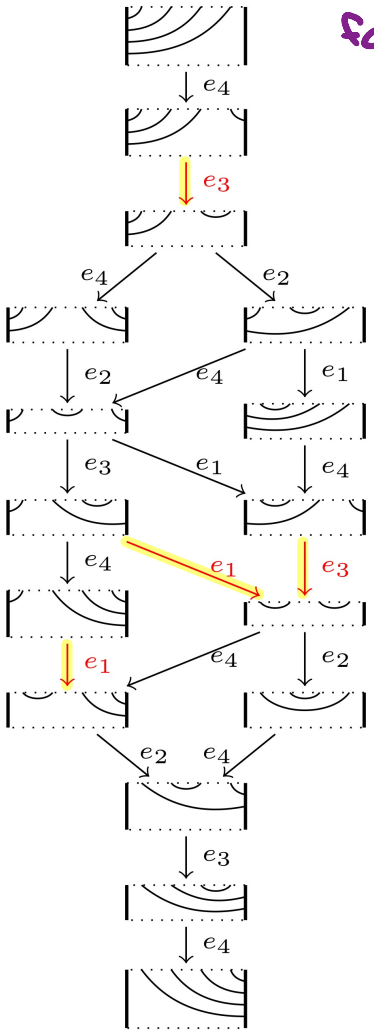




Observe The bottom of each diagram is fixed across $Bl_k d$, and $Bl_k \subset Bl_k d$ doesn't depend on the bottom, so we label this module by half diagrams.

Caution: Track parity of wall connections to determine when z appears!

{even → odd} versus {odd → even}



favorite / most vexing **standard module**

w/ 0 prop. stands (Generic case for TL_k)

$\xrightarrow{e_i}$ indicates a coef of $\mathbb{Z}$.

Q. For what choices of z does this module split / fail to be simple?

Caution: Submodule structure is not necessarily obvious from this diagram structure...

Rewind

Back in the classical setting...

Schur-Weyl duality gave us a correspondance between...

Classical Temperley-Lieb: ↓

$\mathbb{T}_k$-modules $\xleftrightarrow{SWD}$ $U_q\mathfrak{sl}_2$-modules

partitions w/ ≤ 2 parts

"$\subseteq$" (keep whatever factors through quotient) ↓

Actions on $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ (k factors, n rows)

$\mathbb{H}_k$-modules $\xleftrightarrow{SWD}$ $U_q\mathfrak{sl}_n$-modules

finite type A Hecke ↑

partitions w/ $\leq n$ parts

2-boundary case:

TL_k-modules $\longleftrightarrow$ (SWD) $\longleftrightarrow$ $U_q\mathfrak{sl}_2$-modules

special box arrangements w/ < 2 "parts"

$\downarrow$ "$\subseteq$" (keep whatever factors through quotient)

$\uparrow$ "$\supseteq$"

Actions on $M \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n \otimes N$ (k factors; n)

H_k-modules $\longleftrightarrow$ (SWD) $\longleftrightarrow$ $U_q\mathfrak{sl}_n$-modules

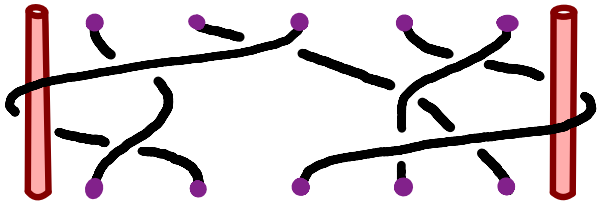
special box arrangements w/ $< n$ "parts"

(D-Ram – arXiv: 1804.10296)

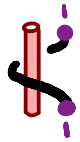
In paper: $q_i = t_i^{1/2}$

Two boundary Hecke algebras $H_k = H_k(q, q_0, q_k)$

aka Affine Hecke algebra of type $(C, C^\vee)$

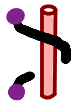


Generated by

$T_0 =$ 

$T_i =$ 

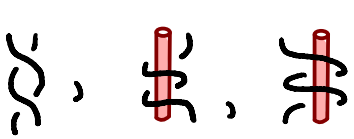
$i = 1, \ldots, n-1$

$T_k =$ 

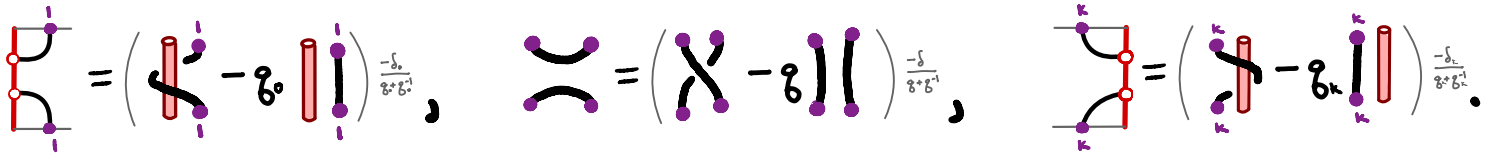
w/ relations

$$(T_j - q_j)(T_j + q_j^{-1}) = 0.$$

$q_1 = \cdots = q_{k-1} = q$

↳ reduces 

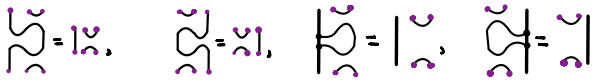
Define:



Thm $TL_k \cong H_k / \circledast$

and TL_k-modules are H_k-modules that factor through $/\circledast$.

$\circledast$ $e_i e_{i\pm1} e_i = e_i, \quad 1 \le i \le k-1$



(see paper for formulas)

What do we coin? ← (arXiv:2009.02812)

① H_k: Deformed, decorated type C

→ Representation theory governed by type-C combinatorics

→ intertwining operators

② Braids give very useful perspective!

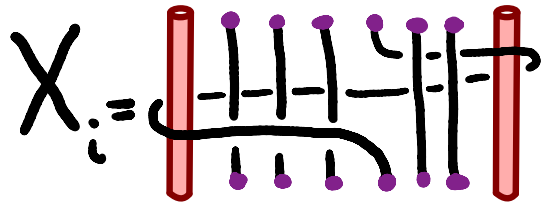
For example...

Two-pole braids have nice Jucys-Murphy elts

→ Calibrated modules

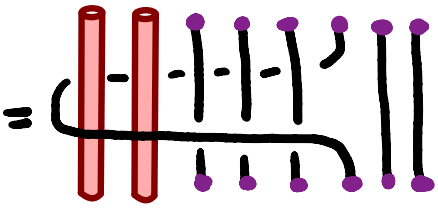
Classify modules by weight of X_i-actions.

$X_i =$



"W_i" in paper

$=$



Calibrated H_k-modules

W = type BC Weyl group (signed permutations)

point $(c_1, c_2, \dots, c_k)$ — set of roots (depends on $\vec{c}$) — set of signed permutations that depend on J

$$H^{(\vec{c}, J)} = \mathbb{C}\{v_\omega \mid \omega \in F(\vec{c}, J) \subseteq W\}$$

deal with one coset of $\mathbb{Z}^k$ at a time.

$$\forall\, X_i v_\omega = -q^{2c_{\omega^*(i)}} v_\omega, \quad i = 1, \dots, k.$$

(essentially forces actions of T_i's)

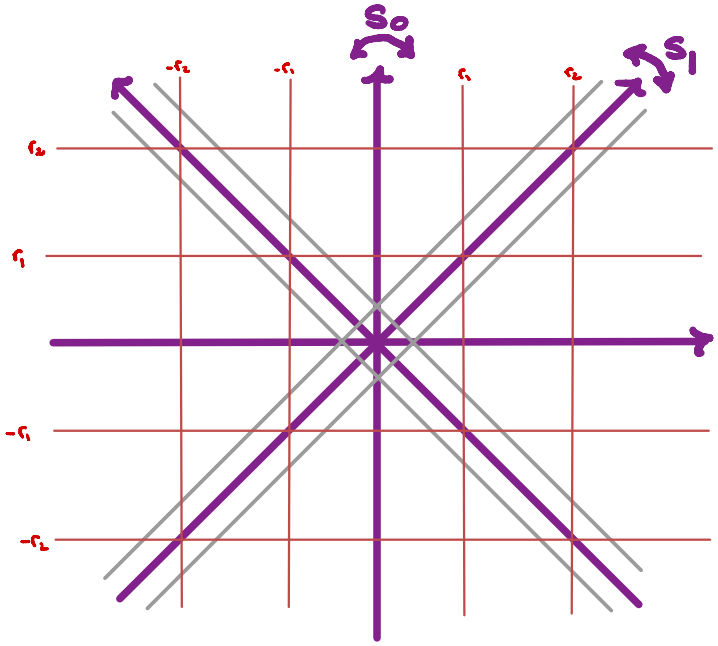
Example: $k=2$

$\vec{c} = (c_1, c_2)$

$W = \langle s_0, s_1 \mid s_i^2 = 1,\ (s_0 s_1)^4 = 1 \rangle$

$= \{$reflection group of ✳$\}$

$\leftrightarrow$ chambers of ✳



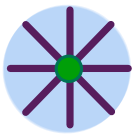
Other important hyperplanes:

$c_2 = c_1 \pm 1,\ -c_1 \pm 1$

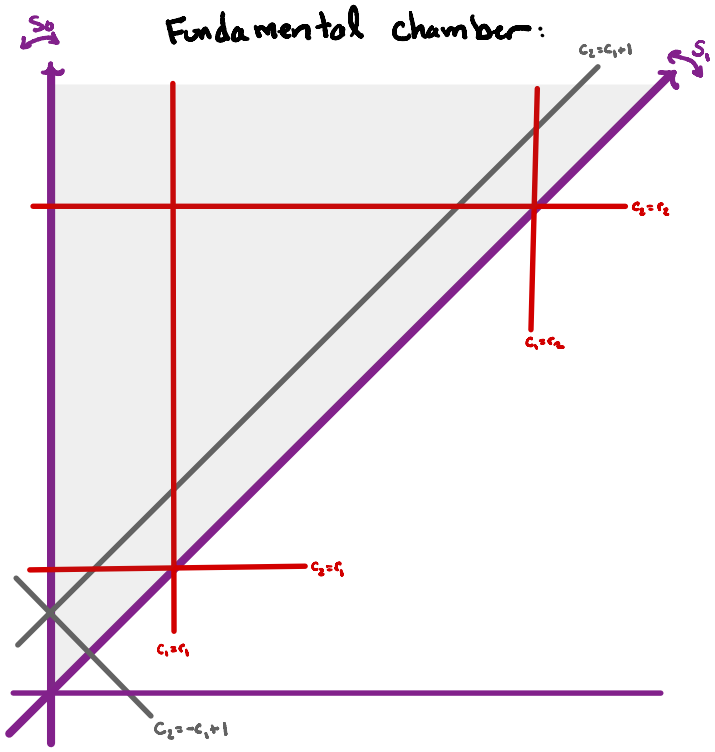
$c_1 c_2 = r_1 r_2$ (encodes parameters $q^{2r_1} = q_k/q_0$, $q^{2r_2} = -q_k q_0$)

At any point, hyperplanes locally partitions space.

$J \leftrightarrow$ parts



Fundamental chamber:



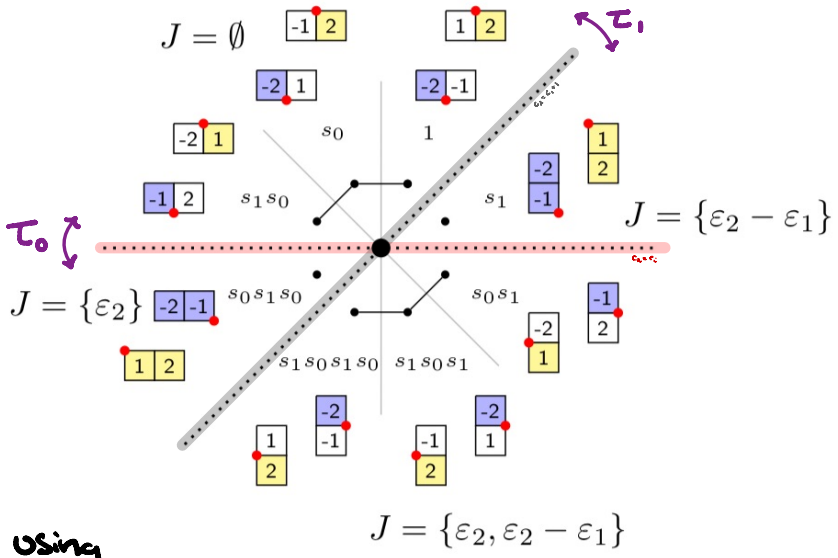
A little more detail...

↳ Solid and dotted hyperplanes have to do when intertwiners in H_k are not defined or not invertible.

↳ J is a choice of positive roots, coming from hyperplanes or point intersects

↳ For "nice" points, there is a nice encoding of $(\vec{c}, J)$ and alcoves using generalized skew shapes and standard tableaux / fillings.

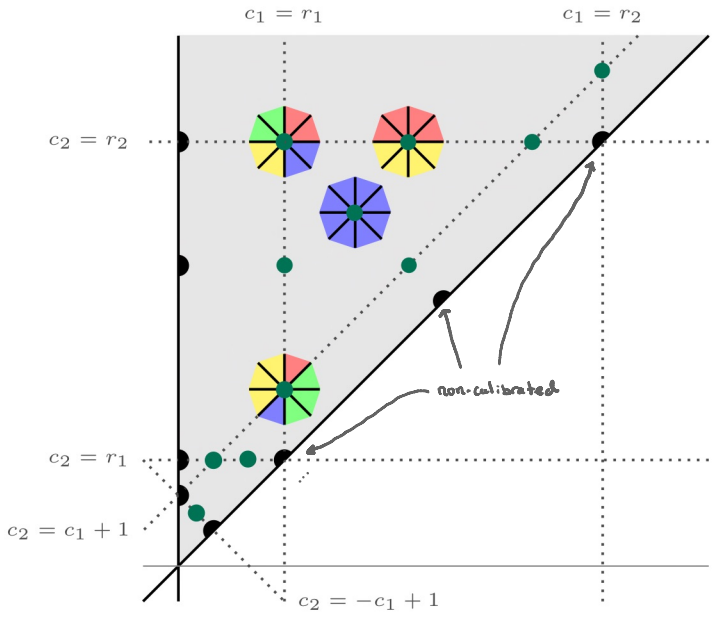
(Arrange boxes on rungs labeled by $\mathbb{Z}$ according to the point $\vec{c}$, and choose relative positions according to J.)



Thm (D.-Ram)

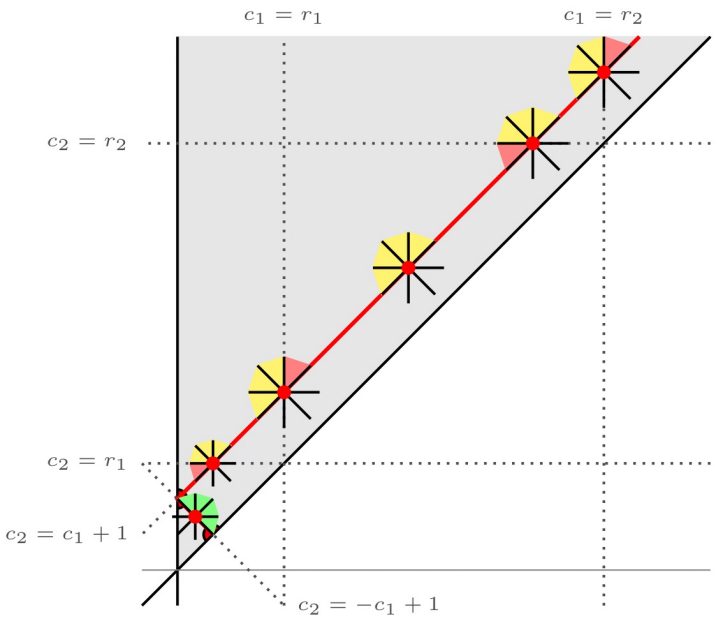
(1) Simple calibrated H_k-mods are indexed by $(\vec{c}, J)$, where $\vec{c}$ is interior to the fundamental chamber.

(arXiv: 1804.10296)



(2) Simple calibrated TL_k-mods are H_k-mods sitting on $\vec{c} = (c_1, c_1+1, c_1+2, \ldots, c_1+k-1)$ w/ J on the positive side.

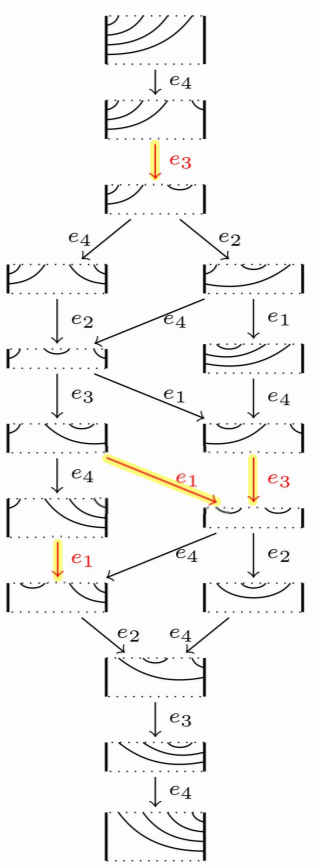
(arXiv: 2009.02812)



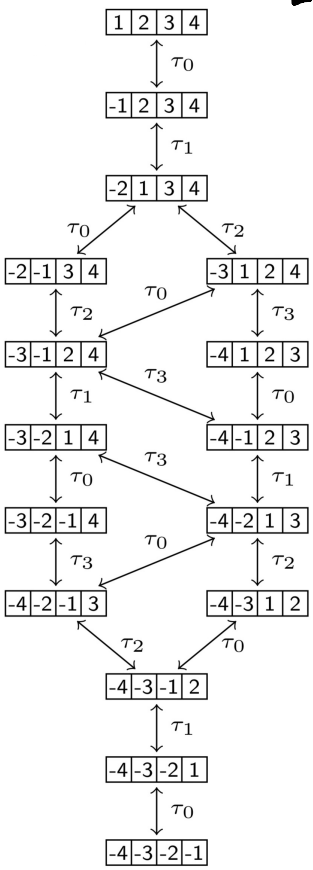
Diagrammatics of
- De Gier - Nichols
- Green, Martin, Parker, King...

Principal series modules

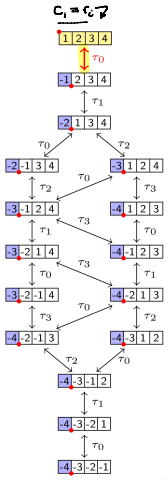
- D., Ram



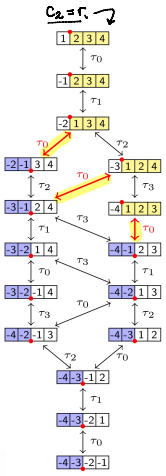
Generic weight ↘



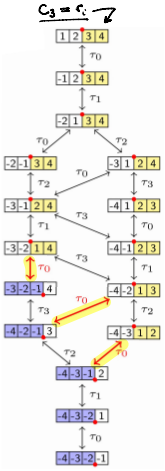
$C_1 = \tau_0 \tau_2$



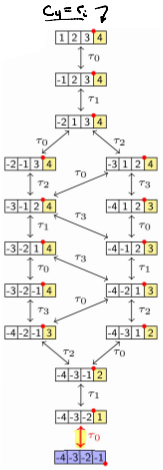
$C_2 = \tau_1 \tau_3$



$C_3 = \tau_0 \tau_1 \tau_3$



$C_4 = \tau_0 \tau_2 \tau_3$



Thanks!