An Algebraic Approach to Voting Theory

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Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

1st :	A	A	B	B	C	C
2nd :	B	C	A	C	A	B
3rd :	C	B	C	A	B	A
	13	0	19	0	17	5

So who should be president?

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	13	0	19	0	17	5

So who should be president?

Method 1: Vote for your first choice.

A	B	C
13	19	22

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13	19	22

Carly wins!

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13	19	22

Carly wins!

Method 2: Tell us your full ranking, and we'll pair them off.

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	13	0	19	0	17	5

So who should be president?

Method 1: Vote for your first choice.

A	B	C
13	19	22

Carly wins!

Method 2: Tell us your full ranking, and we'll pair them off.

$A > B$	$B > C$	$A > C$
30	32	32

$A < B$	$B < C$	$A < C$
24	22	22

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

1^{st} :	A	A	B	B	C	C
2^{nd} :	B	C	A	C	A	B
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$A > B$	$B > C$	$A > C$
30	32	32
$A < B$	$B < C$	$A < C$
24	22	22

$A > B, \quad B > C, \quad A > C.$

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

1^{st} :	A	A	B	B	C	C
2^{nd} :	B	C	A	C	A	B
3^{rd} :	C	B	C	A	B	A
	13	0	19	0	17	5

So who should be president?

Method 1: Vote for your first choice.

A	B	C
13	19	22

Carly wins!

Method 2: Tell us your full ranking, and we'll pair them off.

$A > B$	$B > C$	$A > C$
30	32	32
$A < B$	$B < C$	$A < C$
24	22	22

$A > B$, $B > C$, $A > C$.

Alice wins!

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

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2^{nd} :	B	C	A	C	A	B
3^{rd} :	C	B	C	A	B	A
	13	0	19	0	17	5

So who should be president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

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So who should be president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

A	B	C
13	19	22

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Alice loses.

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So who should be president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

A	B	C
13	19	22

Alice loses.

Round 2:

B	C
32	22

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

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So who should be president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

A	B	C
13	19	22

Alice loses.

Round 2:

B	C
32	22

Becca wins!

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

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So who should be president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

A	B	C
13	19	22

Alice loses.

Round 2:

B	C
32	22

Becca wins!

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

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So who should be president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

A	B	C
13	19	22

Alice loses.

Round 2:

B	C
32	22

Becca wins!

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

A	B	C
$13 + 36t$	$19 + 18t$	$22 + 0t$

$(0 \leqslant t \leqslant 1)$

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

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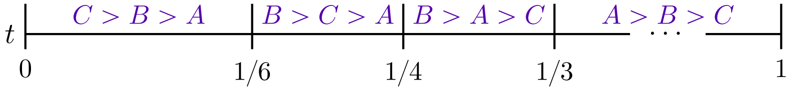
	13	0	19	0	17	5

So who should be president?

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

	A	B	C
$(0 \leqslant t \leqslant 1)$	$13 + 36t$	$19 + 18t$	$22 + 0t$
$t = 0$:	13	19	22
$t = 0.25$:	22	23.5	22
$t = 0.5$:	31	28	22
$t = 0.75$:	40	32.5	22
$t = 1$:	49	37	22



In 2000, in Florida, 2,912,790 people voted for Bush, 2,912,253 voted for Gore, 97,488 voted for Nader, and 40,575 for other. Neglecting the "others", suppose we had asked the rest of the voters for their full rankings of the top three candidates.

1^{st} :	B	G	G	N
2^{nd} :	G	N	B	G
3^{rd} :	N	B	N	B
	2.902mil	1.421mil	1.481mil	0.118mil

So who should have been president?

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So who should have been president?

Method 1: Vote for your first choice.

B	G	N
2.90mil	2.90mil	0.11mil

Bush won.

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2.90mil	2.90mil	0.11mil

Bush won.

Method 2: Tell us your full ranking, and we'll pair them off.

$B > G$	$G > N$	$B > N$
2.90mil	5.80mil	4.32mil
$B < G$	$G < N$	$B < N$
3.02mil	0.12mil	1.60mil

$G > B, \quad G > N, \quad B > N.$

Gore wins.

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So who should have been president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

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Nader loses.

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So who should have been president?

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Round 2:

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2.90mil	3.02mil

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2.90mil	3.02mil

Gore wins.

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B	G	N
$2.902 + 1.421\ t$	$2.902 + 3.020\ t$	$0.118 + 1.481\ t$

$(0 \leqslant t \leqslant 1)$

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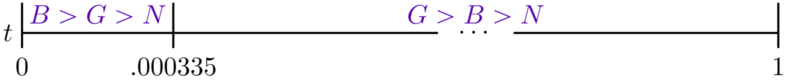
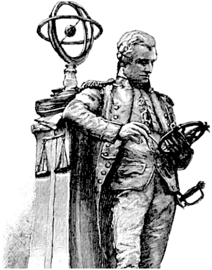
So who should have been president?

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

	A	B	C
$(0 \leqslant t \leqslant 1)$	$2.902 + 1.421\ t$	$2.902 + 3.020\ t$	$0.118 + 1.481\ t$
$t = 0$:	2.90mil	2.90mil	0.19mil
$t = 0.25$:	3.26mil	3.66mil	0.49mil
$t = 0.5$:	3.61mil	4.41mil	0.86mil
$t = 0.75$:	3.97mil	5.17mil	1.23mil
$t = 1$:	4.32mil	5.92mil	1.60mil





Jean-Charles, Chevalier de Borda
1733–1799
Mariner and scientist.
1770: formulated a ranked voting system, the "Borda count". Used by the French Academy of Sciences, until Napolean.

Nicolas de Caritat, Marquis de Condorcet
1743–1794
Philosopher and mathematician.
In 1785, wrote an essay on probability of decisions made on a majority vote, describing likelihood of good jury outcomes; and Condorcet's paradox, which shows that majority preferences can become intransitive with three or more options

Dr. Donald G. Saari (1940–)
Professor of Mathematics and Economics.
1999: Used geometric methods to model voting data as vector spaces, and decompose them based on how they affect various tallying methods.

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Kernel: Doesn't affect any fair voting system.
Borda: Influences both point-based and pairwise systems.
Condorcet: Introduces Condorcet paradox.
Reversal: Influences point-based systems, but not pairwise.

	Kernel	Borda			Condorcet	Reversal		
		$\mathbf{b_A}$	$\mathbf{b_B}$	$\mathbf{b_C}$		$\mathbf{r_A}$	$\mathbf{r_B}$	$\mathbf{r_C}$
ABC	1	1	0	-1	1	1	-2	1
ACB	1	1	-1	0	-1	1	1	-2
BAC	1	0	1	-1	-1	-2	1	1
BCA	1	-1	1	0	1	1	1	-2
CAB	1	0	-1	1	1	-2	1	1
CBA	1	-1	0	1	-1	1	-2	1

							Points $\mathbf{w} = (1, t, 0)$			$A > B$ vs. $A < B$	$B > C$ vs. $B < C$	$C > A$ vs. $C < A$
1^{st} :	A	A	B	B	C	C						
2^{nd} :	B	C	A	C	A	B						
3^{rd} :	C	B	C	A	B	A	A	B	C			
Ker	1	1	1	1	1	1	$2+2t$	$2+2t$	$2+2t$	0	0	0
$\mathbf{b}_A$	1	1	0	-1	0	-1	2	-1	-1	4	0	-4
$\mathbf{b}_B$	0	-1	1	1	-1	0	-1	2	-1	-4	4	0
$\mathbf{b}_C$	-1	0	-1	0	1	1	-1	-1	2	0	-4	4
Cond	1	-1	-1	1	1	-1	0	0	0	2	2	2
$\mathbf{r}_A$	1	1	-2	1	-2	1	$2-4t$	$-1+2t$	$-1+2t$	0	0	0
$\mathbf{r}_B$	-2	1	1	1	1	-2	$-1+2t$	$2-4t$	$-1+2t$	0	0	0
$\mathbf{r}_C$	1	-2	1	-2	1	1	$-1+2t$	$-1+2t$	$2-4t$	0	0	0

Don't worry about negatives: negative votes are fixed by the kernel.

1^{st} :	A	A	B	B	C	C	Points $\mathbf{w} = (1, t, 0)$			$A > B$ vs. $A < B$	$B > C$ vs. $B < C$	$C > A$ vs. $C < A$
2^{nd} :	B	C	A	C	A	B						
3^{rd} :	C	B	C	A	B	A	A	B	C			
Ker	1	1	1	1	1	1	$2+2t$	$2+2t$	$2+2t$	0	0	0
$\mathbf{b}_A$	1	1	0	-1	0	-1	2	-1	-1	4	0	-4
$\mathbf{b}_B$	0	-1	1	1	-1	0	-1	2	-1	-4	4	0
$\mathbf{b}_C$	-1	0	-1	0	1	1	-1	-1	2	0	-4	4
Cond	1	-1	-1	1	1	-1	0	0	0	2	2	2
$\mathbf{r}_A$	1	1	-2	1	-2	1	$2\text{-}4t$	$\text{-}1+2t$	$\text{-}1+2t$	0	0	0
$\mathbf{r}_B$	-2	1	1	1	1	-2	$\text{-}1+2t$	$2\text{-}4t$	$\text{-}1+2t$	0	0	0
$\mathbf{r}_C$	1	-2	1	-2	1	1	$\text{-}1+2t$	$\text{-}1+2t$	$2\text{-}4t$	0	0	0

Don't worry about negatives: negative votes are fixed by the kernel.

Note: The reversal space is trivial *precisely* when $t = 1/2$.

	Points $\mathbf{w} = (1, t, 0)$ A	B	C	$A > B$ vs. $A < B$	$B > C$ vs. $B < C$	$C > A$ vs. $C < A$
Ker	$2 + 2t$	$2 + 2t$	$2 + 2t$	0	0	0
$\mathbf{b}_A$	2	-1	-1	4	0	-4
$\mathbf{b}_B$	-1	2	-1	-4	4	0
$\mathbf{b}_C$	-1	-1	2	0	-4	4
Cond	0	0	0	2	2	2
$\mathbf{r}_A$	$2 - 4t$	$-1 + 2t$	$-1 + 2t$	0	0	0
$\mathbf{r}_B$	$-1 + 2t$	$2 - 4t$	$-1 + 2t$	0	0	0
$\mathbf{r}_C$	$-1 + 2t$	$-1 + 2t$	$2 - 4t$	0	0	0

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Ker	$2 + 2t$	$2 + 2t$	$2 + 2t$	0	0	0
$\mathbf{b}_A$	2	-1	-1	4	0	-4
$\mathbf{b}_B$	-1	2	-1	-4	4	0
$\mathbf{b}_C$	-1	-1	2	0	-4	4
Cond	0	0	0	2	2	2
$\mathbf{r}_A$	$2 - 4t$	$-1 + 2t$	$-1 + 2t$	0	0	0
$\mathbf{r}_B$	$-1 + 2t$	$2 - 4t$	$-1 + 2t$	0	0	0
$\mathbf{r}_C$	$-1 + 2t$	$-1 + 2t$	$2 - 4t$	0	0	0

Don't worry about negatives: negative votes are fixed by the kernel.

Note: The reversal space is trivial *precisely* when $t = 1/2$.

Theorem (Saari '00) Given a full ranking of n candidates, the reversal space is trivial precisely for weight

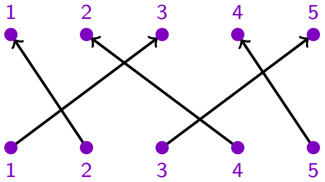
$$\mathbf{w} = \left(1,\ \frac{n-2}{n-1},\ \ldots,\ \frac{2}{n-1},\ \frac{1}{n-1},\ 0\right).$$

Permutations and the symmetric group

A permutation is a bijective (one-to-one and onto) function

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}.$$

Permutation diagrams:



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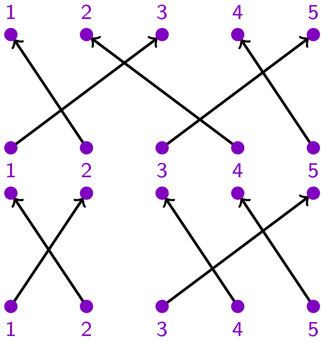

Permutations "multiply" by stacking and resolving.

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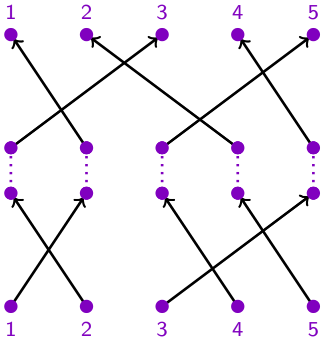

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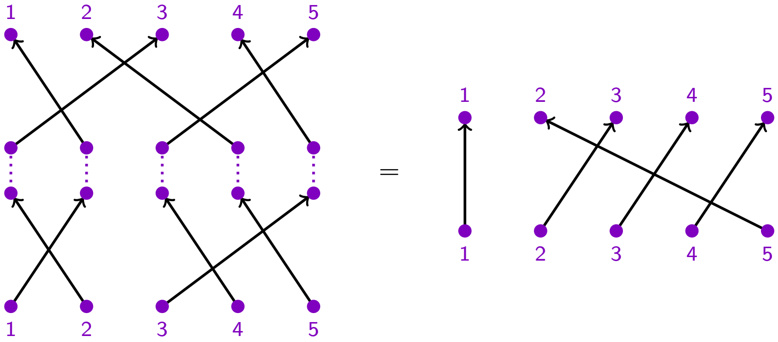
$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}.$$

Permutation diagrams:



Permutations "multiply" by stacking and resolving.

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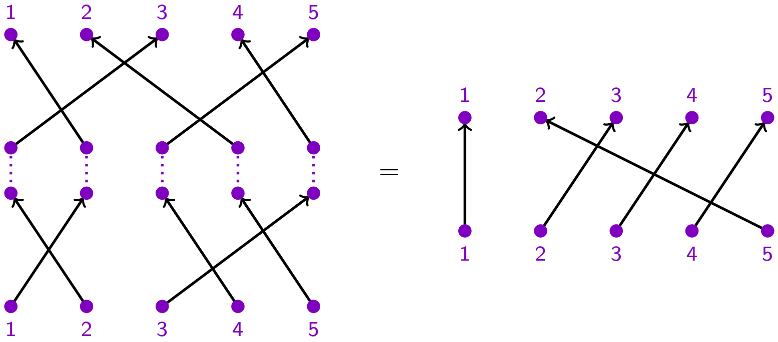

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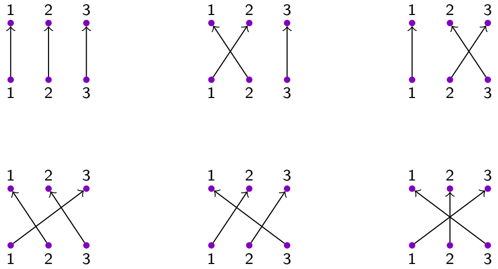
The symmetric group S_n is the group of permutations of $1, \ldots, n$ with multiplication given by function composition.

Some examples:

S_1 :

S_2 :

S_3 :



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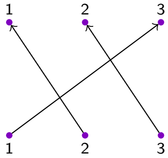
Example: Permutation representation of the symmetric group.



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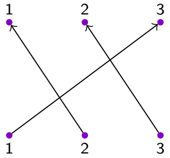

Pick a basis for $\mathbb{Q}^3$:

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

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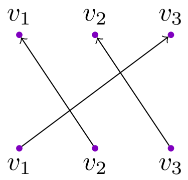

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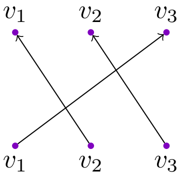

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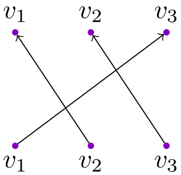

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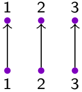
Aside: we actually have a representation of the group ring

$$\mathbb{Q}S_n = \left\{ \sum_{\sigma \in S_n} r_\sigma \sigma \mid r_\sigma \in \mathbb{Q} \right\},$$ with multiplication like polynomials.



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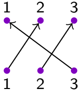

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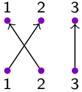

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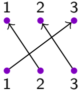

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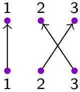

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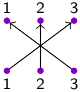

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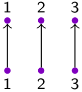

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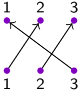

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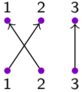

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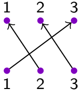

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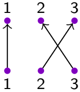

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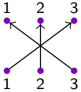

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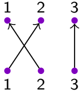
For example,



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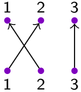

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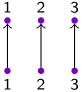
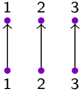


$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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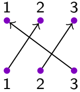

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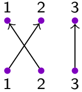

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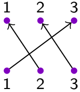

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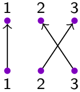

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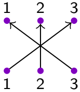

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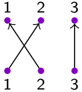

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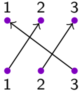

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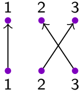

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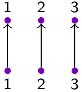

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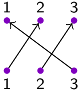

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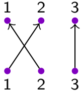

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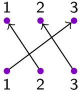

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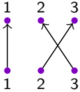

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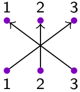

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Notice that the permutation representation has an invariant subspace $\mathbb{Q}\{v_1 + v_2 + v_3\}$, since

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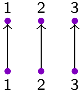
for all permutation matrices M.



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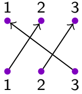

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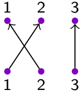

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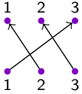

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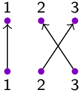

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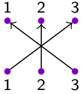

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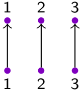

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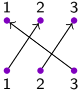

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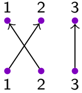

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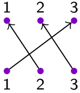

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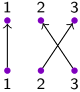

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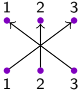

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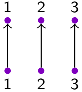

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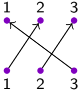

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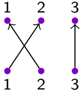

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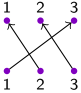

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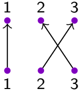

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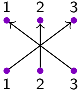

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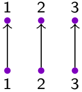

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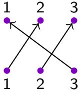

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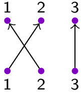

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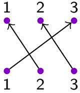

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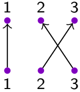

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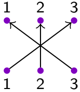

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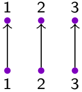

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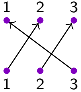

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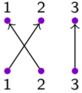

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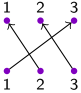

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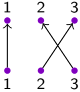

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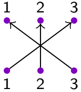

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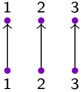

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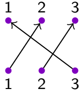

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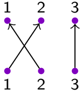

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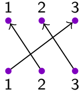

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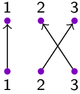

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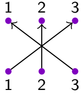

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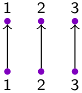

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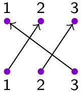

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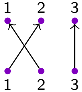

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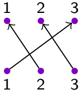

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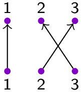

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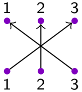

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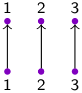

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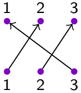

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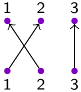

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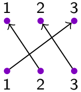

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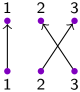

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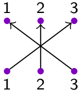

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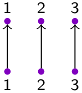

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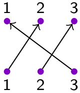

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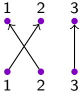

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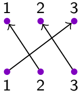

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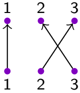

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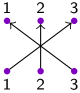

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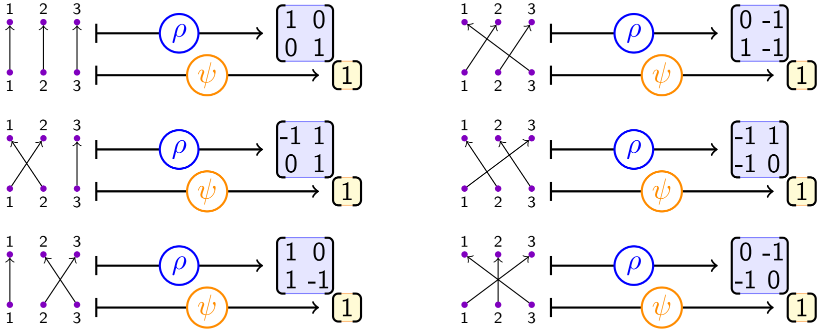

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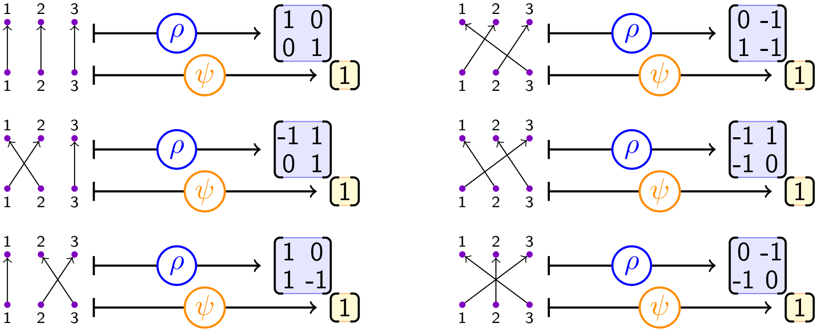

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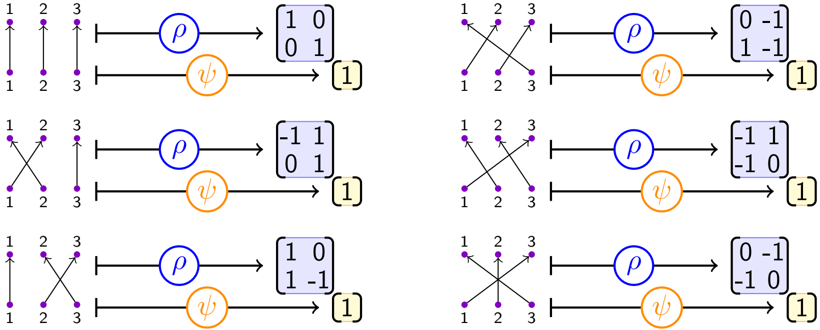

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We say ρ and ψ are simple because neither has any invariant subspaces.

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Let n be a non-negative integer.
A partition λ of n is a non-ordered list of positive integers which sum to n.

Example: the partitions of 3 are (3), $(2,1)$, and $(1,1,1)$.

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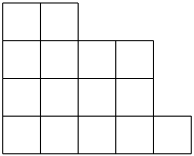
We draw partitions as n boxes left-justified, where the parts are the number of boxes in a row (reading from the bottom):

$$\lambda = (5, 4, 4, 2) =$$



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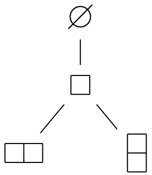

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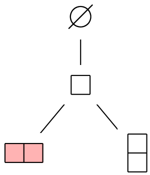

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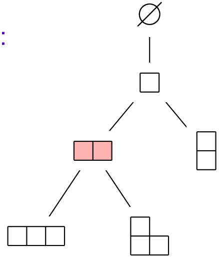

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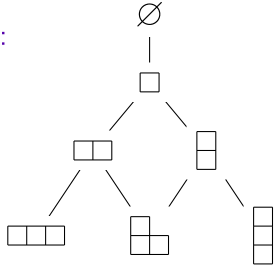

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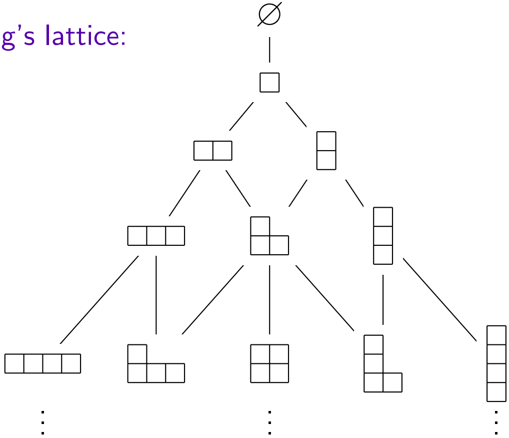

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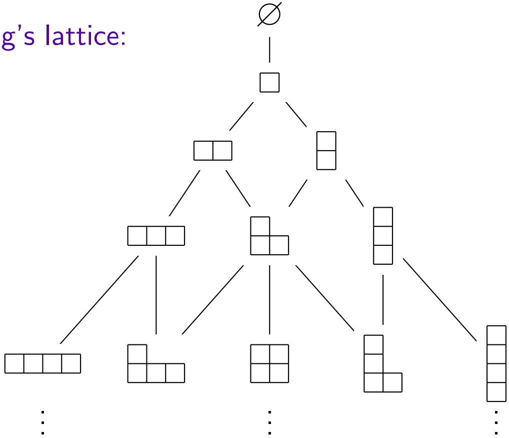

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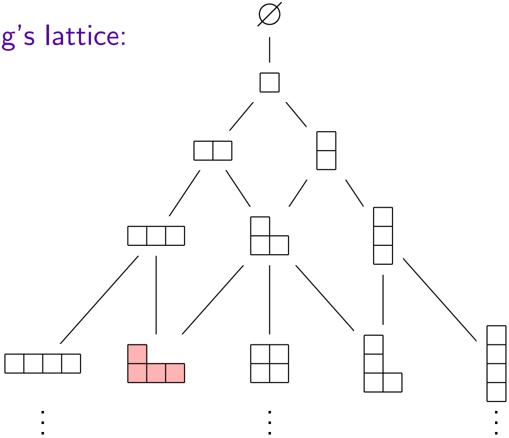

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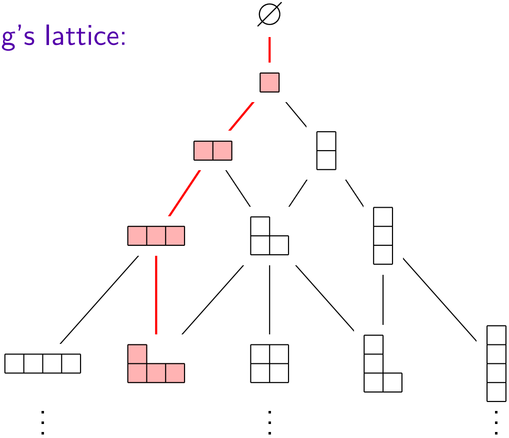

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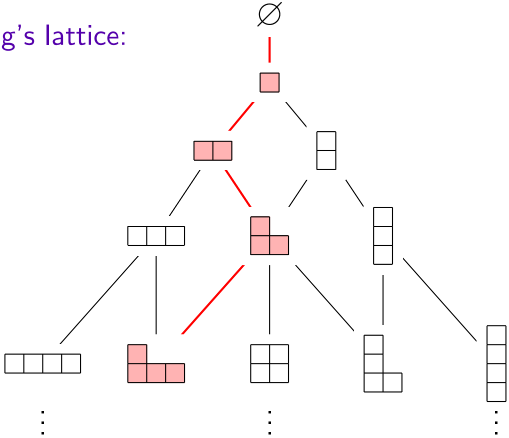

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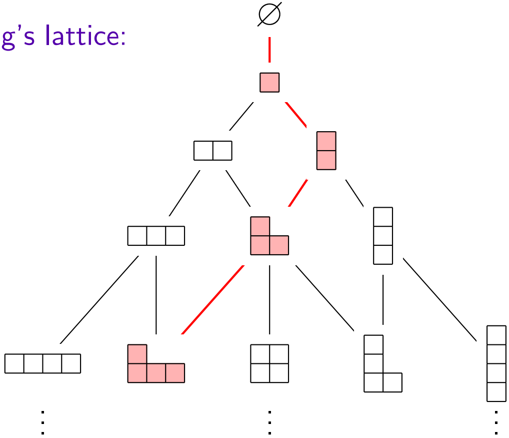

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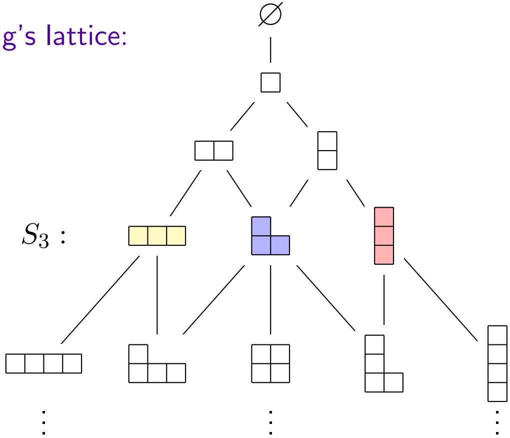

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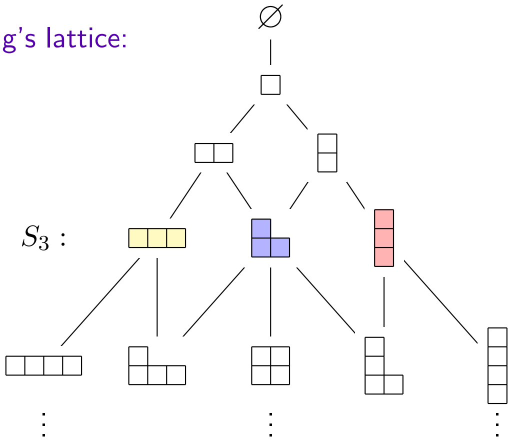

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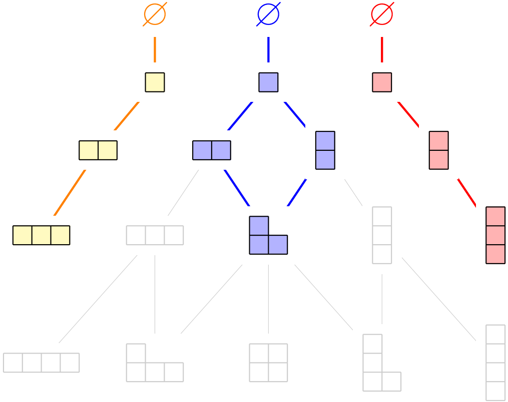

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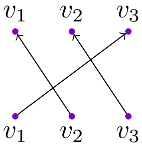
Why do *we* care about representations of S_n?



$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Pick a basis for $\mathbb{Q}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Map each permutation to the matrix that permutes the basis vectors in the same way.

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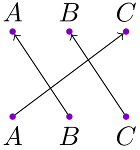

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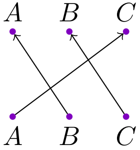

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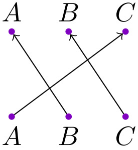

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Theorem. The permutation representation decomposes into one simple 1-dimensional (trivial) representation and one simple $(n-1)$-dimensional (reflection) representation.

The permutation representation models the outcome space.
More: all voting data spaces are symmetric group "modules".

In other words:

Permutations naturally move around voting and outcome spaces.

(Permute the candidate's names, or their places on the ballot.)

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- addition,
  If individual precincts add up votes, and then combine results, that should be the same as if the tallying happened all in one place.
- scaling, and
  If everyone's vote counted 5 times, the outcome should be the same.
- permutations.
  Changing the order that candidates appear on the ballot ideally shouldn't change the outcome.

Big representation theory theorems

Maschke's Theorem: All S_n modules decompose uniquely(ish) into simple modules.

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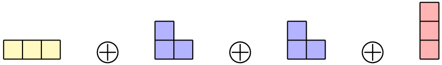

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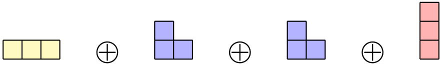

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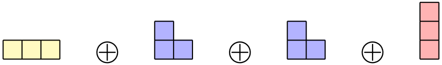

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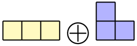

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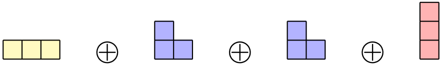

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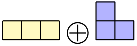

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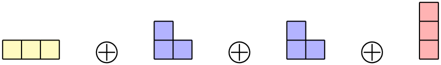

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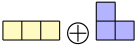

Ties $\oplus$ Relative positions

First result: If you decide a winner based on a points system, Condorcet cycles get lost in the tally.

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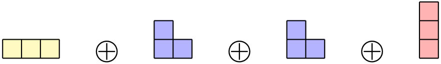

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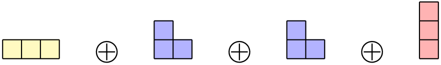

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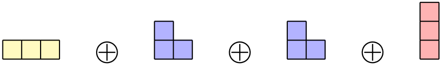.

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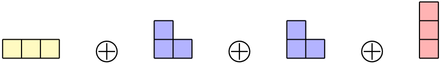

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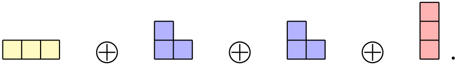
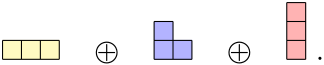
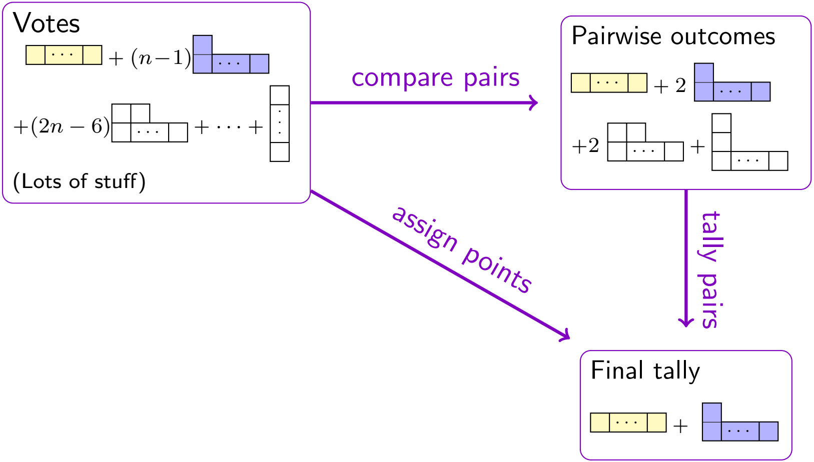


But by further analysis, one can compute that the image is at most



In fact, the information lost is precisely the "reversal" space.

Collect full rankings of preferences for n candidates...



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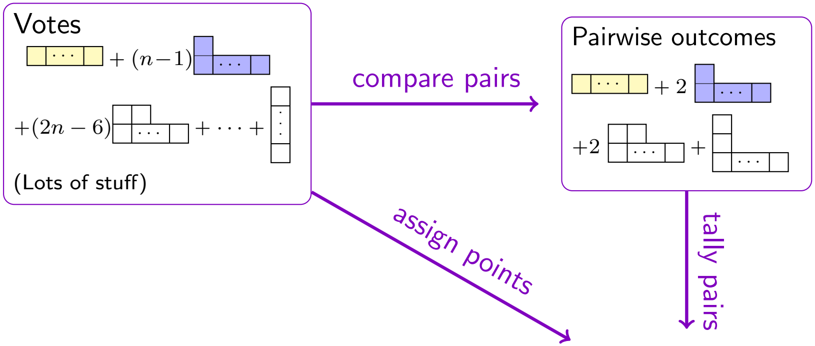

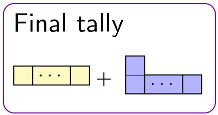

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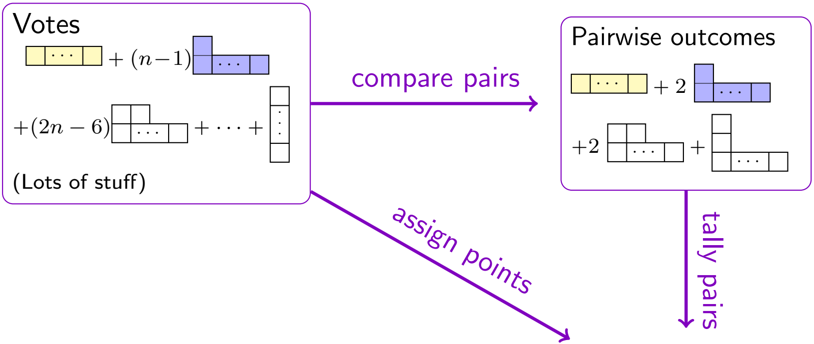

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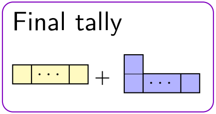

Result 1 (Saari, **D.**): The point-based system that agrees most with the pairwise outcomes is the (modified) Borda count, with weight

$$\mathbf{w} = \left(1,\ \frac{n-2}{n-1},\ \ldots,\ \frac{2}{n-1},\ \frac{1}{n-1},\ 0\right)$$

(as expected).

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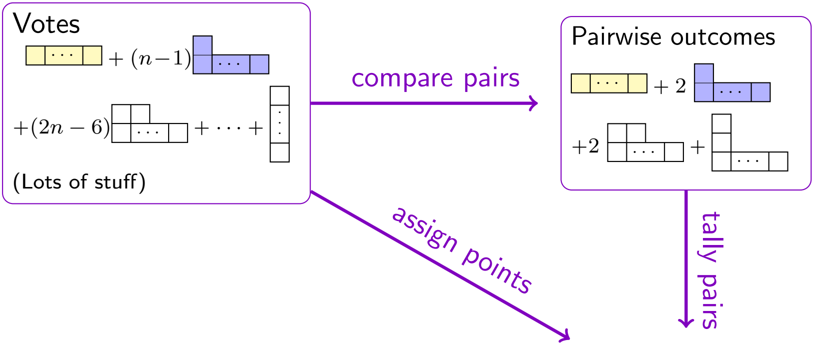

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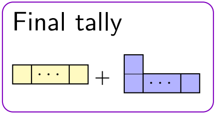

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Other results. . .

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(Average the remaining points amongst the last-place candidates.)

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Committees (Barcelo, Bernstein, Bockting-Conrad, McNichols, Nyman, Viel; '18): How to tally votes for committees with representation from several departments (using representation theory of $S_m \wr S_n$).

What do we want, and why do we care?

- Mathematicians: More data preservation is better.
- Idealists: Everyone should have a fair say.
- Pragmatists: Simpler voting systems are easier to implement.
- Cynics: Stupid things happen when people, en mass, are forced to game their votes.
- (Over-educated) Conspiracy Theorists: Our voting system is provably about as bad as it can be without *everyone* noticing.
- Kenneth Arrow: No voting system is ideal, so. . .
- Mathematicians (again): Oh, come *on*!

Some references. . .

- Many many publications of Donal Saari, particularly around 1999–2000. Also, “Decisions and Elections; Explaining the Unexpected”, Cambridge University Press, 2001.

- “Voting, the symmetric group, and representation theory”, by **D.**, Eustis, Minton, and Orrison. American Mathematical Monthly 116 (2009), no. 8, 667–687.

- “Algebraic voting theory and representations of $S_m \wr S_n$”, by Barcelo, Bernstein, Bockting-Conrad, McNichols, Nyman, Viel. (preprint: arXiv:1807.03743)

- “How not to be wrong”, by Jordan Ellenberg.
Chapter 17: “There is no such thing as public opinion.”

And even though our system is non-ideal as is. . .

Vote!

http://vote.nyc.ny.us