Permutations, Partitions, & Posets:
A taste of combinatorial representation theory

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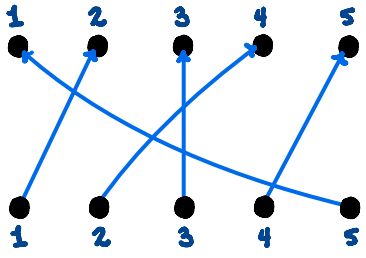
Permutations
& the symmetric group

A **permutation** is a bijective (1-to-1) function.

We can draw

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

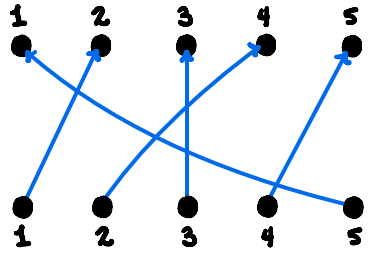
as a **permutation diagram**:

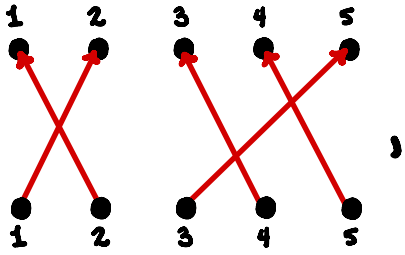
$\sigma =$ 

means

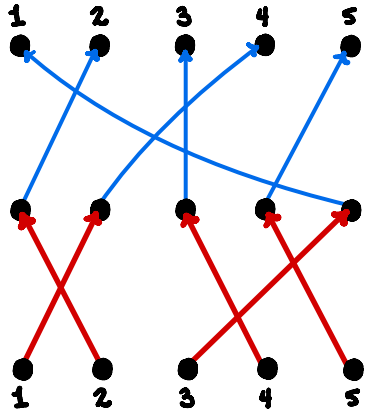
$1 \mapsto 2$
$2 \mapsto 4$
$3 \mapsto 3$
$4 \mapsto 5$
$5 \mapsto 1$

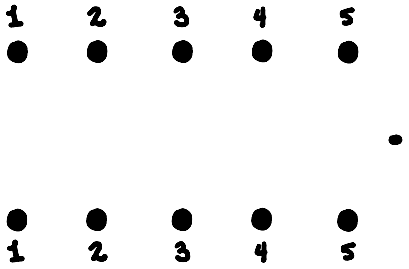
Functions "multiply" via composition:

Ex. If $\sigma =$ 

and $\tau =$ 

,

then $\sigma\tau =$ 

$=$ 

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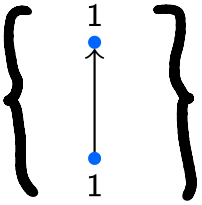
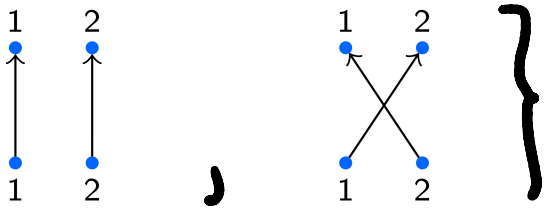
The **symmetric group**(*) S_n is the set of permutations of $1, 2, \ldots, n$, together with "multiplication" given by function composition.

(*) A "group" is a set G together with an associative multiplication $*: G \times G \to G$ satisfying a couple of conditions:
(i) There is an identity element 1, satisfying $1*g = g$ and $g*1 = g$ for all g in G; and
(ii) Every element $g \in G$ has an inverse $g^{-1} \in G$, satisfying $g*g^{-1} = 1$ and $g^{-1}*g = 1$.
Ex: $\mathbb{Q}_{>0}$ with multiplication
$\mathbb{Q}$ with addition
Non ex: $\mathbb{Q}$ with multiplication

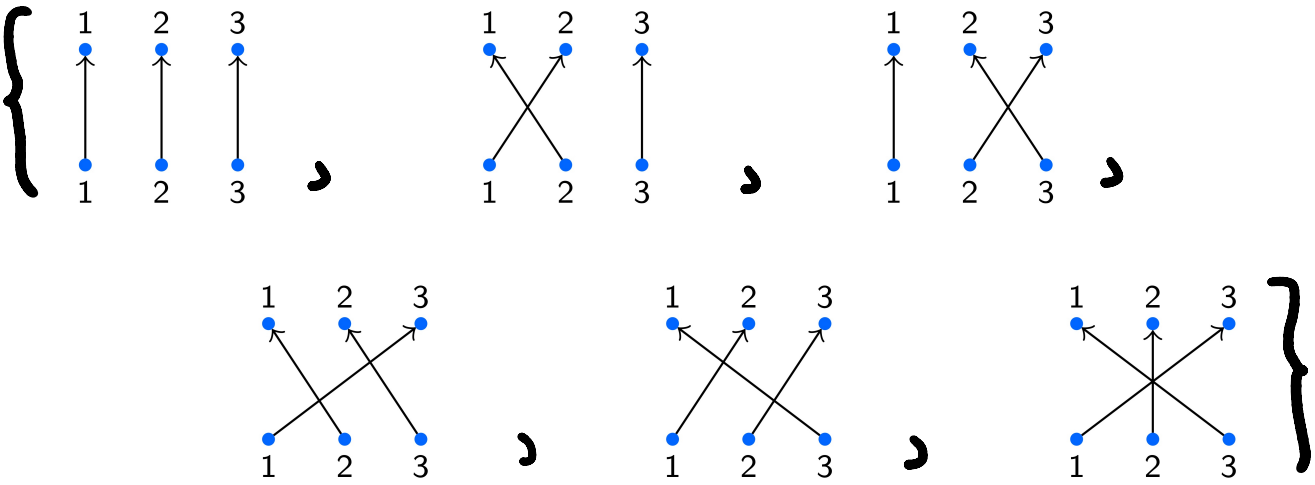
Some examples of S_n:

$S_1 =$ { 

}

$S_2 =$ { 

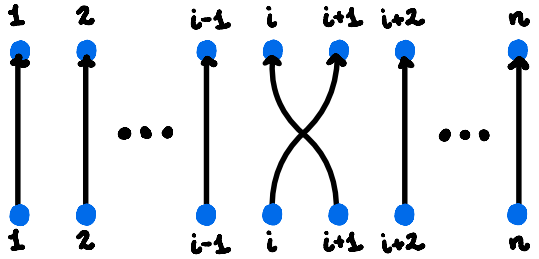
}

$S_3 =$ 

Building blocks of S_n:

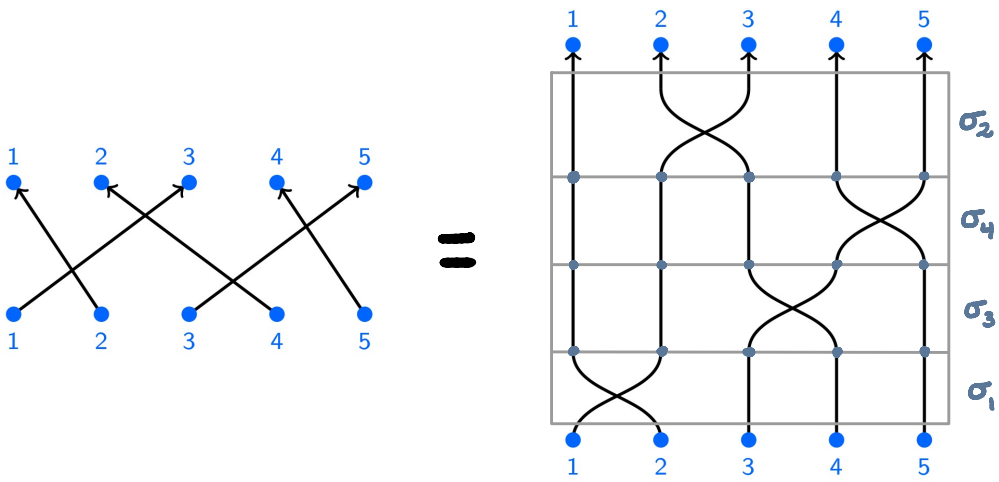
"generators" $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$.

Let σ_i be the permutation that swaps $i \leftrightarrow i+1$ (and fixes everything else)



"adjacent transposition"

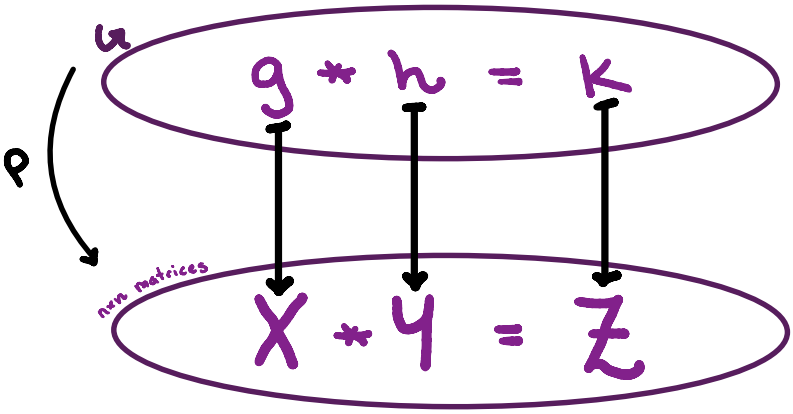
I can build any permutation out of $\sigma_1, \sigma_2, \ldots, \sigma_n$:



Representations

"homomorphism"

A representation "ρ" of a group G is a map from G to a set of matrices that follow the same multiplication rules as the elements of G:



$$\rho(g) * \rho(h) = \rho(g * h).$$

Ex. The permutation representation of S_3 (same idea for any n)

Fix a favorite basis of $\mathbb{C}^3$:

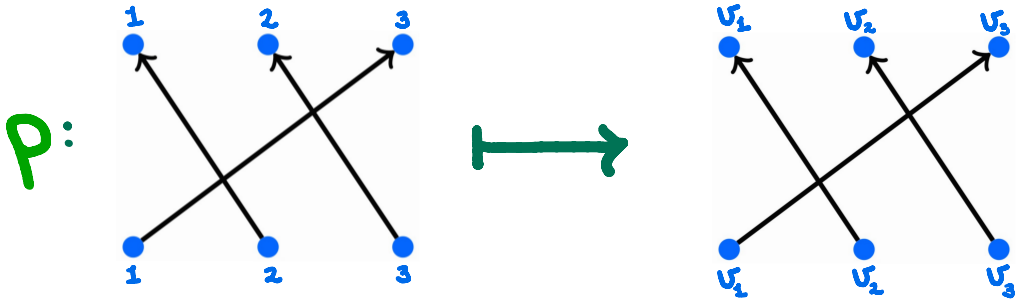
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $\rho: S_3 \longrightarrow \{3\times 3 \text{ matrices over } \mathbb{C}\} = "M_3(\mathbb{C})"$ (technically $\longrightarrow GL_3(\mathbb{C}) = \{\text{invrble } 3\times 3 \text{ matrices over } \mathbb{C}\}$)

send a permutation σ to a matrix that permutes the fixed basis vectors* correspondingly...

* Remember: a linear function (matrix) is uniquely determined by where the basis goes.

For example,



Building the matrix:

Once you have fixed a basis $v_1, v_2, \dots, v_n$ of $\mathbb{C}^n$, the matrix for a linear transformation

$$f: \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

is

$$\begin{pmatrix} | & | & & | \\ f(v_1) & f(v_2) & \cdots & f(v_n) \\ | & | & & | \end{pmatrix}$$

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Challenge 1: Compute the matrices for all $\sigma \in S_3$. (Answers below)

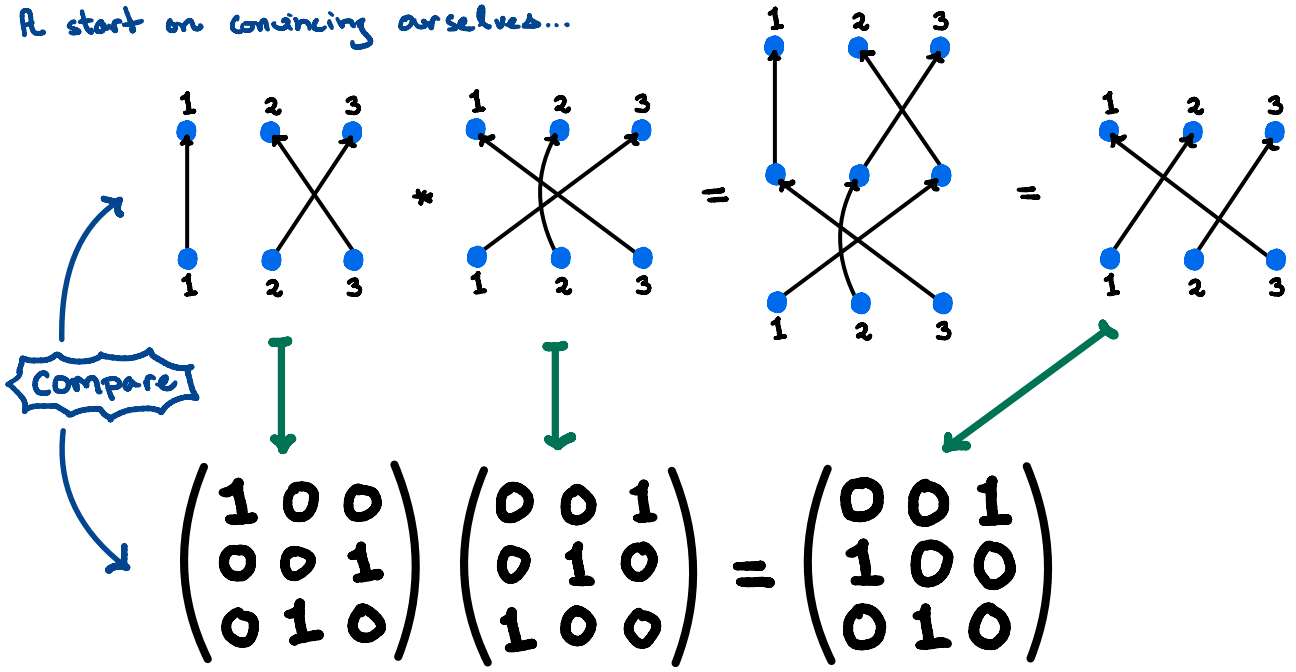
Challenge 2: Let $v_1, \dots, v_n$ be a basis of $\mathbb{C}^n$, and define

$$\rho: S_n \to M_n(\mathbb{C}) \text{ by } \rho(\sigma) v_i = v_{\sigma(i)}.$$

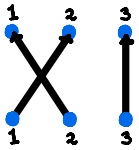
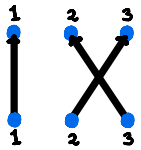
Check that $\rho(\sigma)\rho(\tau) = \rho(\sigma\tau)$ for all $\sigma, \tau \in S_n$ by checking that the LHS & RHS agree on all basis vectors v_i.

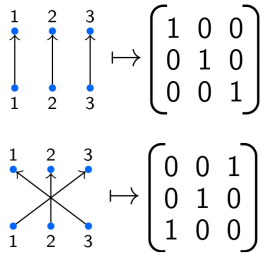
(Why is this sufficient?)

A start on convincing ourselves...



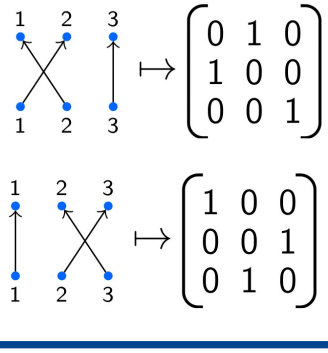
Some notes:

① Because $\sigma_1 =$  and $\sigma_2 =$  generate all permutations in S_3, and ρ preserves multiplication, I only really need $\rho(\sigma_1)$ & $\rho(\sigma_2)$.



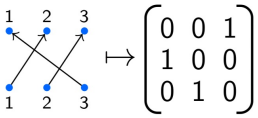
$$\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

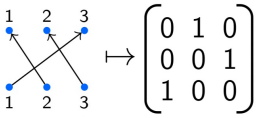


$$\mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



$$\mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



$$\mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

② Every matrix fixes the subspace

$$\mathbb{C}\{v_1 + v_2 + v_3\}.$$

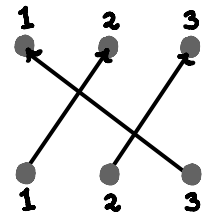
"invariant subspace"

↳ for all $\sigma \in S_n$, $\rho(\sigma)$ just rearranges $\{v_1, v_2, v_3\}$, and addition is commutative.

Let's change basis and see what happens...

$$w_1 = v_1 - v_2, \quad w_2 = v_2 - v_3, \quad w_3 = v_1 + v_2 + v_3.$$

Ex:

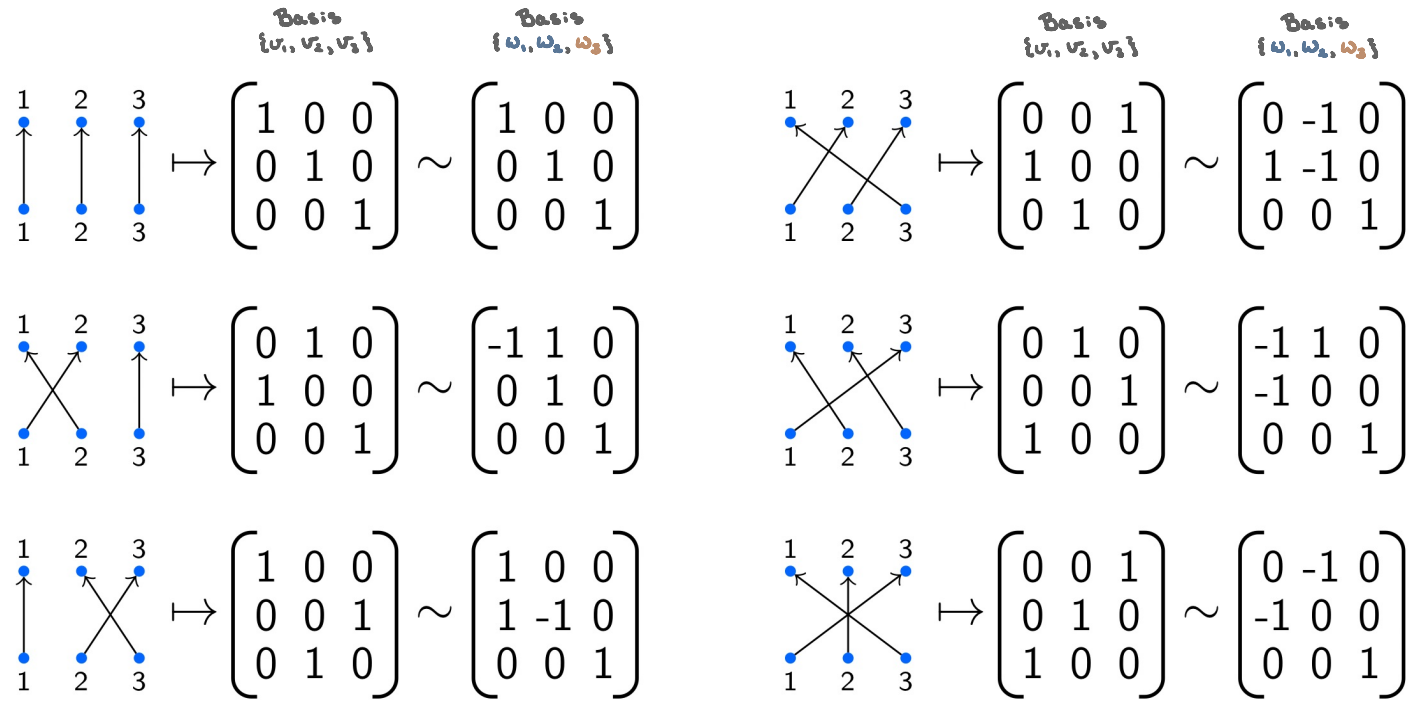


$$w_1 \mapsto v_2 - v_3 = w_2$$

$$w_2 \mapsto v_3 - v_1 = -w_1 - w_2$$

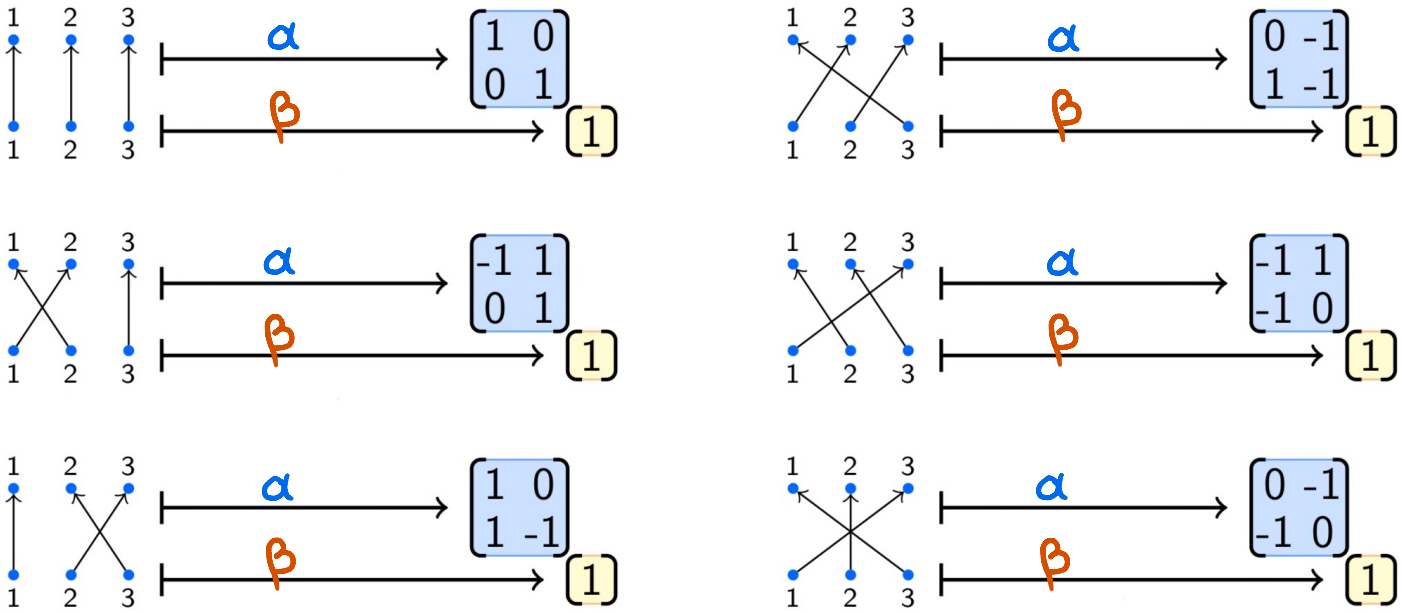
$$w_3 \mapsto v_2 + v_3 + v_1 = w_3$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



By changing basis, we can see that the permutation representation ρ splits/decomposes into two simple representations α & β:

(simple → neither splits any further)



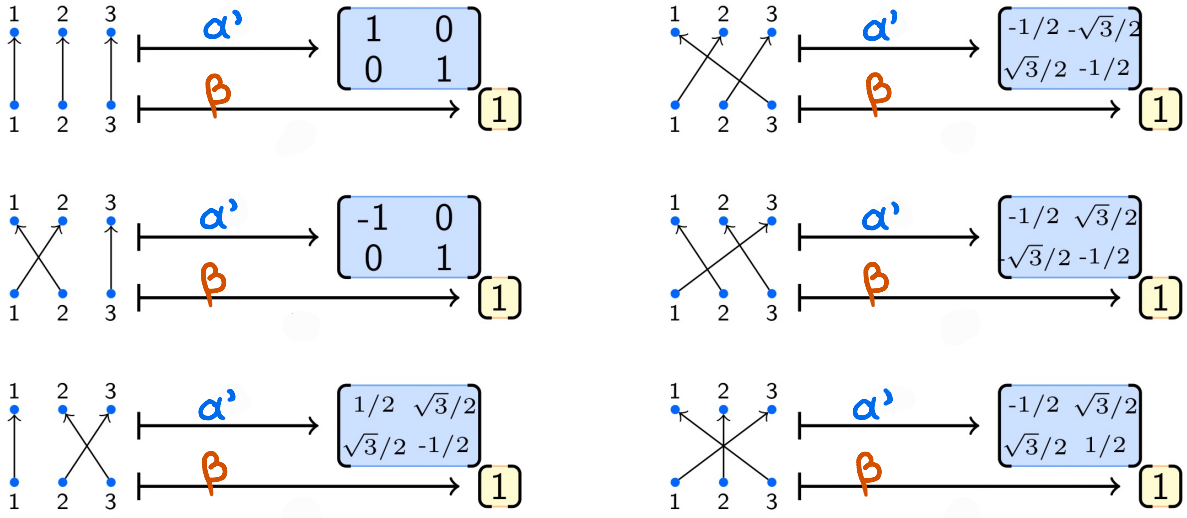
We write $\rho \cong \alpha \oplus \beta$.

↳ "isomorphic"

Tinkering a <u>little</u> more with the basis can additionally lend some symmetry to our decomposition:

$$\omega_1 = \sqrt{3}(v_1 - v_2), \qquad \omega_2 = v_1 + v_2 - 2v_3, \qquad \omega_3 = v_1 + v_2 + v_3.$$

gives



We say $\alpha \cong \alpha'$.

... meaning α & α' are the same, up to a change of basis.

Representations ✓

Combinatorics?

Partitions

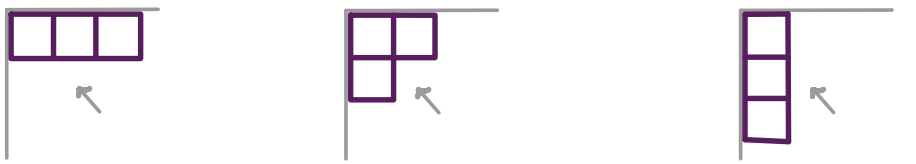
Fix an integer $n \geq 0$.

A **partition** λ of n is a collection of positive integers that sum to n.

Convention: Write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$
with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$
and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Example. There are exactly three partitions of $n=3$:

(3) $(2,1)$ $(1,1,1)$



We draw partitions as collections of boxes packed up and to the left, where λ_i = # boxes in row i.

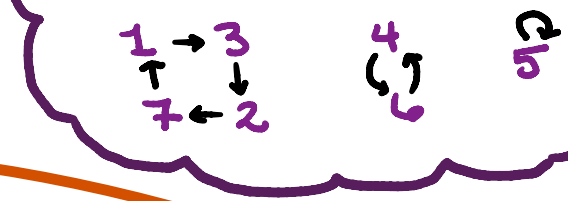
↳ "parts" of the partition

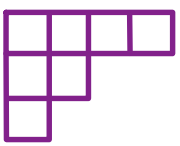
One connection to S_n:

"Cycle notation"

The **cycle type** of σ is the collection of sizes of the cycles.

$(1\ 3\ 2\ 7)(4\ 6)(5)$ means





"Cycle type" is a partition!

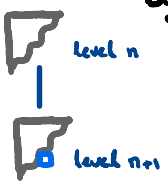
Thm. Two permutations are conjugate if and only if the same cycle type

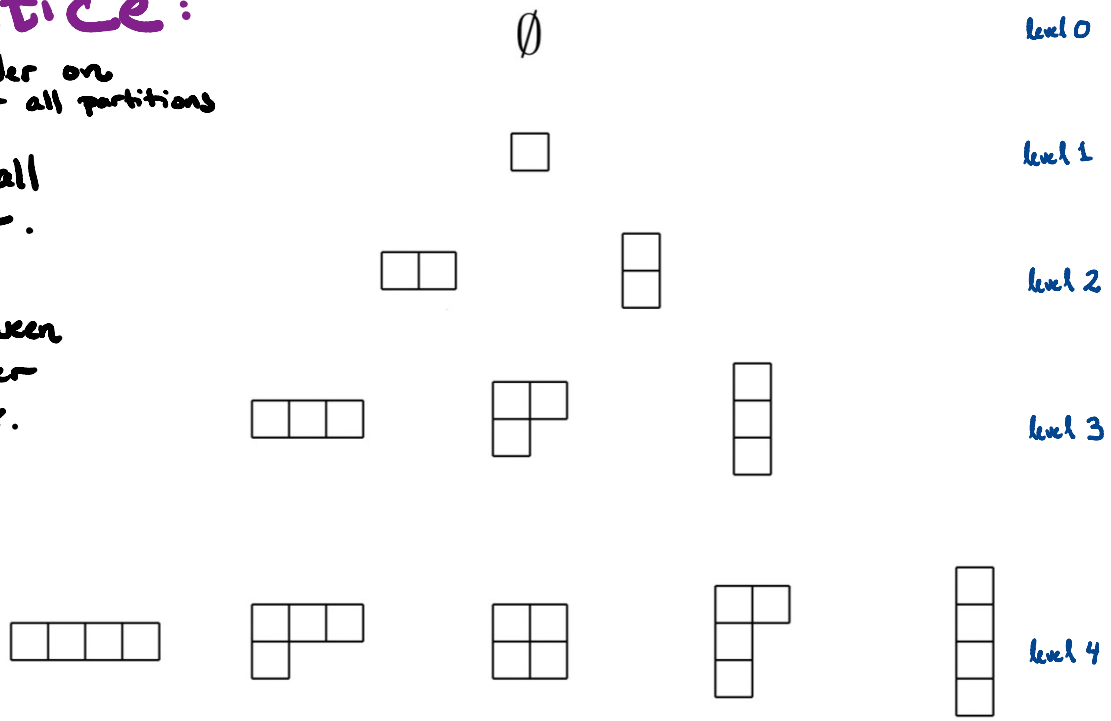
Theorem. Up to isomorphism, the simple representations of S_n are in bijection ("indexed by") with the partitions of n.

Young's Lattice:

partial order on the set of all partitions

- On level n, draw all partitions of n.
- Draw an edge between partitions that differ by one box.





A **λ-tableau** is a downward-moving path from ∅ down to λ.

Challenge 3: Draw level 5.

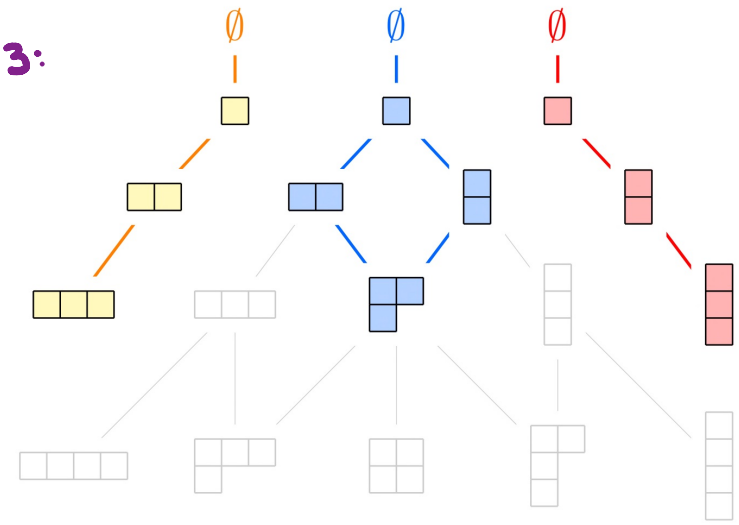
Challenge 4: Can you find a recurrence relation for the number of λ-tableaux (using Young's Lattice)?

(Hint: Pascal's triangle.)

Challenge 5: For $n=1,2,3,4,\ldots$, count the total number of paths from ∅ down to level n, and then back up to ∅ again. Do you see a pattern?

Theorem. If λ is a partition of n, then the representation of n indexed by λ has a basis indexed by λ-tableaux...

Ex. $n=3$:



Moreover, the matrices have formulas given in terms of more combinatorial data!

The content of a box in λ is the "diagonal" number

column(box) − row(box).

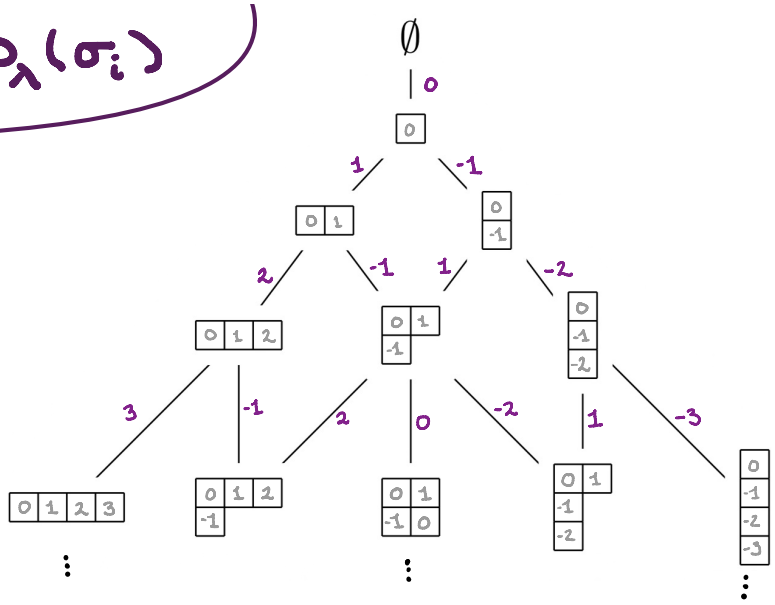
0	1	2	3	4	5	6
-1	0	1	2			
-2	-1	0	1			
-3	-2	-1				
-4	-3	-2				
-5						

To review:

- Each partition λ is secretly a representation; P_λ
- Each path T is secretly a basis vector; v_T
- Any representation is determined by the matrices for the generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$. (adjacent transpositions)

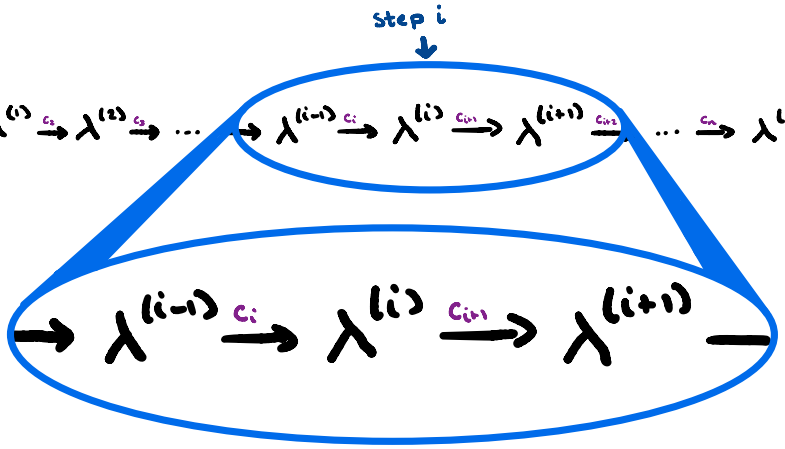
Building a matrix for $\rho_\lambda(\sigma_i)$

Label the edges of Young's Lattice with the content of the box added:



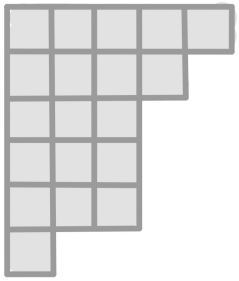
Observe!

For a path $\emptyset \xrightarrow{c_1} \lambda^{(1)} \xrightarrow{c_2} \lambda^{(2)} \xrightarrow{c_3} \cdots \to \lambda^{(i-1)} \xrightarrow{c_i} \lambda^{(i)} \xrightarrow{c_{i+1}} \lambda^{(i+1)} \xrightarrow{c_{i+2}} \cdots \xrightarrow{c_n} \lambda^{(n)} = \lambda \ldots$

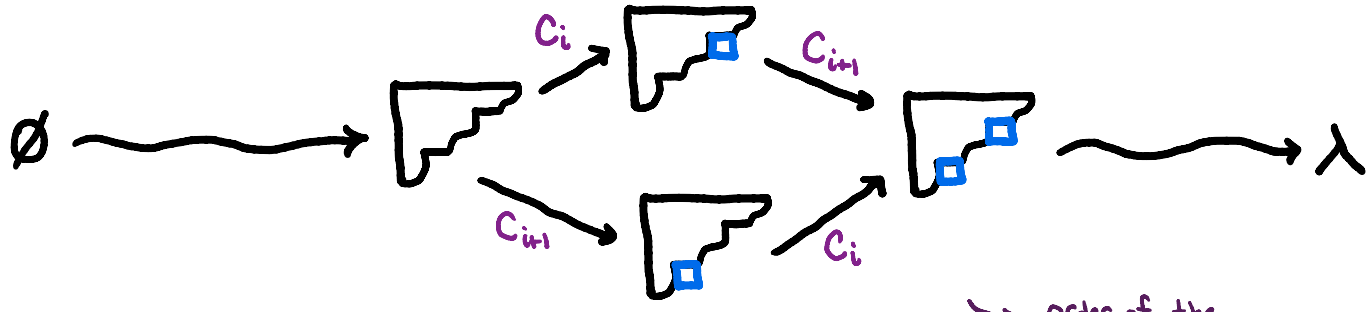


① We must have $c_{i+1} \neq c_i$.

② If $c_{i+1} = c_i + 1$ or $c_{i+1} = c_i - 1$, then the $i+1$ box is next to the i box, and they had to come in that order.



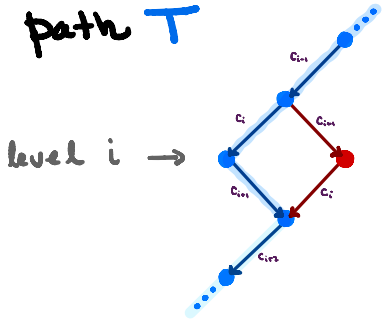
③ If $|c_{i+1} - c_i| > 1$, then boxes i & $i+1$ are far enough to be swapped, and there is exactly one path that is the same everywhere except $\lambda^{(i)}$

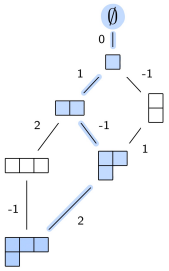


↳ order of the contents swap $c_i \leftrightarrow c_{i+1}$

Still: we've fixed n, λ, i; i, and we want a matrix for $\rho_\lambda(\sigma_i)$

Suppose v_T goes with a path T





and $v_{T'}$ goes with a path T' that only changes at step i (if it exists).

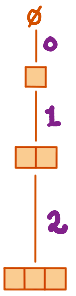
Then in $\rho(\sigma_i)\, v_T$, the coefficient of...

↳ v_T is $\dfrac{1}{c_{i+1} - c_i}$

↳ $v_{T'}$ is $\sqrt{1 - \left(\dfrac{1}{c_{i+1} - c_i}\right)^2}$

↳ any other basis vector is 0.

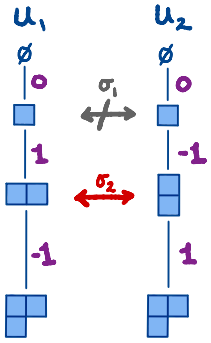
Back to S_3:



$\sigma_1 \mapsto (1/1-0) = (1)$

$\sigma_2 \mapsto (1/2-1) = (1)$

"trivial"



$$\sigma_1 \mapsto \begin{pmatrix} \frac{1}{1-0} & 0 \\ 0 & \frac{1}{-1-0} \end{pmatrix}$$

$$\sigma_2 \mapsto \begin{pmatrix} \frac{1}{-1-1} & \sqrt{1-(1/2)^2} \\ \sqrt{1-(-1/2)^2} & \frac{1}{1-(-1)} \end{pmatrix}$$

"reflection"



$\sigma_1 \mapsto (1/-1-0) = (-1)$

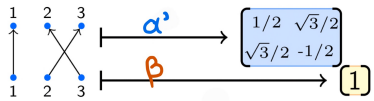
$\sigma_2 \mapsto (1/-2-(-1)) = (-1)$

"sign"
"alternating"

The permutation rep splits as...

Recall! We've seen these before...





Observe:

$(*) \oplus \begin{pmatrix} * & * \\ * & * \end{pmatrix} \oplus (*)$ has 6 "degrees of freedom"

and $|S_3| = 6$ ... not a coincidence!

Instead of just S_n, we can study the **group algebra**

$$\mathbb{C}S_n = \left\{ \sum_{\sigma \in S_n} c_\sigma \sigma \;\middle|\; c_\sigma \in \mathbb{C} \right\}$$

↳ add and multiply like polynomials.

Ex: $\mathbb{C}S_3 = \left\{ \begin{matrix} c_1 + c_2(12) + c_3(123) \\ + c_4(13) + c_5(123) + c_6(132) \end{matrix} \;\middle|\; c_i \in \mathbb{C} \right\}$

is a 6-dimensional vector space that also has a multiplication operation.

ex: $(\frac{1}{2} - 5(12))(-1 + (12) + (123))$

$= -\frac{1}{2} + \frac{1}{2}(12) + \frac{1}{2}(123)$

$+ 5(12) - 5\underbrace{(12)(12)}_{=1} - 5\underbrace{(12)(123)}_{=(23)}$

$+ (\frac{1}{2} - 5) + (\frac{1}{2} + 5)(12) + \frac{1}{2}(123) - 5(23).$

Theorem. "The simple representations are the building blocks of the algebra."

See: Artin-Wedderburn

The map

$$\bigoplus_{\lambda \vdash n} \rho_\lambda : \mathbb{C}S_n \longrightarrow \bigoplus_{\lambda \vdash n} M_{n_\lambda}(\mathbb{C})$$

↓ the partitions λ of n

↳ $= \#$ λ-tableaux

is an isomorphism.

⊛ See Challenge 5...
Can you make a connection?

Our favorite example:

$$\rho_{\square\square\square} \oplus \rho_{\square\square} \oplus \rho_{\square} : \mathbb{C}S_3 \longrightarrow M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$$

$$\sigma \longmapsto \begin{pmatrix} \rho_{\square\square\square}(\sigma) & 0 & 0 & 0 \\ 0 & & \rho_{\square\square}(\sigma) & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & \rho_{\square}(\sigma) \end{pmatrix}$$

In summary

Given a group, ring, algebra, etc. A...

In **representation theory**, we ask...

- What are the  rep'ns of A?
  - "simple," "polynomial," "projective," ...
- How much of A can I build from ?

In **Combinatorial** rep'n theory, we ask...

- Is there a nice puzzle/game I can play to enumerate/encode representation-theoretic data?
  - (list of representations, how big they are, what they look like ...)
- If I've seen this game before, is it because of a deeper connection?
  - eg. What do conjugacy classes have to do with simple representations?
  - eg. There are other objects in algebra & elsewhere whose structure relates to partitions/tableaux. Is there a meaningful connection to S_n too?

See: "Combinatorial Representation Theory", by Barcelo & Ram

Thank you!