Combinatorics and representation theory of diagram algebras.

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Slides available at https://zdaugherty.ccnysites.cuny.edu/research/

Combinatorial representation theory

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Representation theory: Given an algebra A...

- What are the A-modules/representations?

  (*Actions* $A \circlearrowright V$ *and homomorphisms* $\varphi : A \to \mathrm{End}(V)$)
- What are the simple/indecomposable A-modules/reps?
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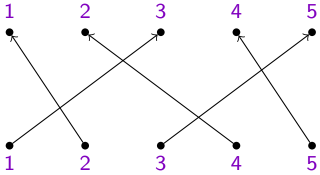
In combinatorial representation theory, we use combinatorial objects to index (construct a bijection to) modules and representations, and to encode information about them.

Motivating example: Schur-Weyl Duality

The **symmetric group** S_k (permutations) as diagrams:



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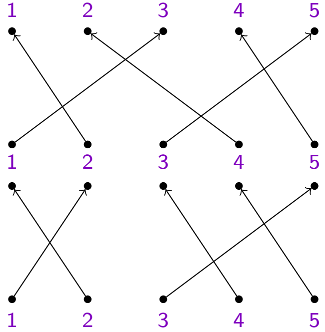

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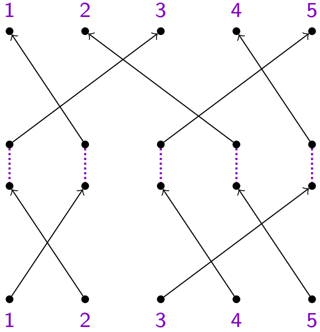

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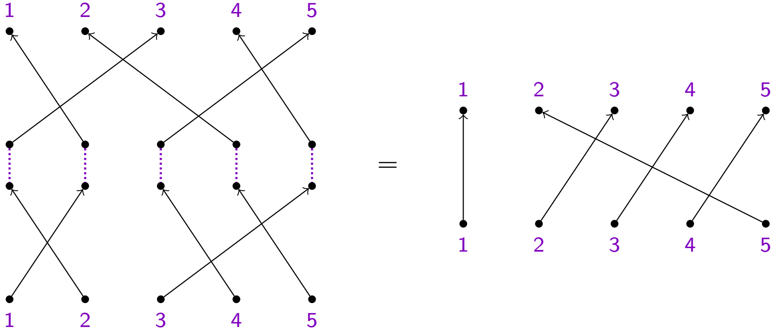

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Motivating example: Schur-Weyl Duality

$\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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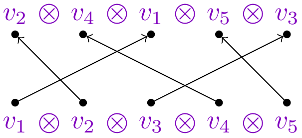
S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



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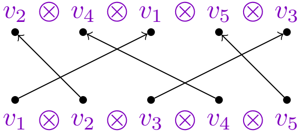
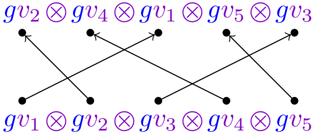
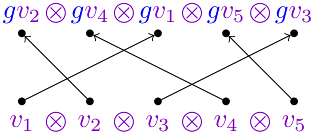


These actions commute!



vs.



Motivating example: Schur-Weyl Duality

Schur (1901): S_k and GL_n have commuting actions on $(\mathbb{C}^n)^{\otimes k}$.

Even better,

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\substack{\text{(all linear maps that}\\ \text{commute with } \mathrm{GL}_n)}} = \underbrace{\pi(\mathbb{C}S_k)}_{\substack{\text{(img of } S_k\\ \text{action)}}} \quad \text{and} \quad \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\substack{\text{(img of } \mathrm{GL}_n\\ \text{action)}}}.$$

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Powerful consequence:

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

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For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \left(G^{\ydiagram{3}} \otimes S^{\ydiagram{3}}\right) \oplus \left(G^{\ydiagram{2,1}} \otimes S^{\ydiagram{2,1}}\right) \oplus \left(G^{\ydiagram{1,1,1}} \otimes S^{\ydiagram{1,1,1}}\right)$$

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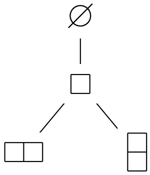

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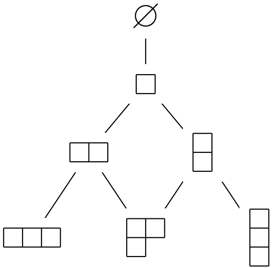

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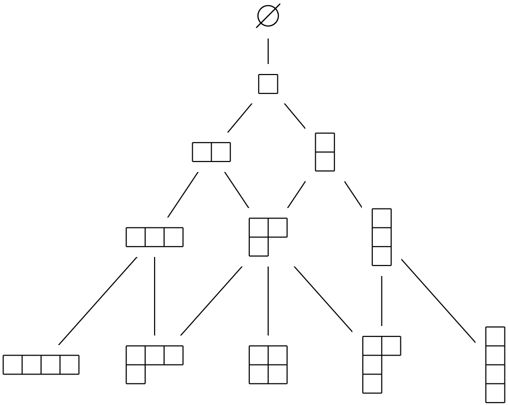

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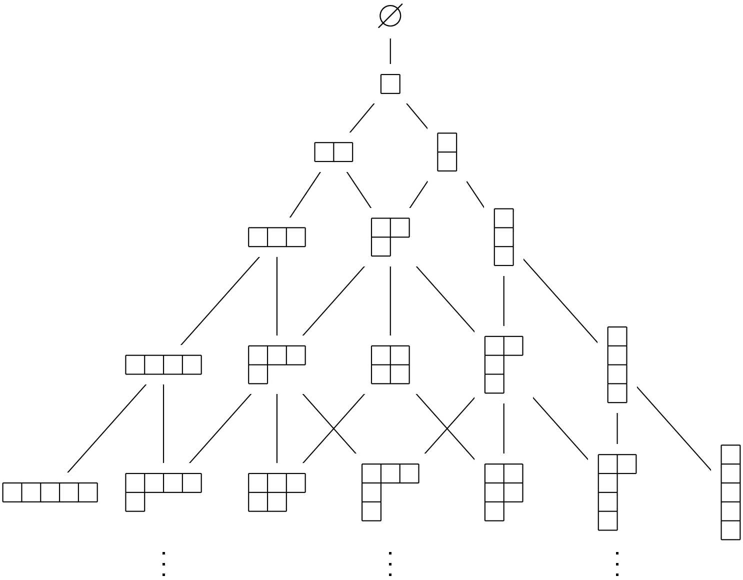

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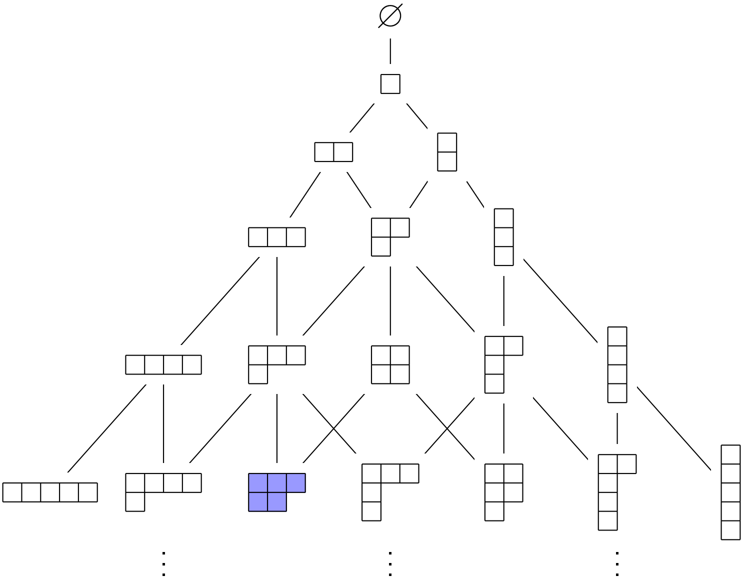

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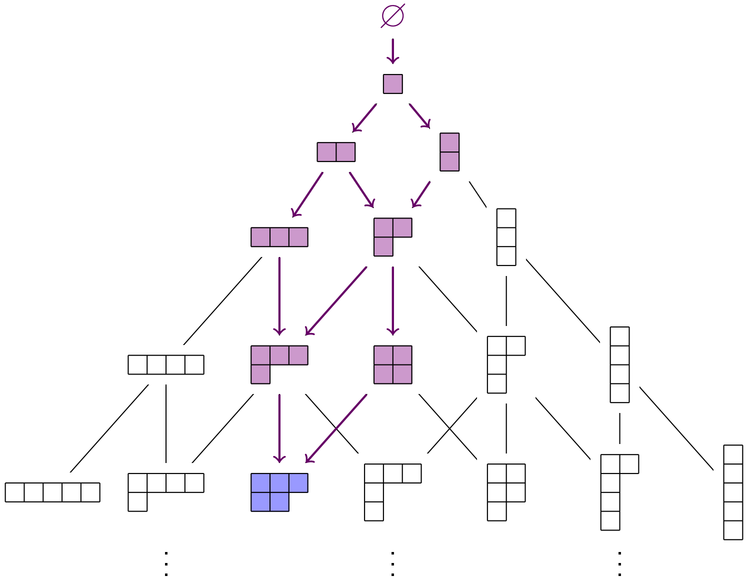
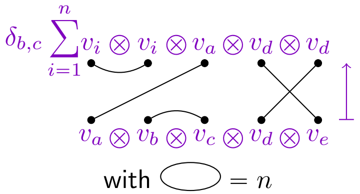


More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize the **Brauer algebra**:



Diagrams encode maps $V^{\otimes k} \to V^{\otimes k}$ that commute with the action of some classical algebra.

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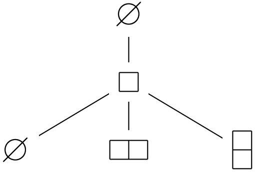

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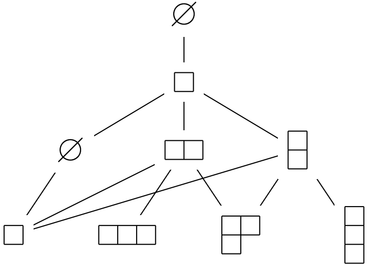

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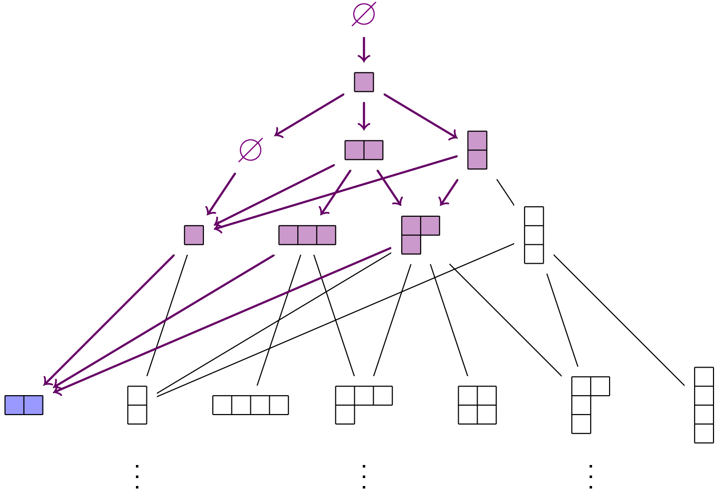

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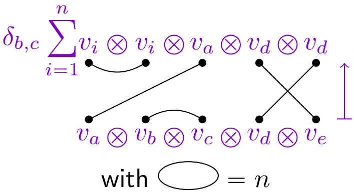
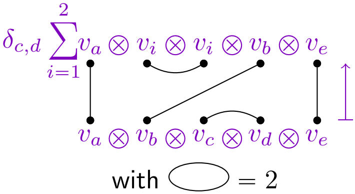


Temperley-Lieb (1971)
GL_2 and SL_2 (and $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$) acting on $(\mathbb{C}^2)^{\otimes k}$ diagonally centralize the **Temperley-Lieb algebra**:



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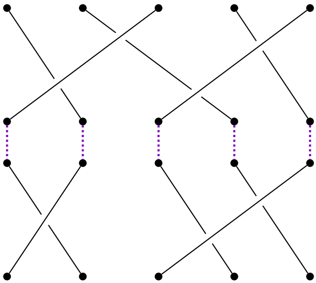
More diagram algebras: braids

The **braid group**:



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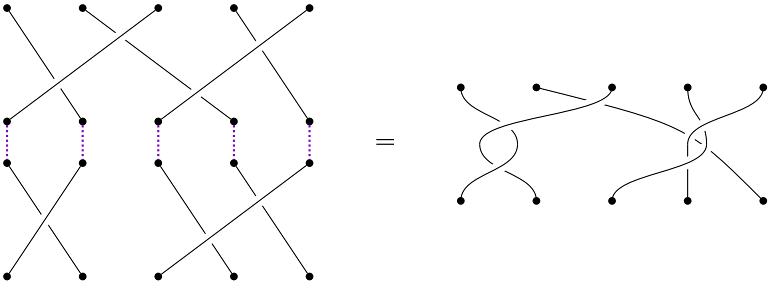

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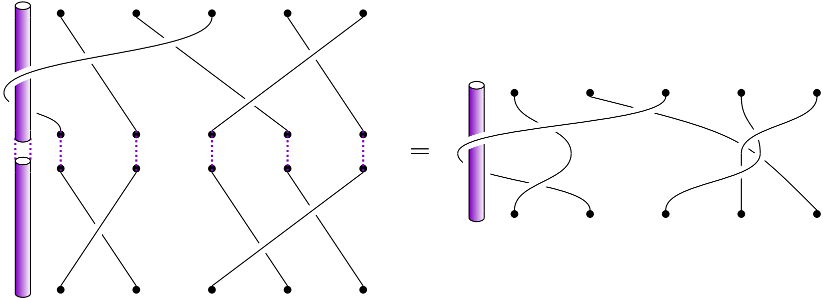
More diagram algebras: braids

The **affine (one-pole) braid group**:



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Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q\mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

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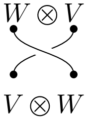
$$\check{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$



that (1) satisfies braid relations, and
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for any $\mathcal{U}$-module V.

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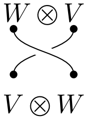

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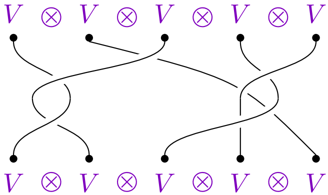
The braid group shares a commuting action with $\mathcal{U}$ on $V^{\otimes k}$:



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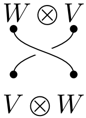

that (1) satisfies braid relations, and
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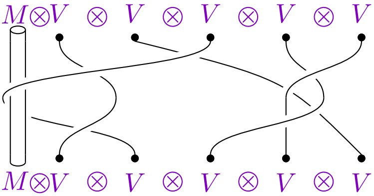
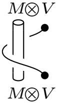
The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k}$:



Around the pole:



$= \check{R}_{MV} \check{R}_{VM}$

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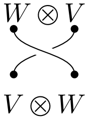

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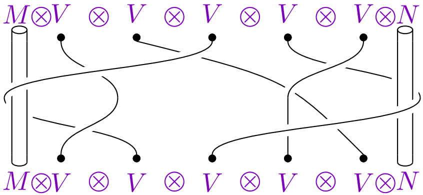
The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k} \otimes N$:



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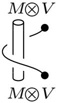
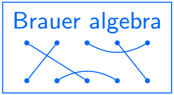
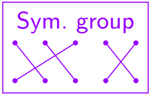


$$= \check{R}_{MV} \check{R}_{VM}$$

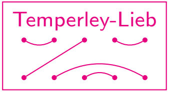
Lie grp/alg

Type B, C, D	Type A	Small Type A
(orthog. & sympl.)	(gen. & sp. linear)	(GL_2 & SL_2)

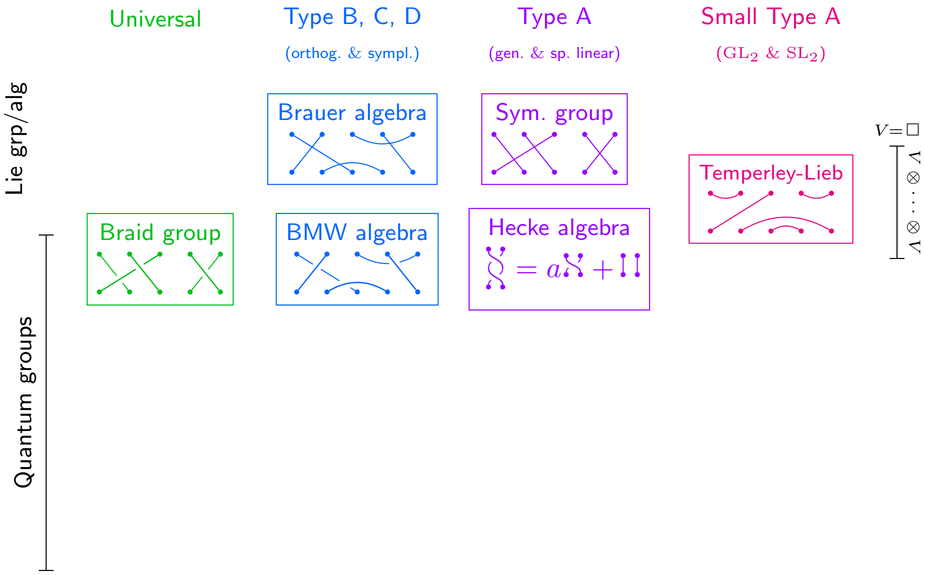




$V = \square$



$V \otimes \cdots \otimes V$



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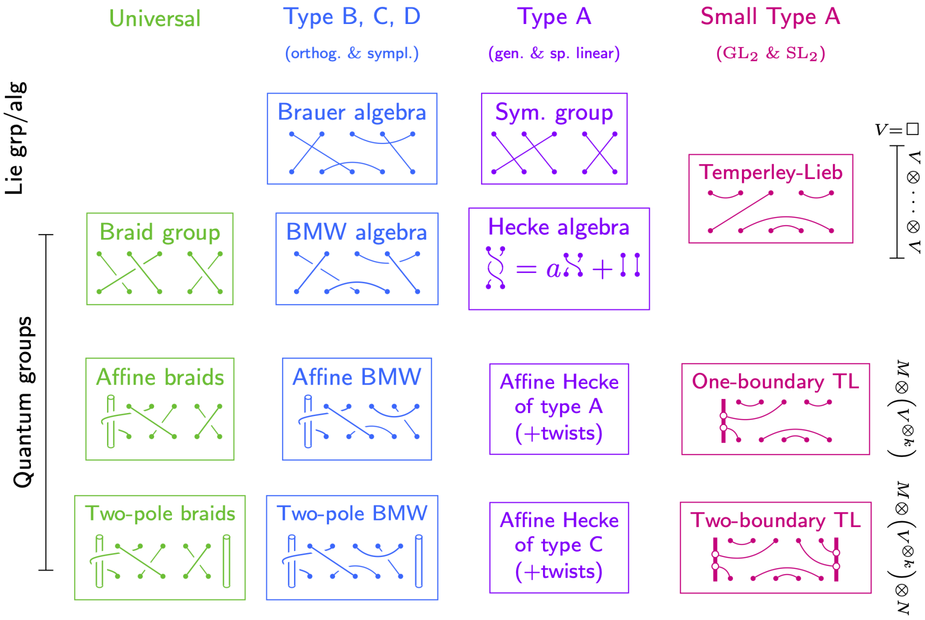
BMW algebra
Hecke algebra



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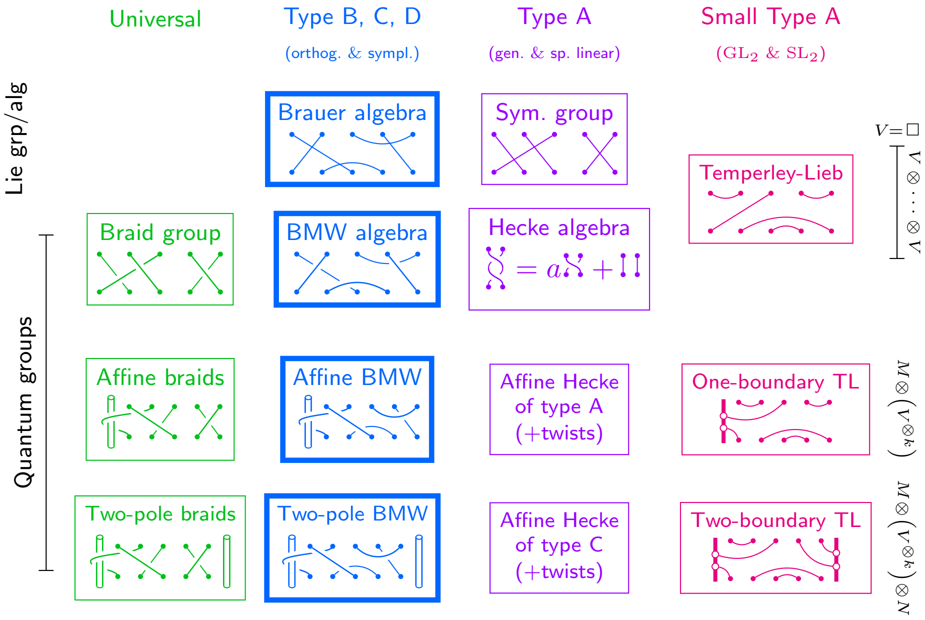
Affine BMW
Affine Hecke of type A (+twists)
One-boundary TL
$M \otimes \left(V^{\otimes k}\right)$
Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two-boundary TL
$M \otimes \left(V^{\otimes k}\right) \otimes N$



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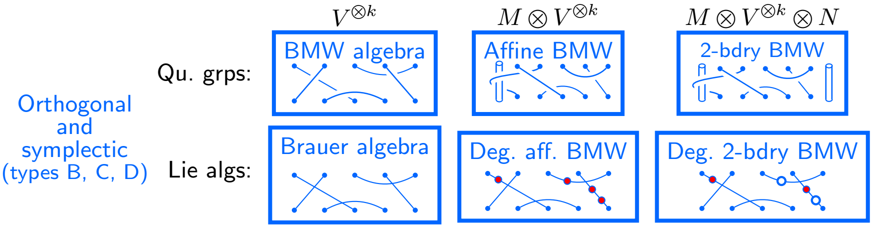
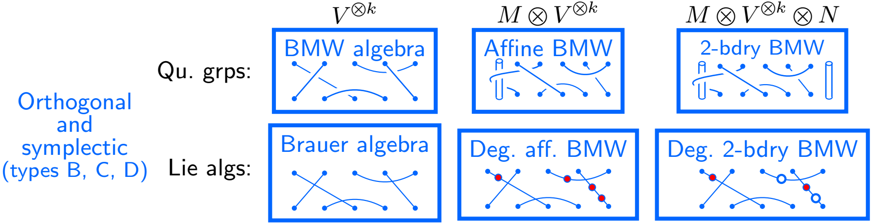
One-boundary TL
$M \otimes (V^{\otimes k})$
Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two-boundary TL
$N \otimes (V^{\otimes k}) \otimes M$


$V^{\otimes k}$
$M \otimes V^{\otimes k}$
$M \otimes V^{\otimes k} \otimes N$
Orthogonal
and
symplectic
(types B, C, D)
Qu. grps:
BMW algebra
Affine BMW
2-bdry BMW
Lie algs:
Brauer algebra
Deg. aff. BMW
Deg. 2-bdry BMW



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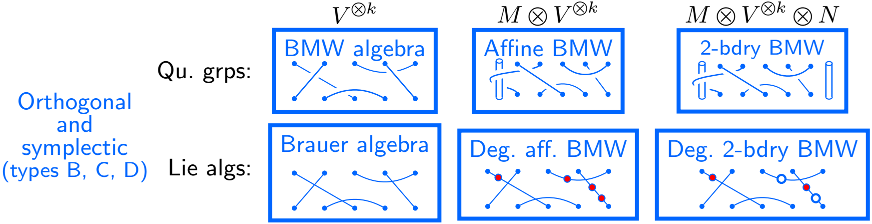

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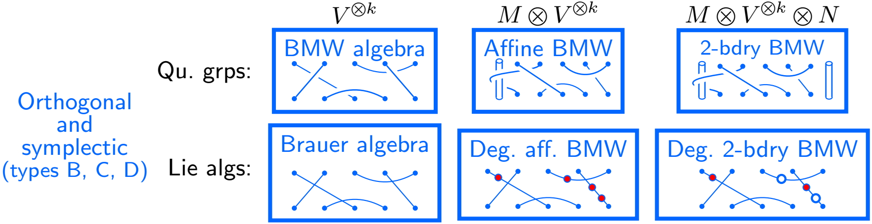

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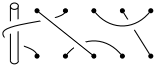
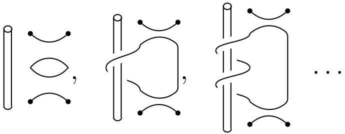
Example: "Admissibility conditions"

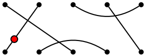
Affine BMW algebra



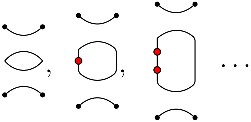
Closed loops:



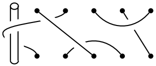
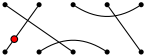
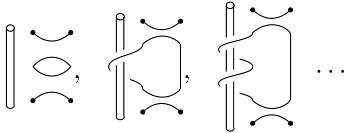
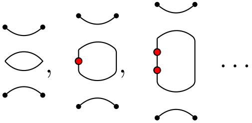
Degenerate affine BMW algebra



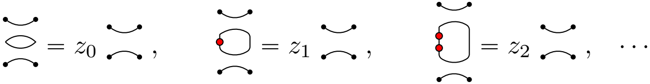
Closed loops:



Example: “Admissibility conditions”

Affine BMW algebra	Degenerate affine BMW algebra
	
Closed loops: 	Closed loops: 

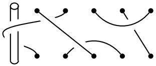
The associated parameters of the algebra, e.g.



aren’t entirely free.

Example: "Admissibility conditions"

Affine BMW algebra



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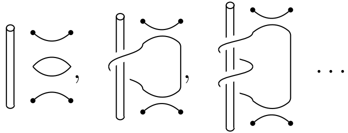
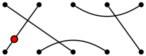
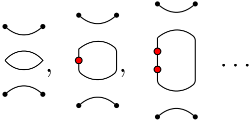


Degenerate affine BMW algebra



Closed loops:



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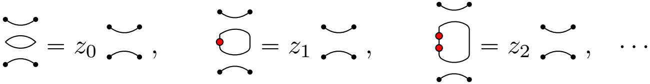
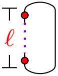
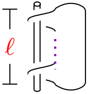


aren't entirely free.

Important insight: As operators on tensor space $M \otimes V \otimes V$,

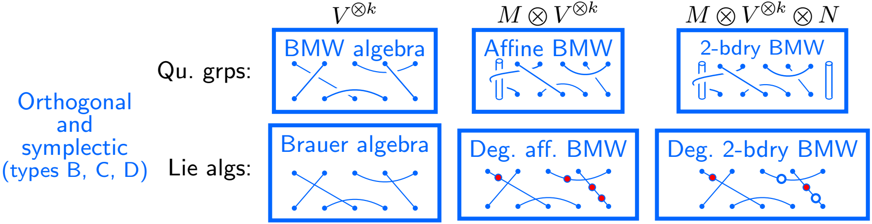


$\in Z(U\mathfrak{g}) \otimes \mathbb{C} \otimes \mathbb{C}$ and



$\in Z(U_q\mathfrak{g}) \otimes \mathbb{C} \otimes \mathbb{C}$.

"Higher Casimir invariants"



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Constructing 2-boundary analogues
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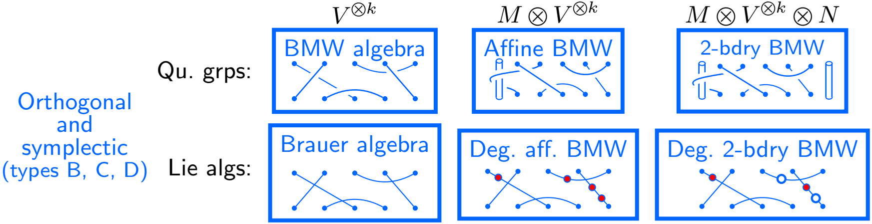

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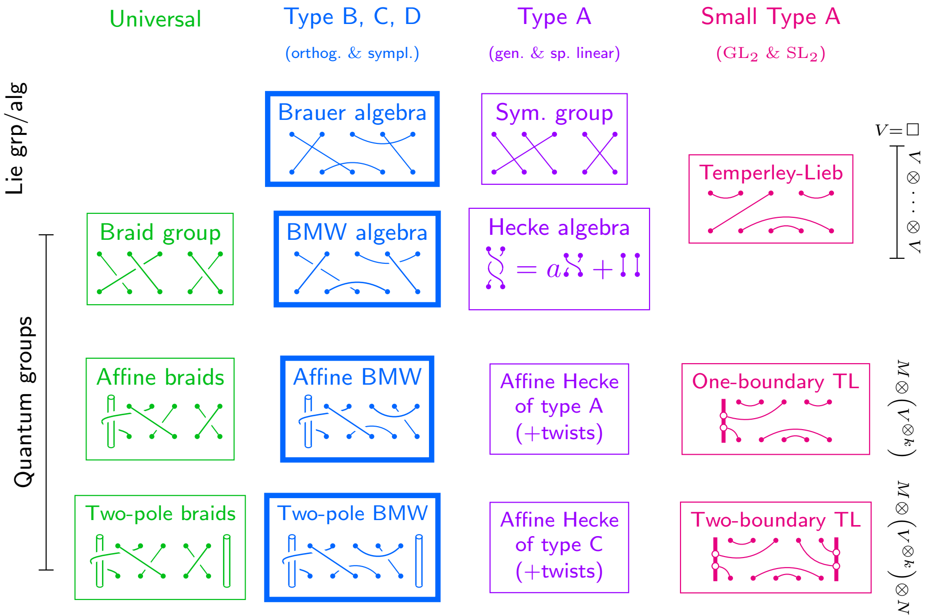
Balagovic et al.:
Signed versions and representations of periplectic Lie superalgebras.



Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL_2 & SL_2)
Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
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$V = \square$
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Braid group
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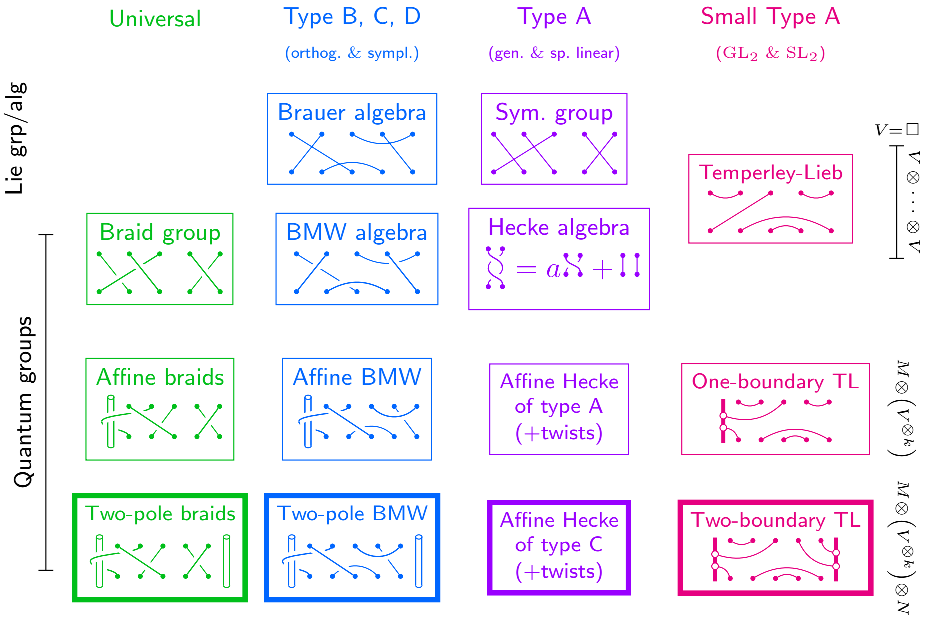

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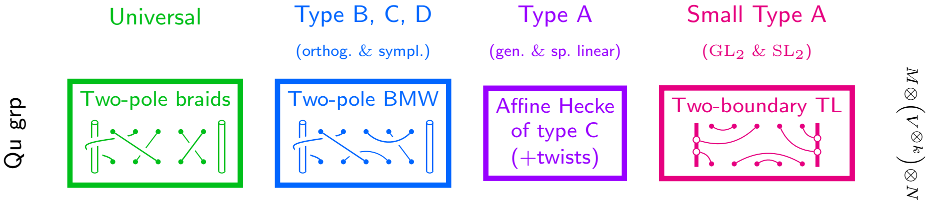
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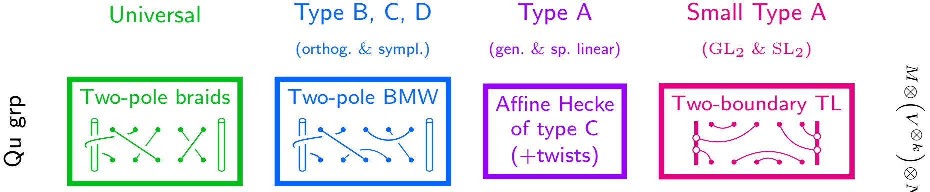

Two boundary algebras (type A)

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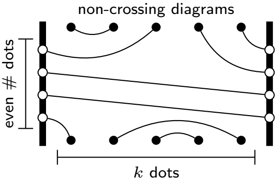

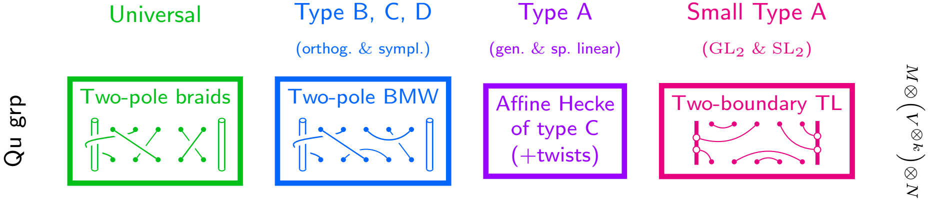

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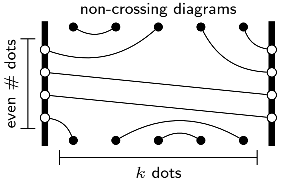

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Qu grp

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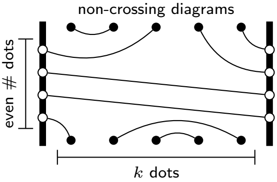

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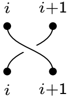
D. (2010): The centralizer of $\mathfrak{gl}_n$ acting on tensor space $M \otimes V^{\otimes k} \otimes N$ displays type C combinatorics for good choices of M, N, and V.

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

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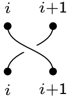
$T_k =$ , $T_0 =$  and $T_i =$ 

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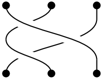
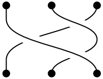
subject to relations

$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1}$,

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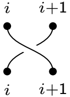
$T_k =$ , $T_0 =$  and $T_i =$ 

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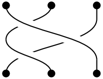
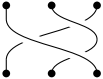
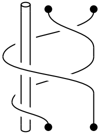
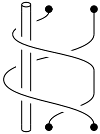
$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$,

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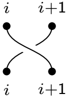
$T_k =$ , $T_0 =$  and $T_i =$ 

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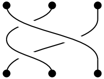
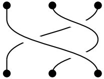
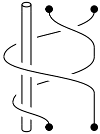
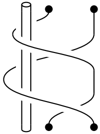
$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1}$,

$T_1T_0T_1T_0 =$  $=$  $= T_0T_1T_0T_1$,

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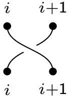
$T_k =$ , $T_0 =$  and $T_i =$ 

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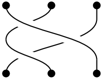
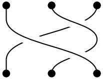
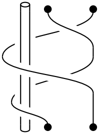
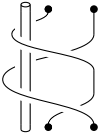
i.e.

$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$,

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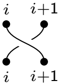
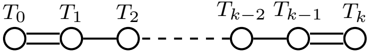
$T_k =$ , $\quad T_0 =$  and $\quad T_i =$ 

for $1 \leqslant i \leqslant k-1$,

subject to relations 

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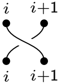
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leqslant i \leqslant k-1$,

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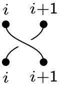
(2) Fix constants $t_0, t_k, t \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

$$\text{and} \quad (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

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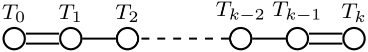
subject to relations 

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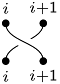
$T_k =$ , $T_0 =$  and $T_i =$ 

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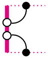
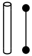
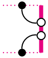
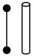
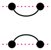
(3) Set

 $= t_0^{1/2}$  $-$  $\qquad (e_0 = t_0^{1/2} - T_0)$

 $= t_k^{1/2}$  $-$  $\qquad (e_k = t_k^{1/2} - T_k)$

 $= t^{1/2}$  $-$  $\qquad (e_i = t^{1/2} - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).

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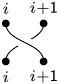
$T_k =$ , $T_0 =$  and $T_i =$  for $1 \leqslant i \leqslant k-1$,

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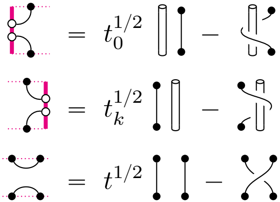

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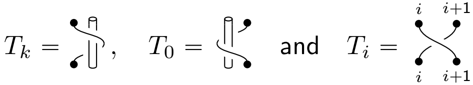

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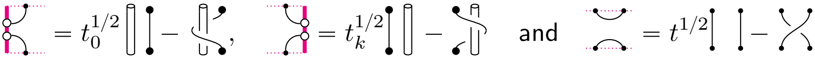

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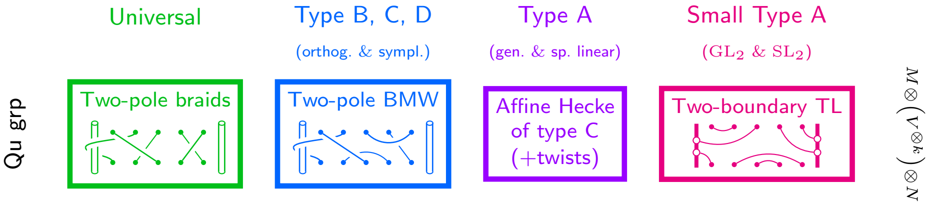

Theorem (D.-Ram)

(1) *Let* $U = U_q\mathfrak{g}$ *for any complex reductive Lie algebras* $\mathfrak{g}$.
Let M, N, *and* V *be finite-dimensional modules.*
The two-boundary braid group B_k *acts on* $M \otimes (V)^{\otimes k} \otimes N$ *and this action commutes with the action of* U.

(2) *If* $\mathfrak{g} = \mathfrak{gl}_n$, *then (for correct choices of* M, N, *and* V*), the affine Hecke algebra of type* C, H_k, *acts on* $M \otimes (V)^{\otimes k} \otimes N$ *and this action commutes with the action of* U.

(3) *If* $\mathfrak{g} = \mathfrak{gl}_2$, *then the action of the two-boundary Temperley-Lieb algebra factors through the T.L. quotient of* H_k.

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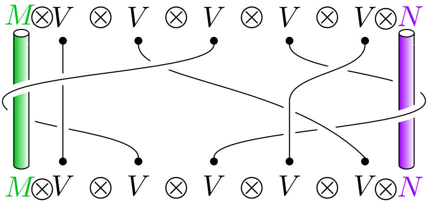
Some results:

(a) A diagrammatic intuition for H_k.

(b) A combinatorial classification and construction of irreducible representations of H_k (type C with distinct parameters) via central characters and generalizations of Young tableaux.

(c) A classification of the representations of TL_k in [dGN08] via central characters, including answers to open questions and conjectures regarding their irreducibility and isomorphism classes.


M⊗V ⊗ V ⊗ V ⊗ V ⊗ V⊗N
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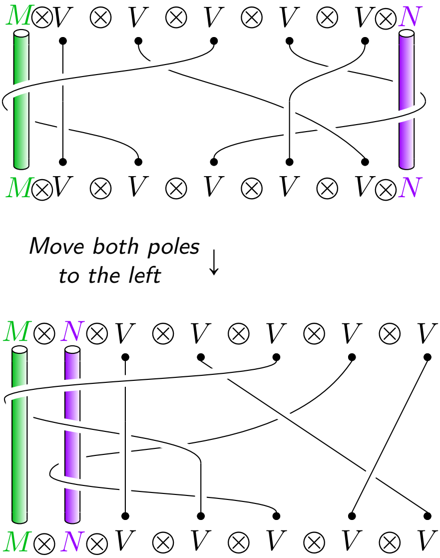

$M \otimes V \otimes V \otimes V \otimes V \otimes V \otimes N$
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Move both poles to the left ↓
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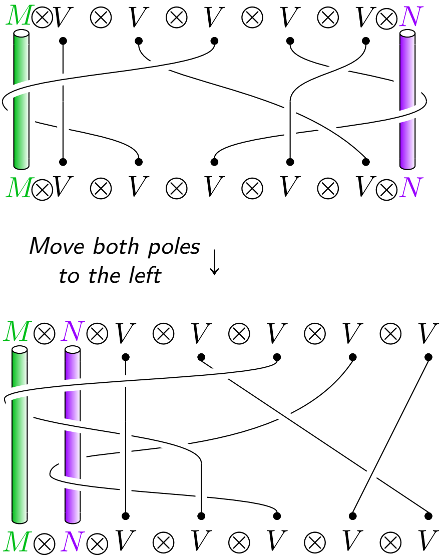
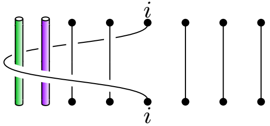
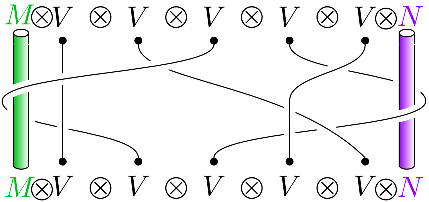


Jucys-Murphy elements:

$Y_i =$ 



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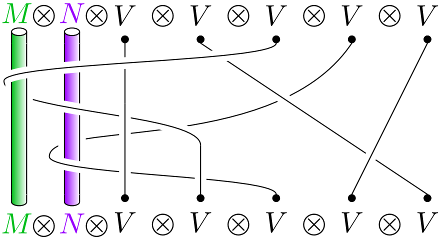

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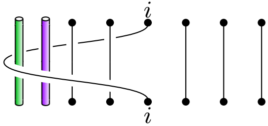
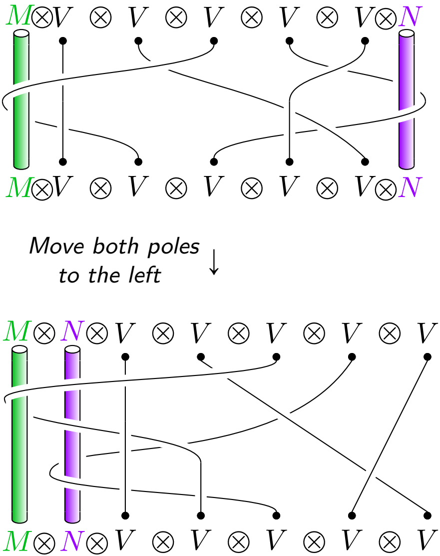
$Y_i =$ 

- Pairwise commute



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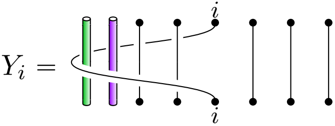

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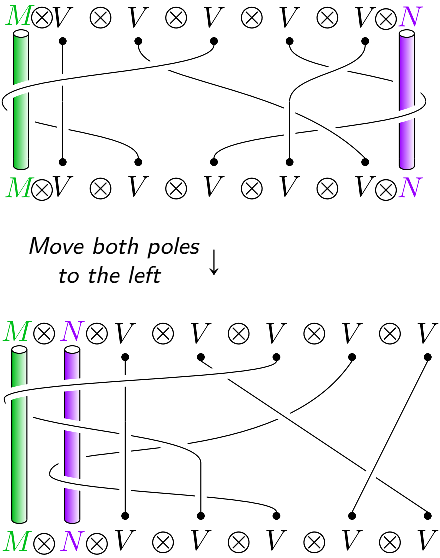
- $Z(\mathcal{H}_k)$ is (type-C) symmetric Laurent polynomials in Z_i's



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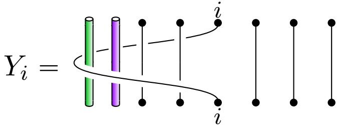

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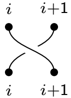
- Central characters indexed by $\mathbf{c} \in \mathbb{C}^k$ (modulo signed permutations)

Back to tensor space operators properties

The eigenvalues of the T_i's must coincide with the eigenvalues of the corresponding R-matrices, which can be computed combinatorially.

$$0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$

$T_0 =$  $\propto \check{R}_{VM}\check{R}_{MV}$ $T_k =$  $\propto \check{R}_{NV}\check{R}_{VN}$ $T_i =$ 

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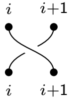
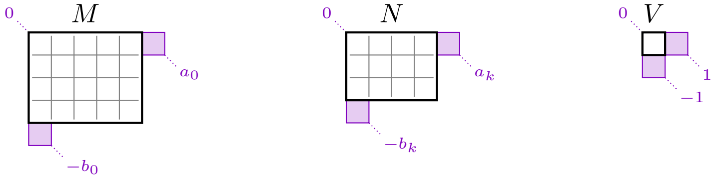
$T_0 =$  $\propto \check{R}_{VM}\check{R}_{MV}$ $T_k =$  $\propto \check{R}_{NV}\check{R}_{VN}$ $T_i =$  $\propto \check{R}_{VV}$



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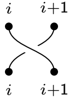
$T_i =$ 

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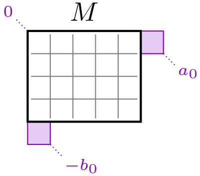
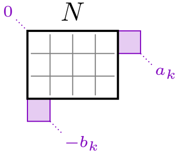


$t_0 = -q^{2(a_0+b_0)}$



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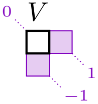

$t = q^2$

Exploring $M \otimes N \otimes L(\square)^{\otimes k}$

Products of rectangles:

$$L((a_0^{b_0})) \otimes L(({a_k}^{b_k})) = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

where Λ is the following set of partitions:

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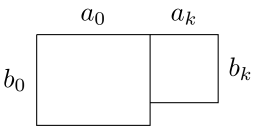

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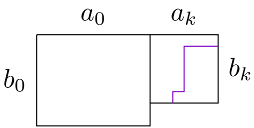

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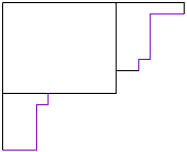

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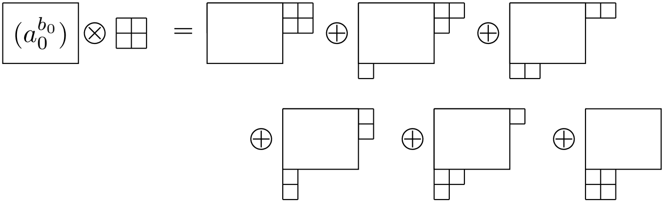

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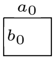
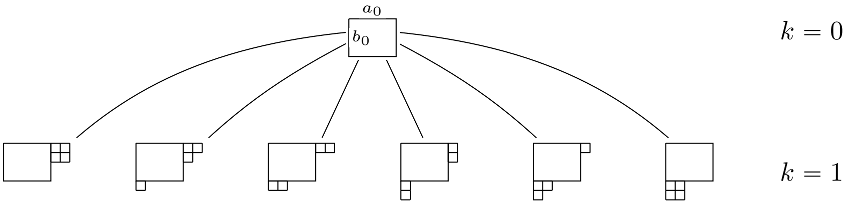


$k = 0$

Exploring $M \otimes N \otimes L(\square)^{\otimes k}$



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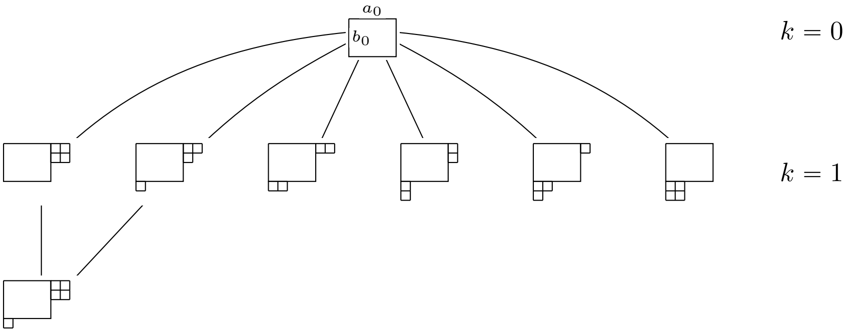

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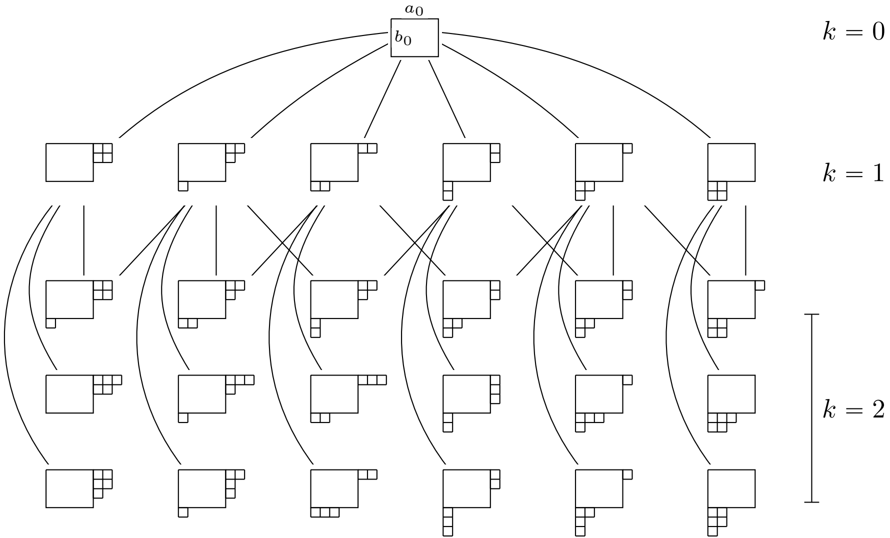
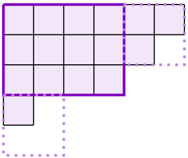
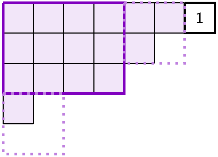


$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right)$$



$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\ydiagram{1}\right)$$



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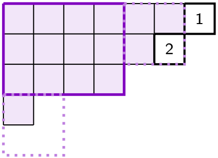
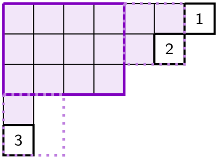
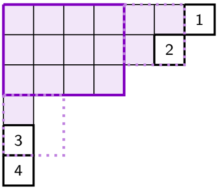


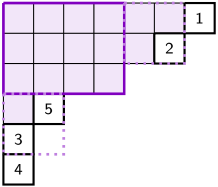
$$L\left(\ydiagram{4,4,4}\right)\otimes L\left(\ydiagram{2,2}\right)\otimes L\left(\ydiagram{1}\right)\otimes L\left(\ydiagram{1}\right)\otimes L\left(\ydiagram{1}\right)$$



$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right)$$

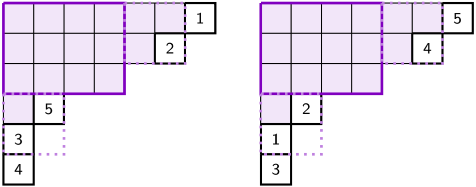


$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right)$$



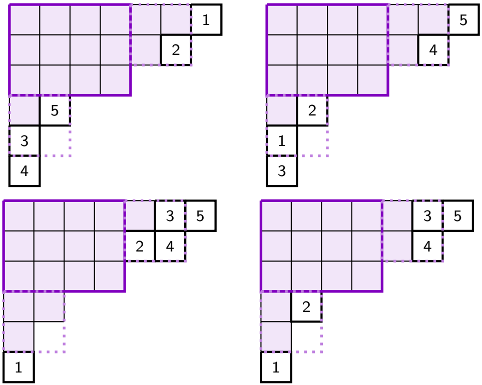
$(*)$ H_k representations in tensor space are labeled by certain partitions λ.

$$L\left(\yng(4,4,4)\right) \otimes L\left(\yng(2,2)\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right)$$



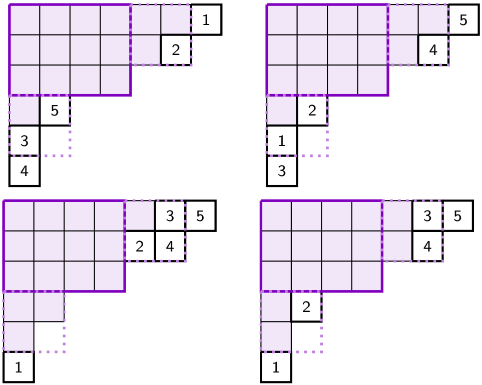
(∗) H_k representations in tensor space are labeled by certain partitions λ.

$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right)$$



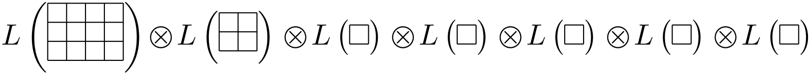
(∗) H_k representations in tensor space are labeled by certain partitions λ.

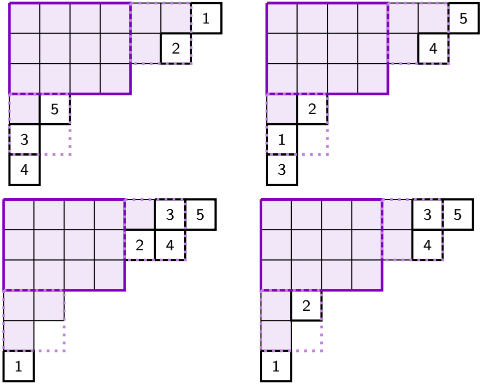
$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right)$$



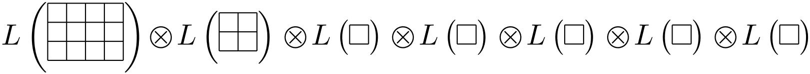
(∗) H_k representations in tensor space are labeled by certain partitions λ.

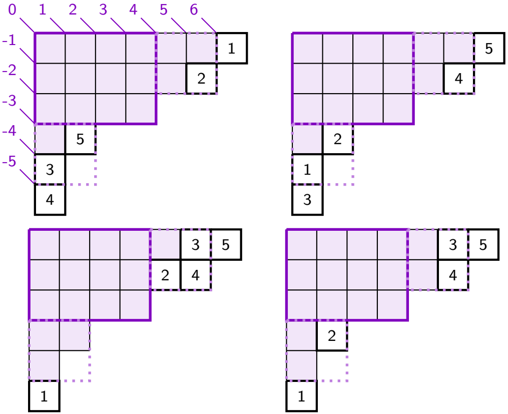
(∗) Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.





- $(*)$ H_k representations in tensor space are labeled by certain partitions λ.
- $(*)$ Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.
- $(*)$ Calibrated (Y_i's are diagonalized)

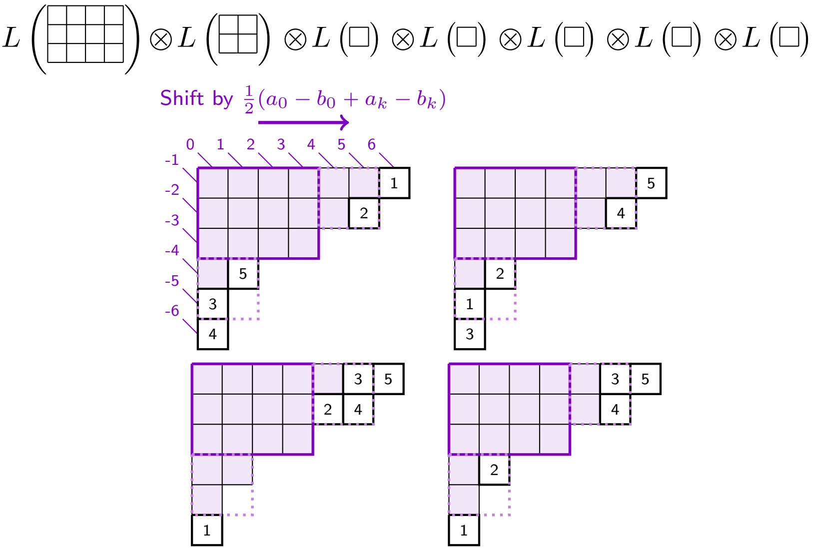




$(*)$ H_k representations in tensor space are labeled by certain partitions λ.

$(*)$ Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.

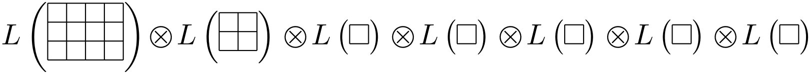
$(*)$ Calibrated (Y_i's are diagonalized)



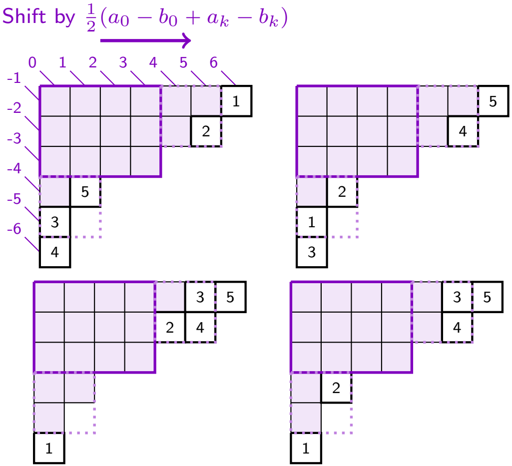
(∗) H_k representations in tensor space are labeled by certain partitions λ.

(∗) Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.

(∗) Calibrated (Y_i's are diagonalized)



$Y_1 \mapsto t^{5.5}$
$Y_2 \mapsto t^{3.5}$
$Y_3 \mapsto t^{-4.5}$
$Y_4 \mapsto t^{-5.5}$
$Y_5 \mapsto t^{-2.5}$

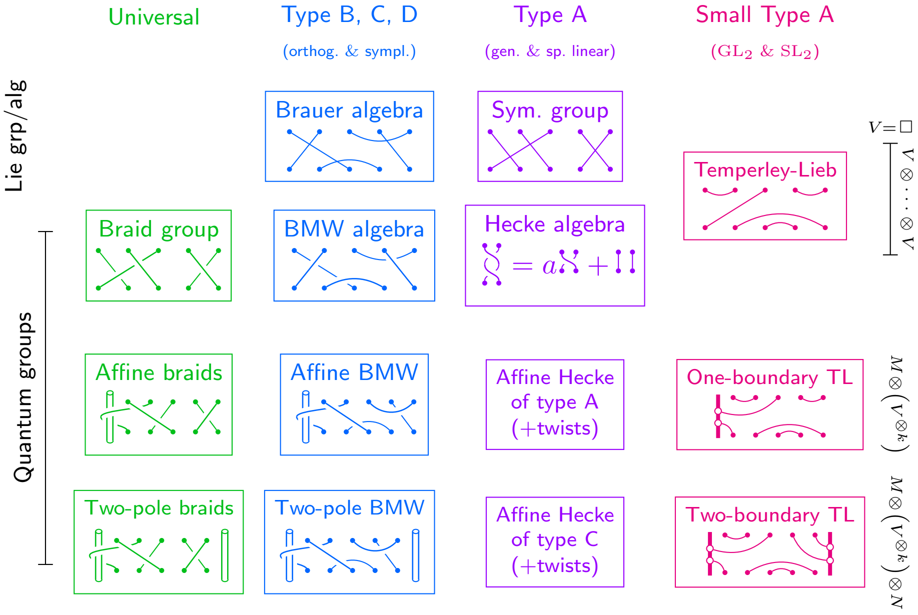


$Y_1 \mapsto t^{5.5}$
$Y_2 \mapsto t^{3.5}$
$Y_3 \mapsto t^{-4.5}$
$Y_4 \mapsto t^{-5.5}$
$Y_5 \mapsto t^{-2.5}$

$Y_1 \mapsto t^{-5.5}$
$Y_2 \mapsto t^{2.5}$
$Y_3 \mapsto t^{4.5}$
$Y_4 \mapsto t^{3.5}$
$Y_5 \mapsto t^{5.5}$

$Y_1 \mapsto t^{5.5}$
$Y_2 \mapsto t^{3.5}$
$Y_3 \mapsto t^{-4.5}$
$Y_4 \mapsto t^{-5.5}$
$Y_5 \mapsto t^{-2.5}$

($*$) H_k representations in tensor space are labeled by certain partitions λ.
($*$) Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.
($*$) Calibrated (Y_i's are diagonalized): Y_i acts by t to the shifted diagonal number of box_i. (Think: signed permutations.)



Thanks! https://zdaugherty.ccnysites.cuny.edu/