Combinatorics and representation theory of diagram algebras.

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Slides available at https://zdaugherty.ccnysites.cuny.edu/research/

Combinatorial representation theory

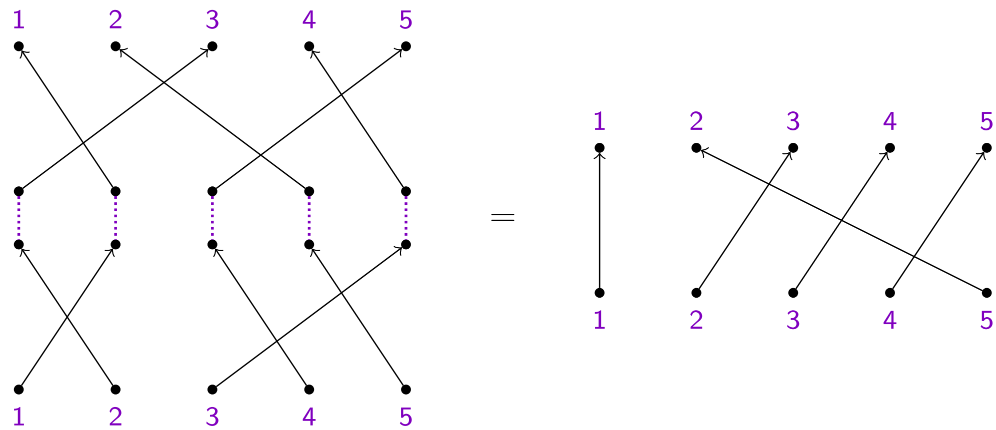
Representation theory: Given an algebra A...

- What are the A-modules/representations?
  (*Actions* $A \circlearrowright V$ *and homomorphisms* $\varphi : A \to \mathrm{End}(V)$)
- What are the simple/indecomposable A-modules/reps?
- What are their dimensions?
- What is the action of the center of A?
- How can I combine modules to make new ones, and what are they in terms of the simple modules?

In combinatorial representation theory, we use combinatorial objects to index (construct a bijection to) modules and representations, and to encode information about them.

Motivating example: Schur-Weyl Duality

The **symmetric group** S_k (permutations) as diagrams:



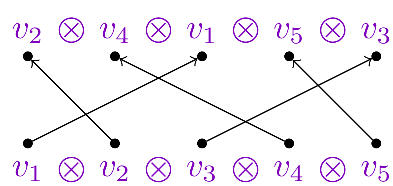
(with multiplication given by concatenation)

Motivating example: Schur-Weyl Duality

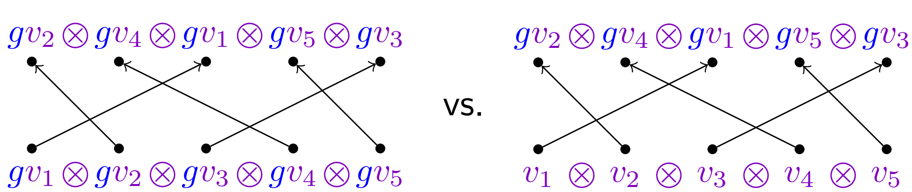
$\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



These actions commute!



Motivating example: Schur-Weyl Duality

Schur (1901): S_k and GL_n have commuting actions on $(\mathbb{C}^n)^{\otimes k}$.
Even better,

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\text{(all linear maps that commute with } \mathrm{GL}_n\text{)}} = \underbrace{\pi(\mathbb{C}S_k)}_{\text{(img of } S_k \text{ action)}} \quad \text{and} \quad \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\text{(img of } \mathrm{GL}_n \text{ action)}}.$$

Powerful consequence:

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

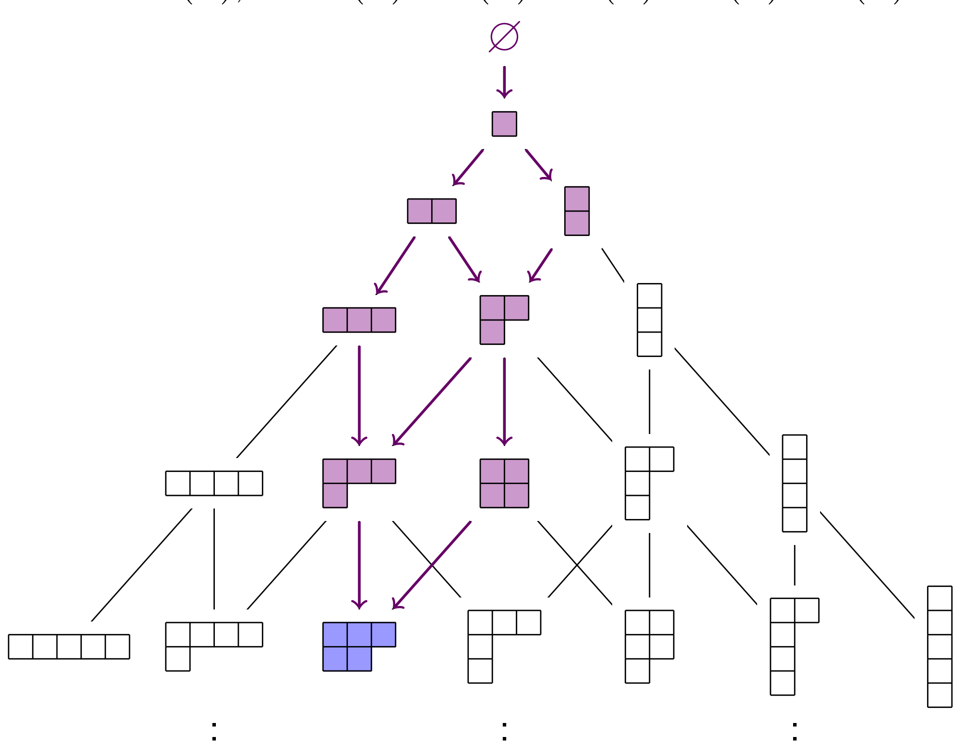
where G^λ are distinct irreducible GL_n-modules
S^λ are distinct irreducible S_k-modules

For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \left(G^{\ydiagram{3}} \otimes S^{\ydiagram{3}}\right) \oplus \left(G^{\ydiagram{2,1}} \otimes S^{\ydiagram{2,1}}\right) \oplus \left(G^{\ydiagram{1,1,1}} \otimes S^{\ydiagram{1,1,1}}\right)$$

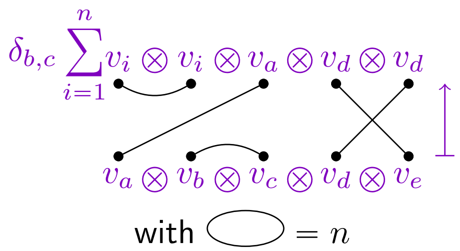
Representation theory of $V^{\otimes k}$

$$V = \mathbb{C} = L(\square), \qquad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots$$

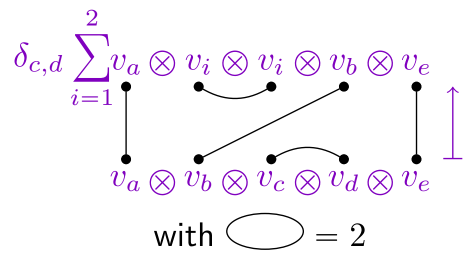


More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize
the **Brauer algebra**:



Temperley-Lieb (1971)
GL_2 and SL_2 (and $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$) acting on $(\mathbb{C}^2)^{\otimes k}$ diagonally centralize the **Temperley-Lieb algebra**:

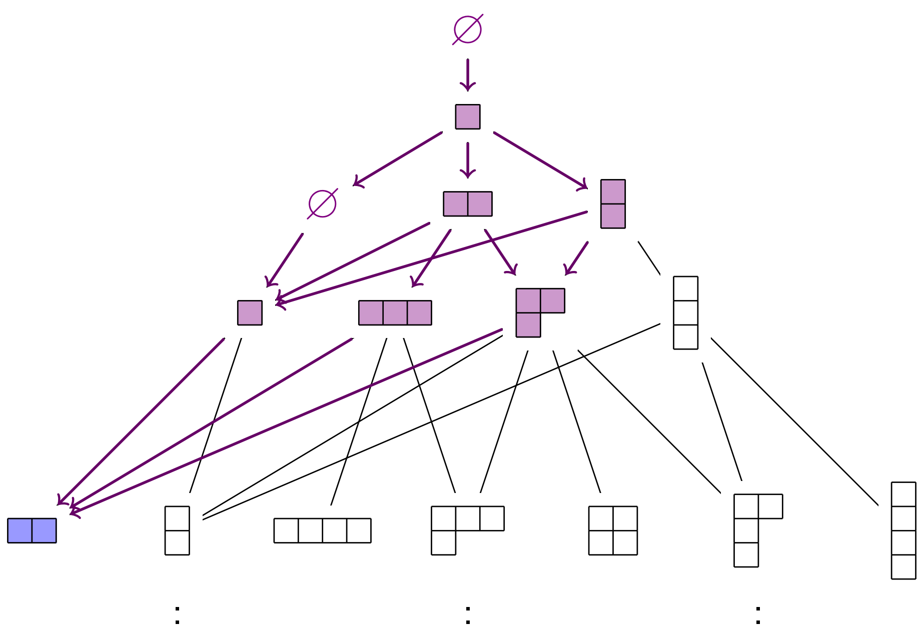


Diagrams encode maps $V^{\otimes k} \to V^{\otimes k}$ that commute with the action of some classical algebra.

More centralizer algebras

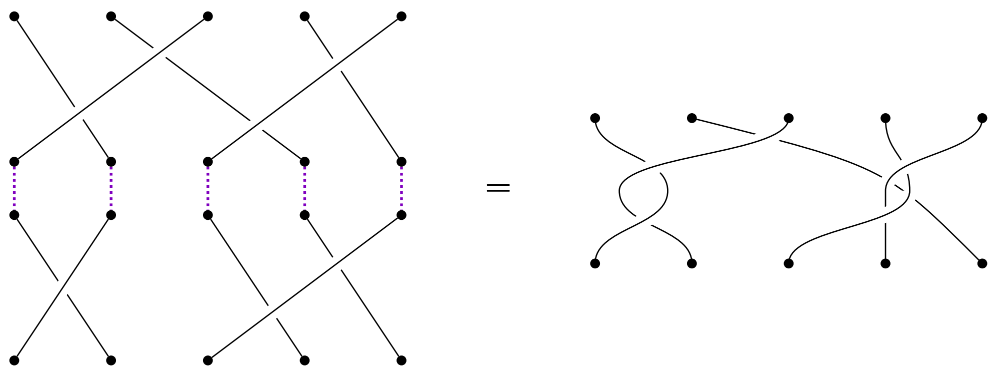
Representation theory of $V^{\otimes k}$, orthogonal and symplectic:

$$V = \mathbb{C} = L(\square), \qquad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots$$



More diagram algebras: braids

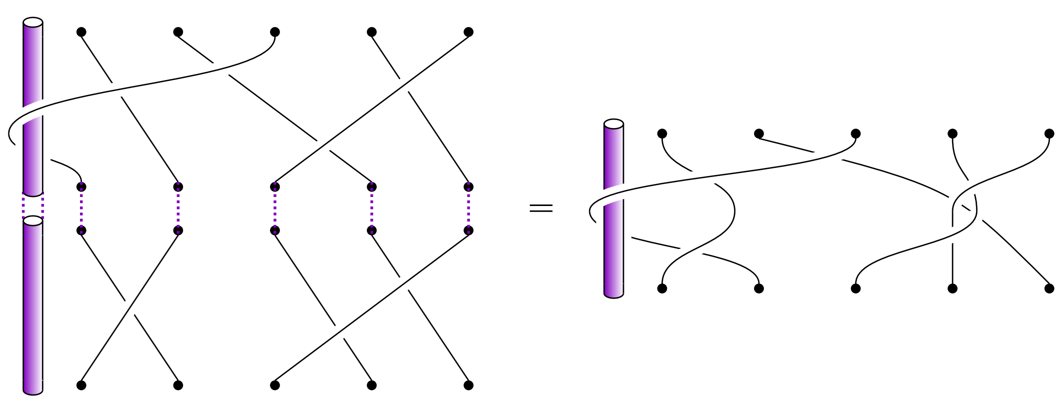
The **braid group**:



(with multiplication given by concatenation)

More diagram algebras: braids

The **affine (one-pole) braid group**:



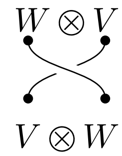
(with multiplication given by concatenation)

Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q\mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

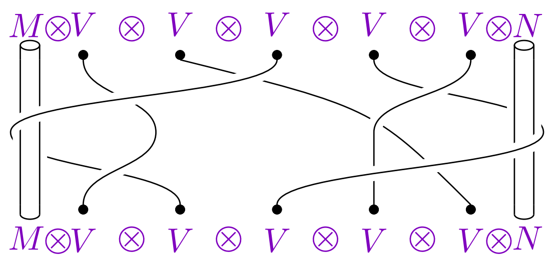
$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\check{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$

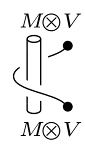


that (1) satisfies braid relations, and
(2) commutes with the action on $V \otimes W$
for any $\mathcal{U}$-module V.

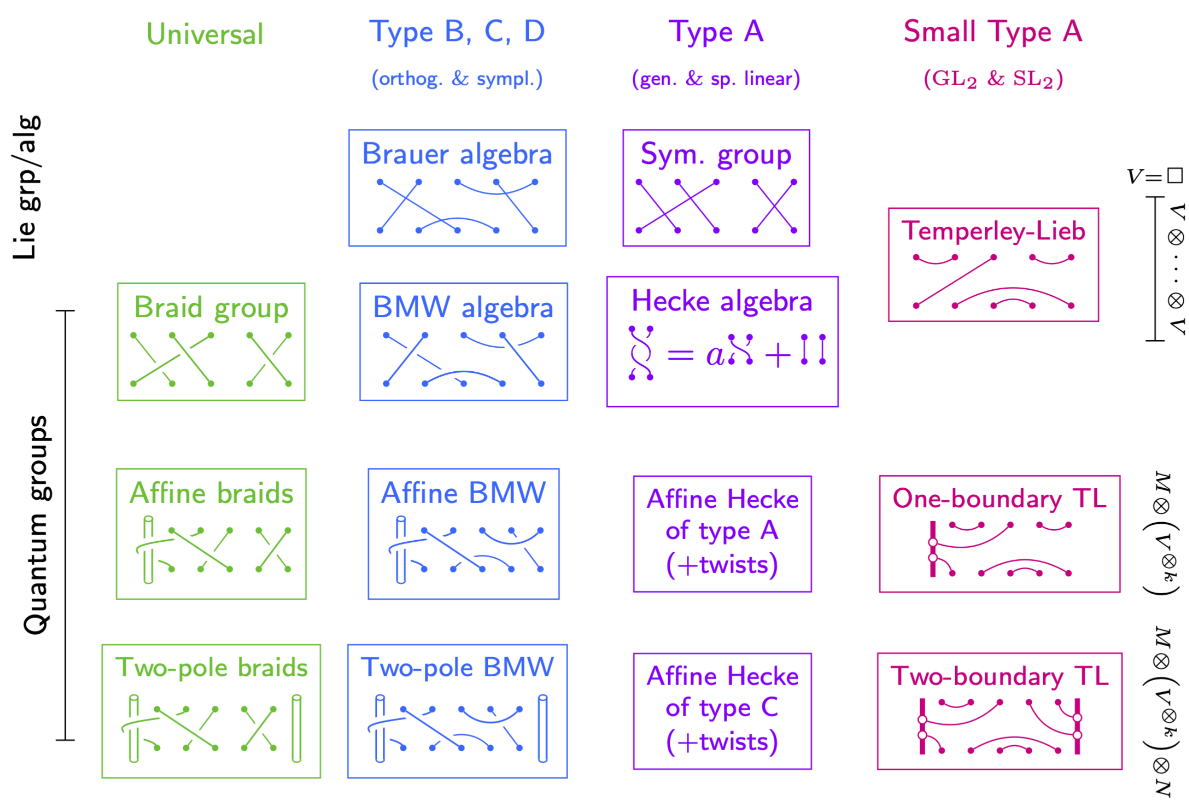
The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k} \otimes N$:

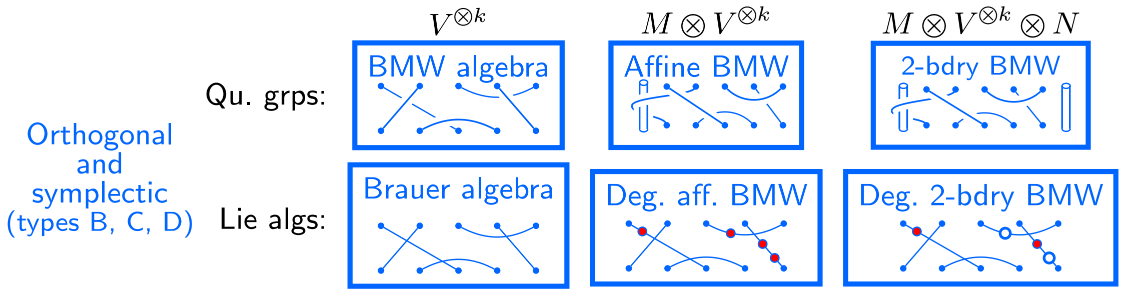


Around the pole:



$= \check{R}_{MV}\check{R}_{VM}$





Nazarov (95): Introduced degenerate affine Birman-Murakami-Wenzl (BMW) algebras, built from Brauer algebras and their Jucys-Murphy elements.

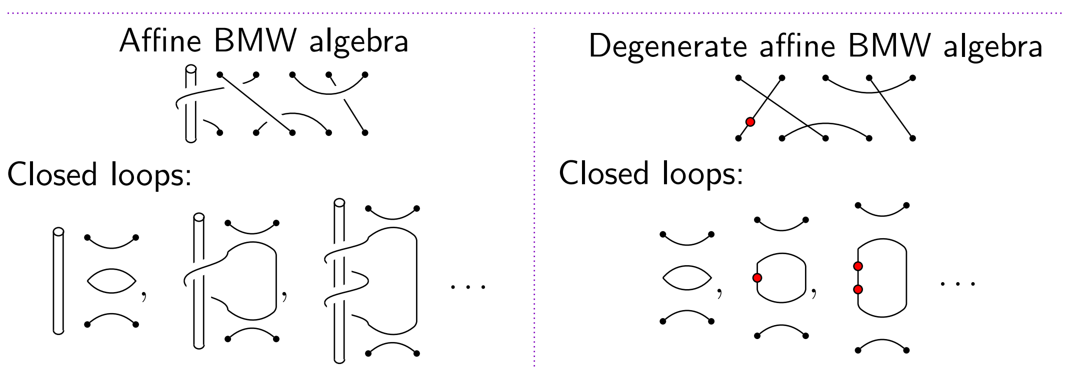
Häring-Oldenburg (98) and Orellana-Ram (04): Introduced the affine BMW algebras. [OR04] gave the action on $M \otimes V^{\otimes k}$ commuting with the action of the quantum groups of types B, C, D.

D.-Ram-Virk: Used these centralizer relationships to study these two algebras simultaneously. Results include computing the centers, handling the parameters associated to the algebras, computing powerful intertwiner operators, etc.

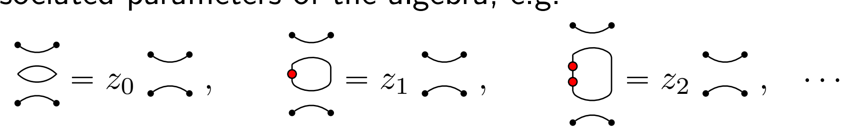
D.-González-Schneider-Sutton: Constructing 2-boundary analogues (in progress.).

Balagovic et al.: Signed versions and representations of periplectic Lie superalgebras.

Example: "Admissibility conditions"

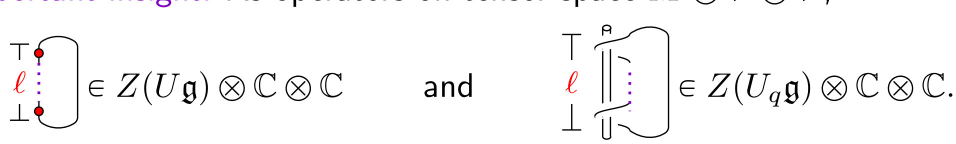


The associated parameters of the algebra, e.g.

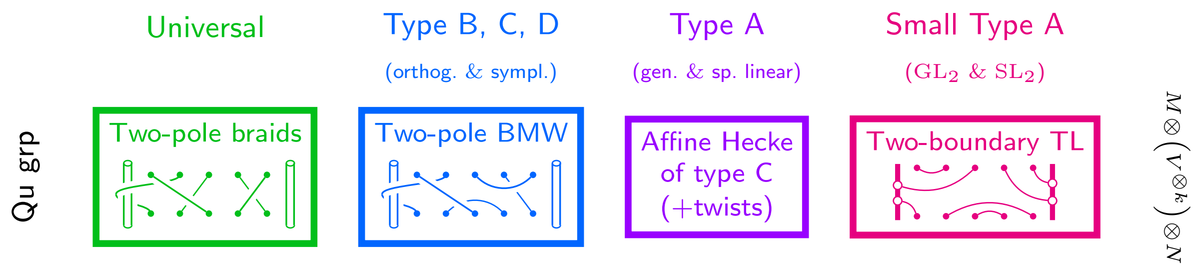


aren't entirely free.

Important insight: As operators on tensor space $M \otimes V \otimes V$,

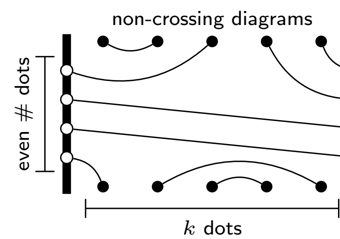


"Higher Casimir invariants"



Two boundary algebras (type A)

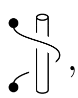
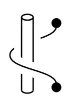
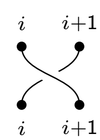
Nienhuis, de Gier, Batchelor (2004): Studying the six-vertex model with additional integrable boundary terms, introduced the two-boundary Temperley-Lieb algebra TL_k:



de Gier, Nichols (2008): Explored representation theory of TL_k using diagrams and established a connection to the affine Hecke algebras of type A and C.

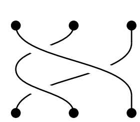
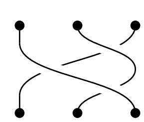
D. (2010): The centralizer of $\mathfrak{gl}_n$ acting on tensor space $M \otimes V^{\otimes k} \otimes N$ displays type C combinatorics for good choices of M, N, and V.

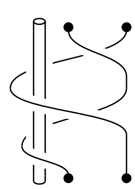
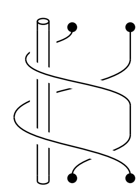
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$  for $1 \leqslant i \leqslant k-1$,

subject to relations

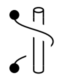
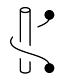
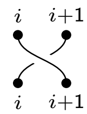
i.e.

$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1}$,

$T_1T_0T_1T_0 =$  $=$  $= T_0T_1T_0T_1$,

and, similarly, $T_{k-1}T_kT_{k-1}T_k = T_kT_{k-1}T_kT_{k-1}$.

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leqslant i \leqslant k-1$,

subject to relations 

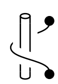
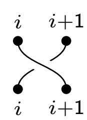
.

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

$$\text{and} \quad (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

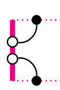
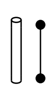
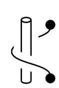
for $1 \leqslant i \leqslant k-1$,

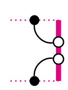
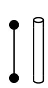
subject to relations 

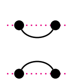
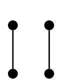
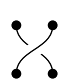
.

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

 $= t_0^{1/2}$  $-$  $\qquad (e_0 = t_0^{1/2} - T_0)$

 $= t_k^{1/2}$  $-$  $\qquad (e_k = t_k^{1/2} - T_k)$

 $= t^{1/2}$  $-$  $\qquad (e_i = t^{1/2} - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).
The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm1} e_i = e_i$ for $i = 1, \ldots, k-1$.

Theorem (D.-Ram)

(1) *Let $U = U_q\mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$. Let M, N, and V be finite-dimensional modules.*

The two-boundary braid group B_k acts on $M \otimes (V)^{\otimes k} \otimes N$ and this action commutes with the action of U.

(2) *If $\mathfrak{g} = \mathfrak{gl}_n$, then (for correct choices of M, N, and V), the affine Hecke algebra of type C, H_k, acts on $M \otimes (V)^{\otimes k} \otimes N$ and this action commutes with the action of U.*

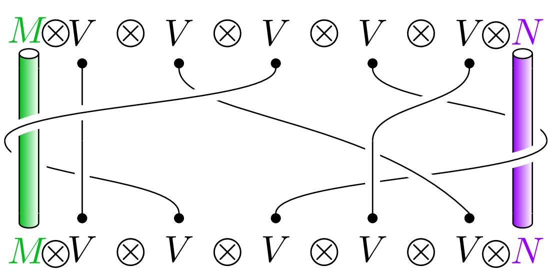
(3) *If $\mathfrak{g} = \mathfrak{gl}_2$, then the action of the two-boundary Temperley-Lieb algebra factors through the T.L. quotient of H_k.*

Some results:

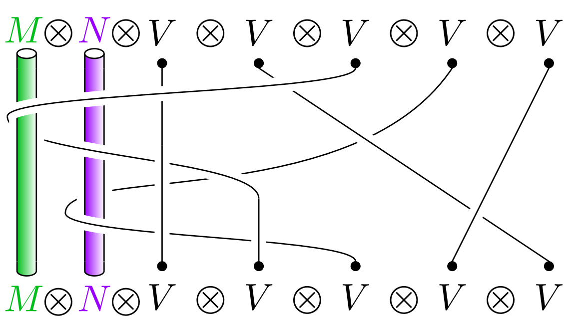
(a) A diagrammatic intuition for H_k.

(b) A combinatorial classification and construction of irreducible representations of H_k (type C with distinct parameters) via central characters and generalizations of Young tableaux.

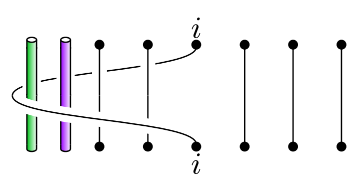
(c) A classification of the representations of TL_k in [dGN08] via central characters, including answers to open questions and conjectures regarding their irreducibility and isomorphism classes.



Move both poles to the left ↓



Jucys-Murphy elements:

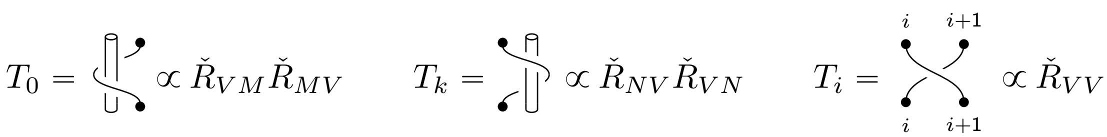
$Y_i =$ 

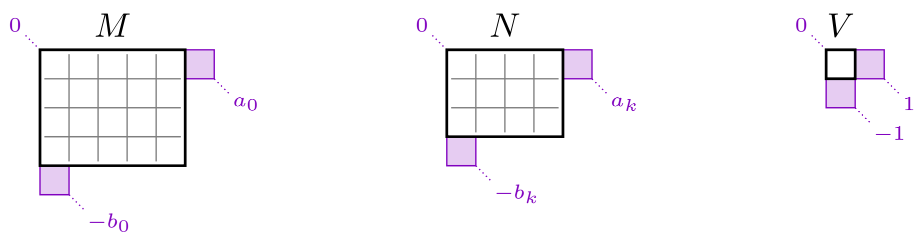
- Pairwise commute
- $Z(\mathcal{H}_k)$ is (type-C) symmetric Laurent polynomials in Z_i's
- Central characters indexed by $\mathbf{c} \in \mathbb{C}^k$ (modulo signed permutations)

Back to tensor space operators properties

The eigenvalues of the T_i's must coincide with the eigenvalues of the corresponding R-matrices, which can be computed combinatorially.

$$0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$





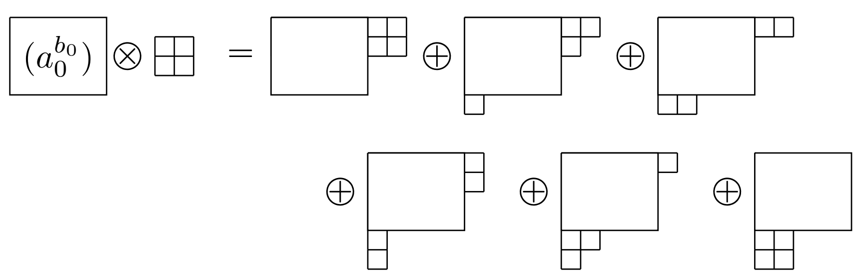
$t_0 = -q^{2(a_0+b_0)}$ $\qquad$ $t_k = -q^{2(a_k+b_k)}$ $\qquad$ $t = q^2$

Exploring $M \otimes N \otimes L(\square)^{\otimes k}$

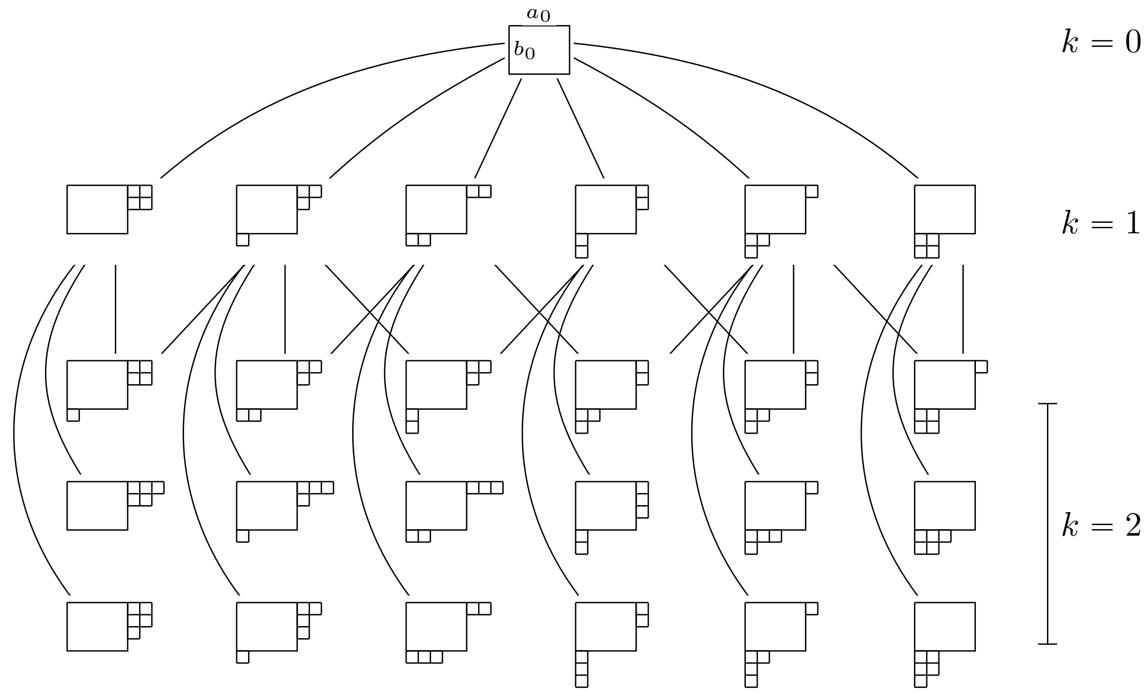
Products of rectangles:

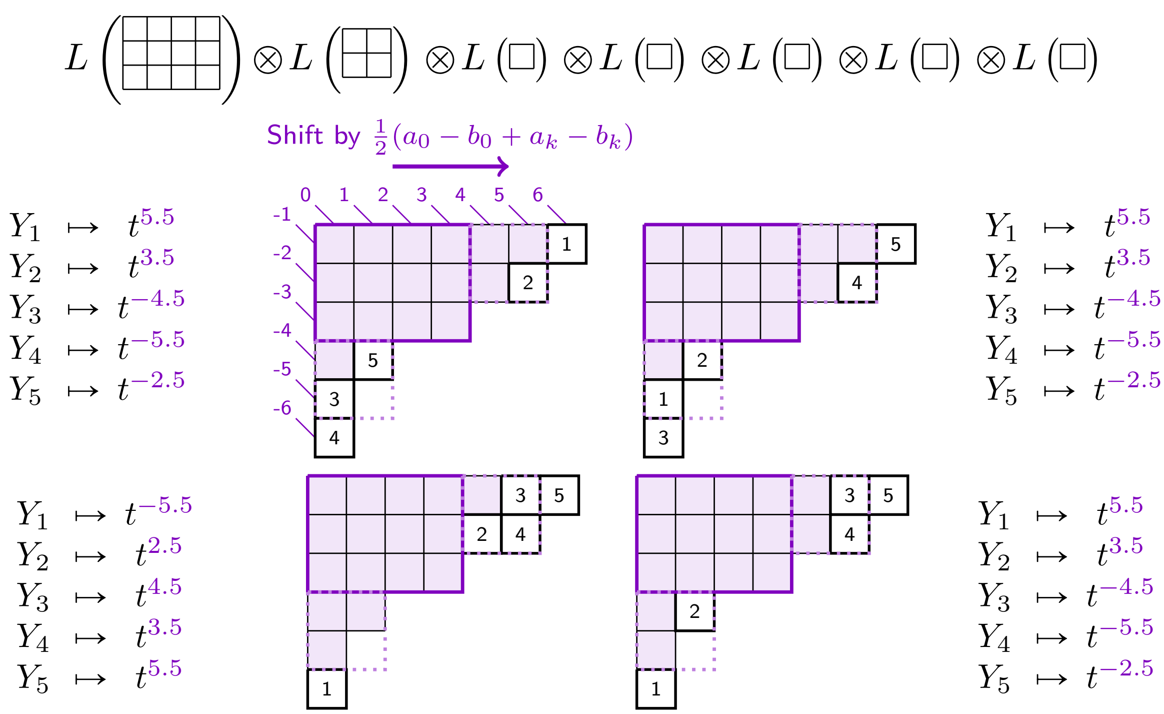
$$L((a_0^{b_0})) \otimes L(({a_k}^{b_k})) = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

where Λ is the following set of partitions...



Exploring $M \otimes N \otimes L(\square)^{\otimes k}$





($*$) H_k representations in tensor space are labeled by certain partitions λ.
($*$) Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.
($*$) Calibrated (Y_i's are diagonalized): Y_i acts by t to the shifted diagonal number of box_i. (Think: signed permutations.)