Representation theory and combinatorics of braid algebras and their quotients.

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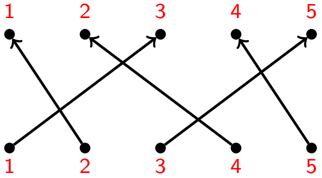
Slides available at https://zdaugherty.ccnysites.cuny.edu/research/

Schur-Weyl Duality

The **symmetric group** S_k (permutations) as diagrams:



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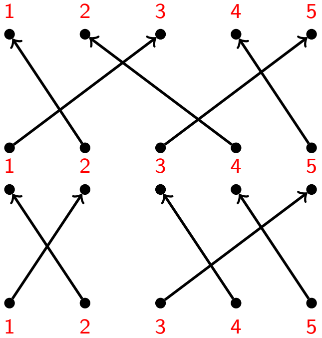

(with multiplication given by concatenation)

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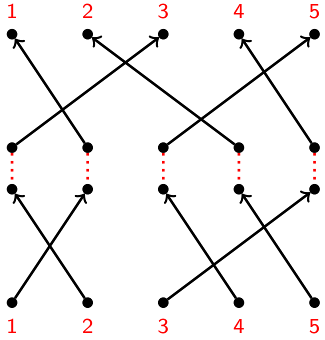

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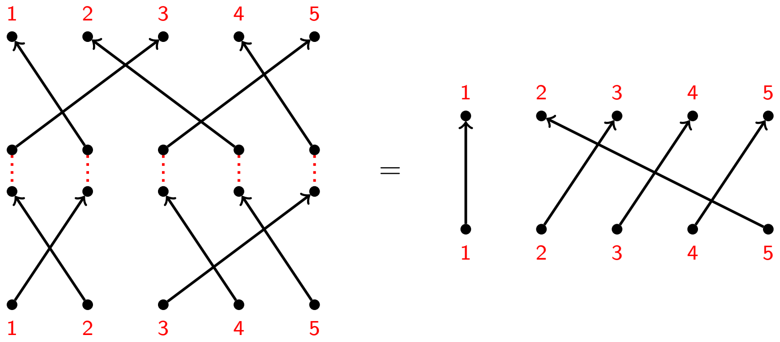

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$\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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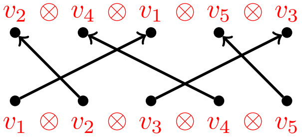

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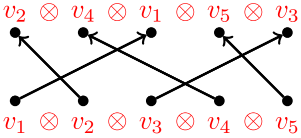
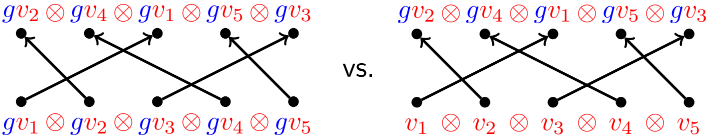


These actions commute.



Schur-Weyl Duality

Schur (1901): S_k and GL_n have commuting actions on $(\mathbb{C}^n)^{\otimes k}$.
Even better,

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\substack{\text{(all linear maps that}\\ \text{commute with } \mathrm{GL}_n\text{)}}} = \underbrace{\pi(\mathbb{C}S_k)}_{\substack{\text{(img of } S_k\\ \text{action)}}} \quad \text{and} \quad \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\substack{\text{(img of } \mathrm{GL}_n\\ \text{action)}}}.$$

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Powerful consequence: a duality between representations

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

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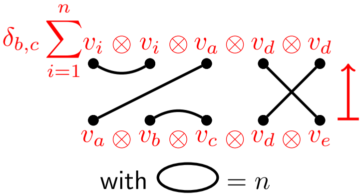
For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \left(G^{\ydiagram{3}} \otimes S^{\ydiagram{3}}\right) \oplus \left(G^{\ydiagram{2,1}} \otimes S^{\ydiagram{2,1}}\right) \oplus \left(G^{\ydiagram{1,1,1}} \otimes S^{\ydiagram{1,1,1}}\right)$$

More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize the **Brauer algebra**:



(Diagrams encoding maps $V^{\otimes k} \to V^{\otimes k}$ that commute with the action of some classical algebra.)

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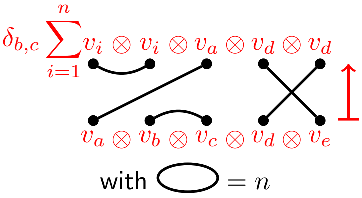
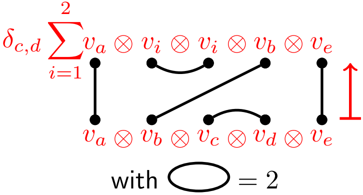


Temperley-Lieb (1971)
GL_2 and SL_2 acting on $(\mathbb{C}^2)^{\otimes k}$ diagonally centralize the **Temperley-Lieb algebra**:



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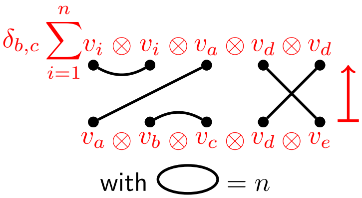

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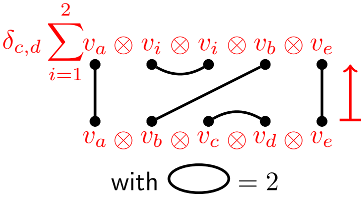

(Diagrams encoding maps $V^{\otimes k} \to V^{\otimes k}$ that commute with the action of some classical algebra.)

Quantum groups and braids

Let $\mathfrak{g}$ be a Lie algebra, and fix $q \in \mathbb{C}$.

One deformation of $\mathfrak{g}$ is the Drinfel'd-Jimbo quantum group $\mathcal{U} = \mathcal{U}_q\mathfrak{g}$.

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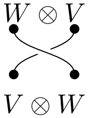
$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\check{\mathcal{R}}_{VW} \colon V \otimes W \longrightarrow W \otimes V$$



that (1) satisfies braid relations, and
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for any $\mathcal{U}$-modules V, W.

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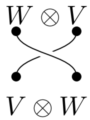

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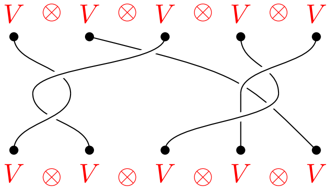
The braid group shares a commuting action with $\mathcal{U}$ on $V^{\otimes k}$:



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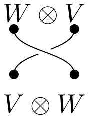

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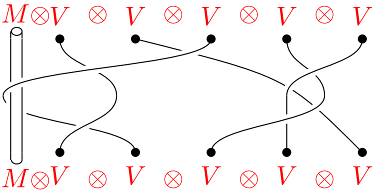
The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k}$:



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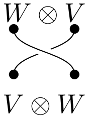

that (1) satisfies braid relations, and
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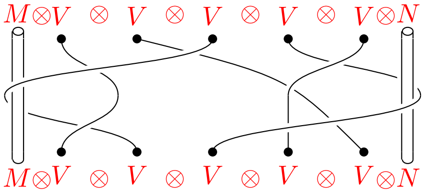
The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k} \otimes N$:



Lie grp/alg

Type B, C, D

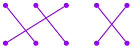
(orthog. & sympl.)

Brauer algebra

Type A

(gen. & sp. linear)

Sym. group



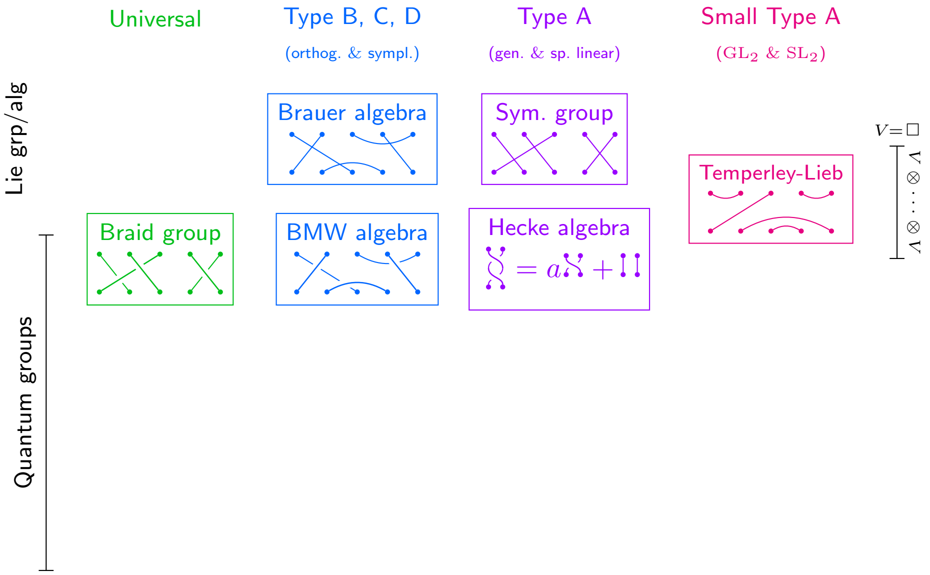
Small Type A

(GL_2 & SL_2)

Temperley-Lieb

$V = \square$

$V \otimes \cdots \otimes V$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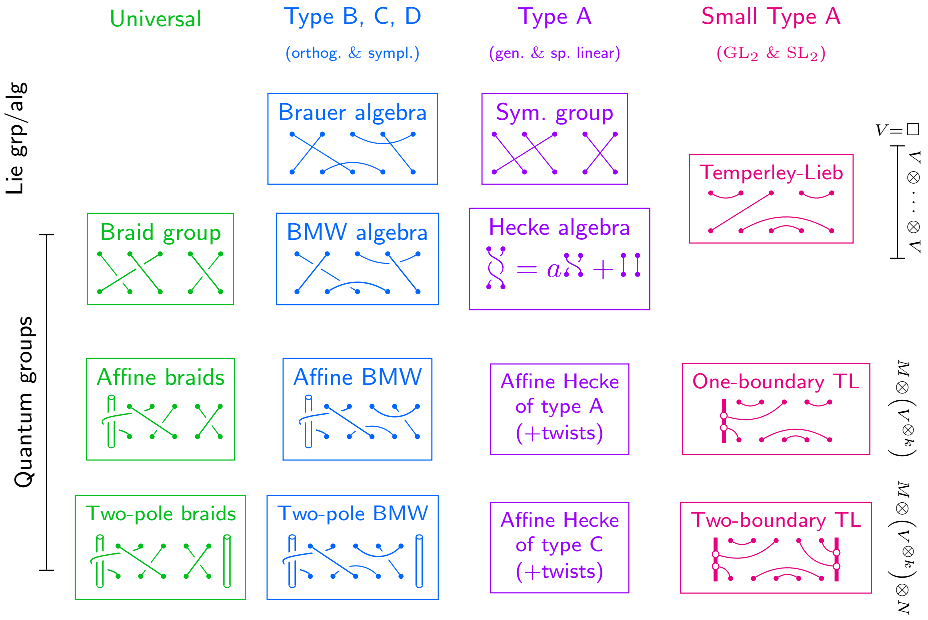

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One-boundary TL
$M \otimes (V^{\otimes k})$
Two-pole braids
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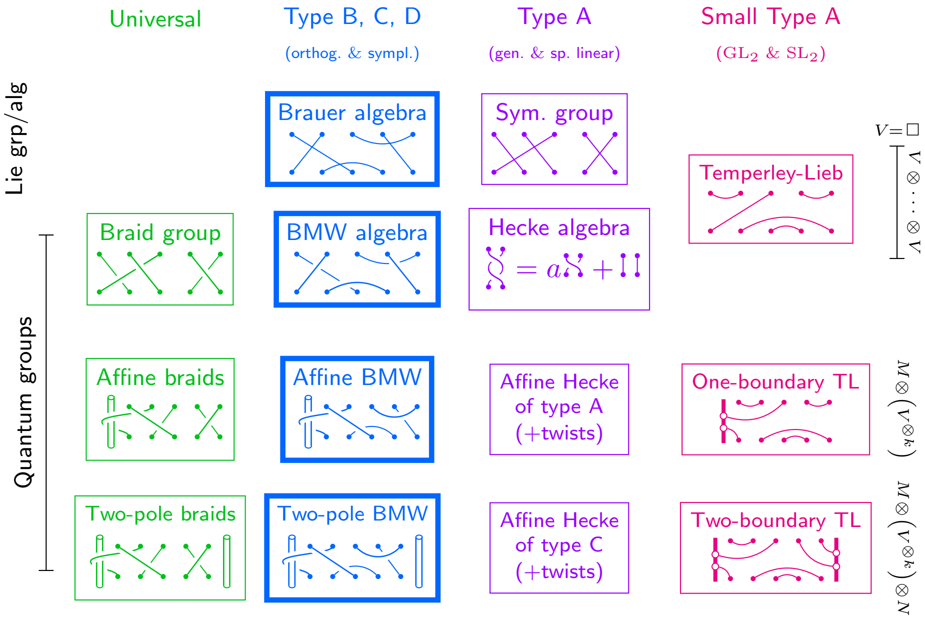
Two-boundary TL
$N \otimes (V^{\otimes k}) \otimes M$



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Two-pole BMW
Affine Hecke
of type C
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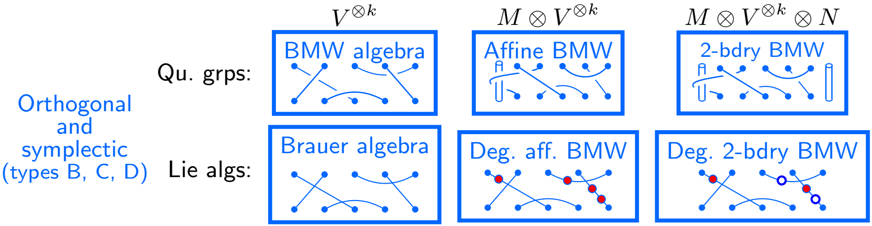

$V^{\otimes k}$
$M \otimes V^{\otimes k}$
$M \otimes V^{\otimes k} \otimes N$
Qu. grps:
BMW algebra
Affine BMW
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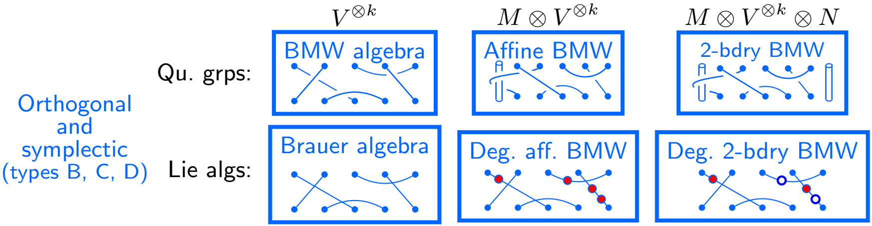
Deg. 2-bdry BMW



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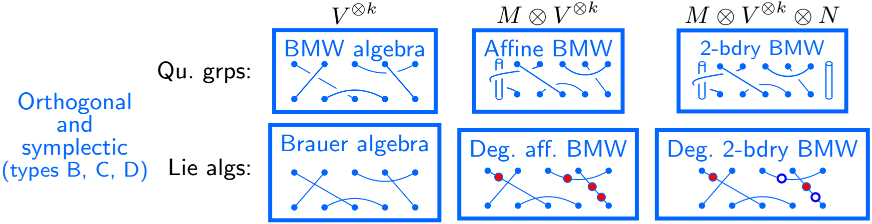

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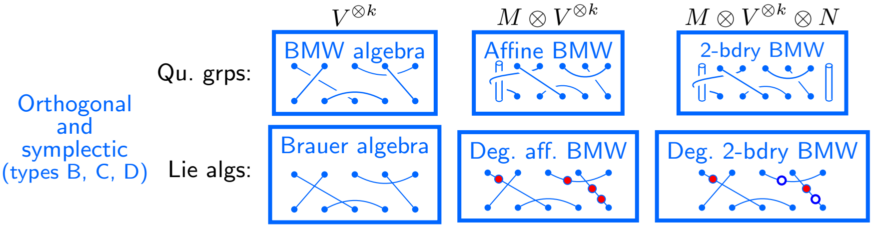

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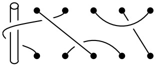
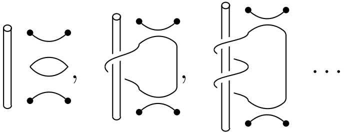
D.-Ram-Virk: Used these centralizer relationships to study these two algebras simultaneously. Results include computing the centers, handling the parameters associated to the algebras, computing powerful intertwiner operators, etc.

Example: “Admissibility conditions”

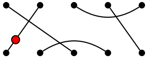
Affine BMW algebra



Closed loops:



Degenerate affine BMW algebra



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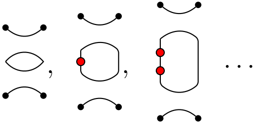

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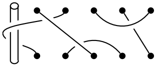

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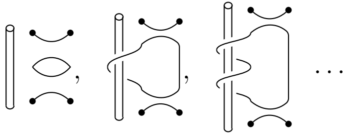

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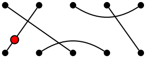

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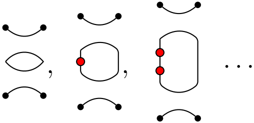
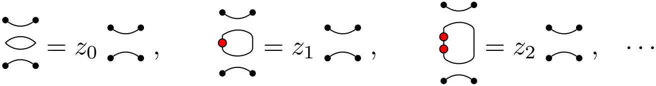


The associated parameters of the algebra, e.g.



aren't entirely free (more so in "cyclotomic quotients").

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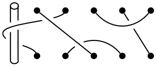

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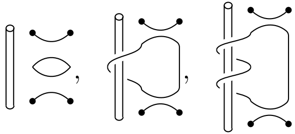

$\dots$

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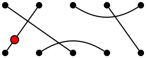

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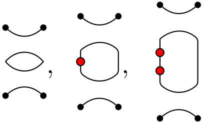
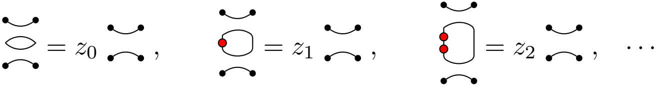


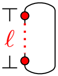
$\dots$

The associated parameters of the algebra, e.g.

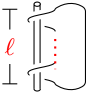


aren't entirely free (more so in "cyclotomic quotients").

Important insight: As operators on tensor space $M \otimes V \otimes V$,

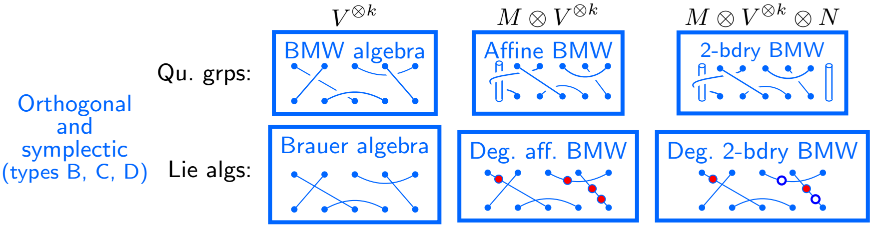


$\in Z(U\mathfrak{g}) \otimes \mathbb{C} \otimes \mathbb{C}$ and



$\in Z(U_q\mathfrak{g}) \otimes \mathbb{C} \otimes \mathbb{C}$.

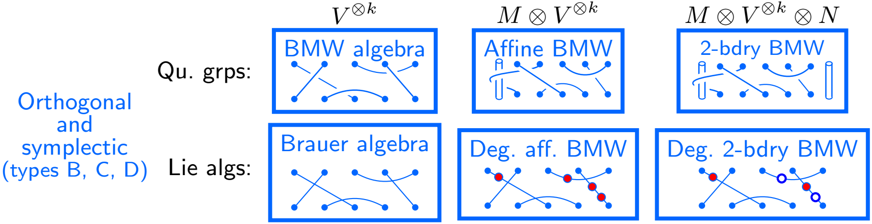
"Higher Casimir invariants"



Nazarov (95): Introduced degenerate affine Birman-Murakami-Wenzl (BMW) algebras, built from Brauer algebras and their Jucys-Murphy elements.

Häring-Oldenburg (98) and Orellana-Ram (04): Introduced the affine BMW algebras. [OR04] gave the action on $M \otimes V^{\otimes k}$ commuting with the action of the quantum groups of types B, C, D.

D.-Ram-Virk: Used these centralizer relationships to study these two algebras simultaneously. Results include computing the centers, handling the parameters associated to the algebras, computing powerful intertwiner operators, etc.



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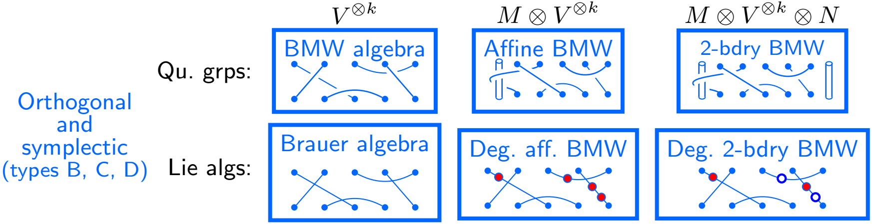
D.-González-Schneider-Sutton:
Constructing 2-boundary analogues
(in progress.).



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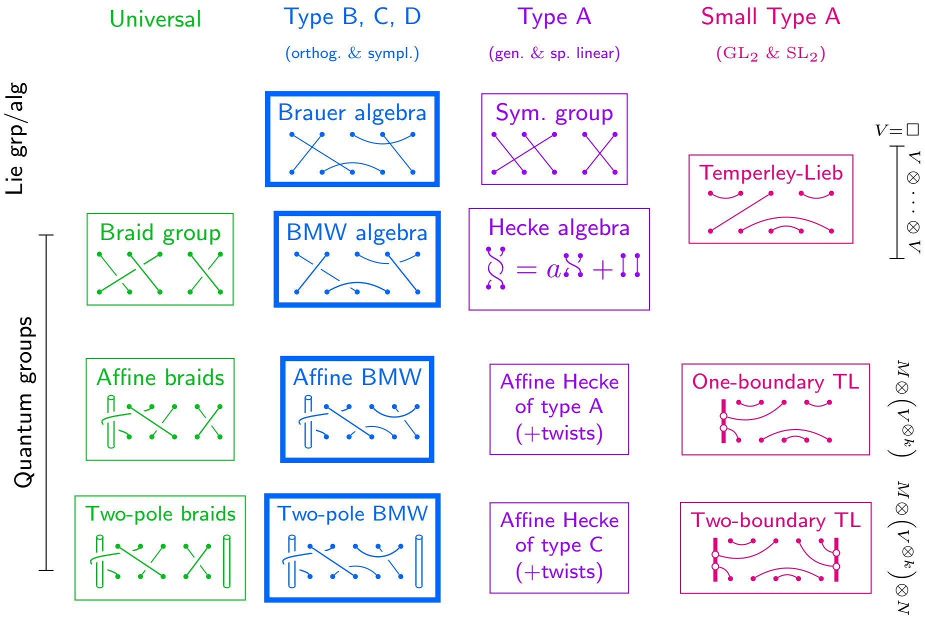
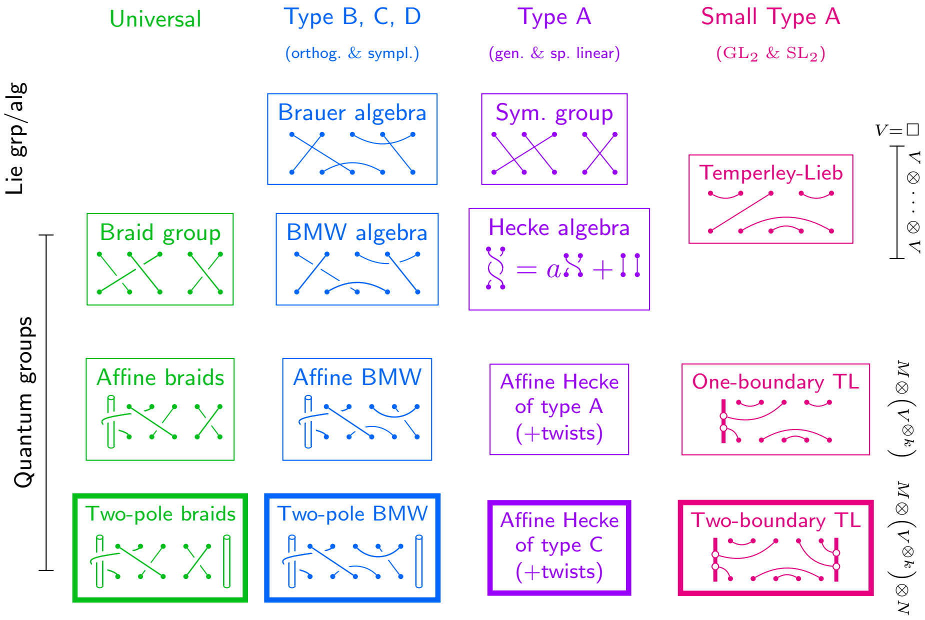
Balagovic et al.: Signed versions and representations of periplectic Lie superalgebras.


Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL_2 & SL_2)
Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
Temperley-Lieb
$V = \square$
$V \otimes \cdots \otimes V$
Braid group
BMW algebra
Hecke algebra
$= a \; + \;$
Affine braids
Affine BMW
Affine Hecke
of type A
(+twists)
One-boundary TL
$M \otimes (V^{\otimes k})$
Two-pole braids
Two-pole BMW
Affine Hecke
of type C
(+twists)
Two-boundary TL
$N \otimes (V^{\otimes k}) \otimes M$



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Braid group
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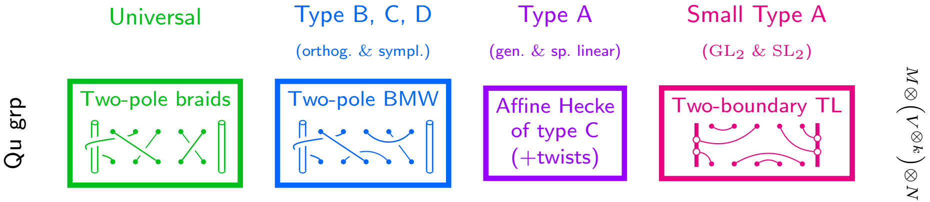
Hecke algebra
$= a$ $+$
Affine braids
Affine BMW
Affine Hecke of type A (+twists)
One-boundary TL
$\left(V^{\otimes k}\right)\otimes M$
Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two-boundary TL
$N\otimes\left(V^{\otimes k}\right)\otimes M$



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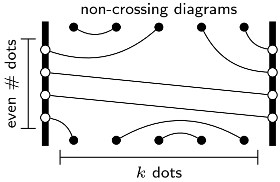
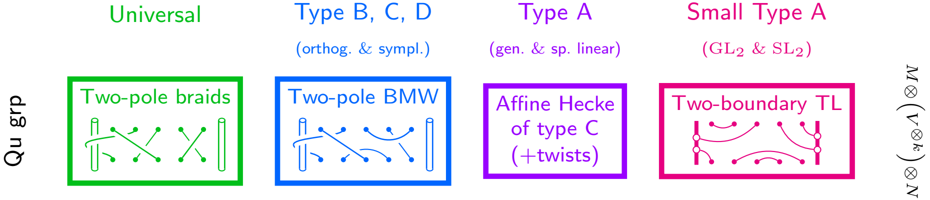
	Universal	Type B, C, D (orthog. & sympl.)	Type A (gen. & sp. linear)	Small Type A (GL_2 & SL_2)	
Qu grp	Two-pole braids	Two-pole BMW	Affine Hecke of type C (+twists)	Two-boundary TL	$N \otimes \left(V^{\otimes k}\right) \otimes M$

Two boundary algebras (type A)

Nienhuis, de Gier, Batchelor (2004): Studying the six-vertex model with additional integrable boundary terms, introduced the two-boundary Temperley-Lieb algebra TL_k:





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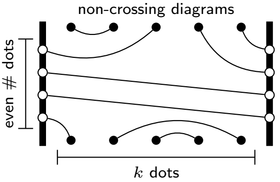

de Gier, Nichols (2008): Explored representation theory of TL_k using diagrams and established a connection to the affine Hecke algebras of type A and C.

Universal	Type B, C, D	Type A	Small Type A
	(orthog. & sympl.)	(gen. & sp. linear)	(GL_2 & SL_2)

Qu grp

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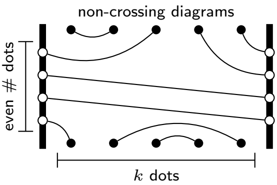

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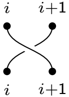
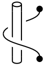
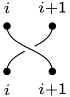
D. (2010): The centralizer of $\mathfrak{gl}_n$ acting on tensor space $M \otimes V^{\otimes k} \otimes N$ displays type C combinatorics for good choices of M, N, and V.

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

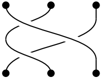
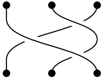
for $1 \leq i \leq k-1$,

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

subject to relations

$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1},$

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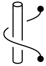
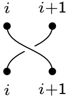
$T_k =$ , $T_0 =$  and $T_i =$ 

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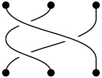
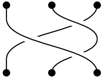
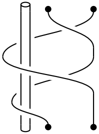
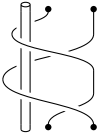
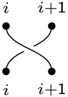
$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

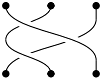
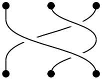
$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$,

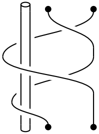
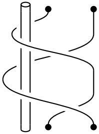
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

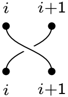
subject to relations

$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$,

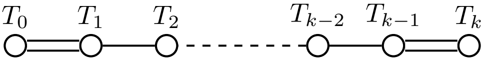
and, similarly, $T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$.

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

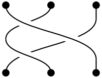
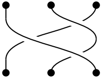
$T_k =$ , $T_0 =$  and $T_i =$ 

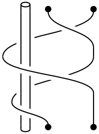
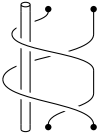
for $1 \le i \le k-1$,

subject to relations



i.e.

$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1}$,

$T_1T_0T_1T_0 =$  $=$  $= T_0T_1T_0T_1$,

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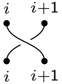
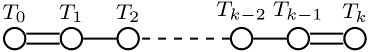
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

subject to relations 

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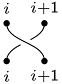
$T_k =$ , $T_0 =$  and $T_i =$ 

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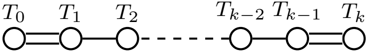
subject to relations 

.

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

$$\text{and} \quad (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

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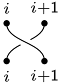
$T_k =$ , $T_0 =$  and $T_i =$ 

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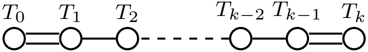
subject to relations 

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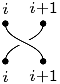
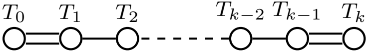
$T_k =$ , $T_0 =$  and $T_i =$  for $1 \le i \le k-1$,

subject to relations  .

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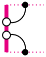
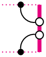
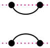
(3) Set

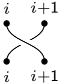
 $= t_0^{1/2}$  $-$  $\qquad (e_0 = t_0^{1/2} - T_0)$

 $= t_k^{1/2}$  $-$  $\qquad (e_k = t_k^{1/2} - T_k)$

 $= t^{1/2}$  $-$  $\qquad (e_i = t^{1/2} - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

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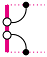
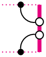
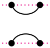
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

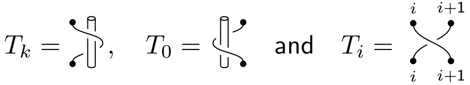
 $= t_0^{1/2}$  $-$  $\quad (e_0 = t_0^{1/2} - T_0)$

 $= t_k^{1/2}$  $-$  $\quad (e_k = t_k^{1/2} - T_k)$

 $= t^{1/2}$  $-$  $\quad (e_i = t^{1/2} - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).
The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm1} e_i = e_i$ for $i = 1, \ldots, k-1$.

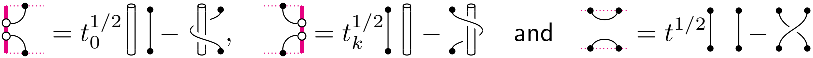
(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by



for $1 \le i \le k-1$.

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set



so that $e_j^2 = z_j e_j$. The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm1} e_i = e_i$ for $i = 1, \ldots, k-1$.

Universal	Type B, C, D (orthog. & sympl.)	Type A (gen. & sp. linear)	Small Type A (GL_2 & SL_2)

Qu grp

$M \otimes (V^{\otimes k}) \otimes N$

Theorem (D.-Ram)

(1) *Let* $U = U_q\mathfrak{g}$ *for any complex reductive Lie algebras* $\mathfrak{g}$*.*
Let M*,* N*, and* V *be finite-dimensional modules.*

The two-boundary braid group B_k *acts on* $M \otimes (V)^{\otimes k} \otimes N$ *and this action commutes with the action of* U*.*

(2) *If* $\mathfrak{g} = \mathfrak{gl}_n$*, then (for correct choices of* M*,* N*, and* V*),*
the affine Hecke algebra of type C*,* H_k*, acts on* $M \otimes (V)^{\otimes k} \otimes N$ *and this action commutes with the action of* U*.*

(3) *If* $\mathfrak{g} = \mathfrak{gl}_2$*, then the action of the two-boundary Temperley-Lieb algebra factors through the T.L. quotient of* H_k*.*

Theorem (D.-Ram)

(1) *Let* $U = U_q\mathfrak{g}$ *for any complex reductive Lie algebras* $\mathfrak{g}$.
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(2) *If* $\mathfrak{g} = \mathfrak{gl}_n$, *then (for correct choices of* M, N, *and* V*), the affine Hecke algebra of type* C, H_k, *acts on* $M \otimes (V)^{\otimes k} \otimes N$ *and this action commutes with the action of* U.

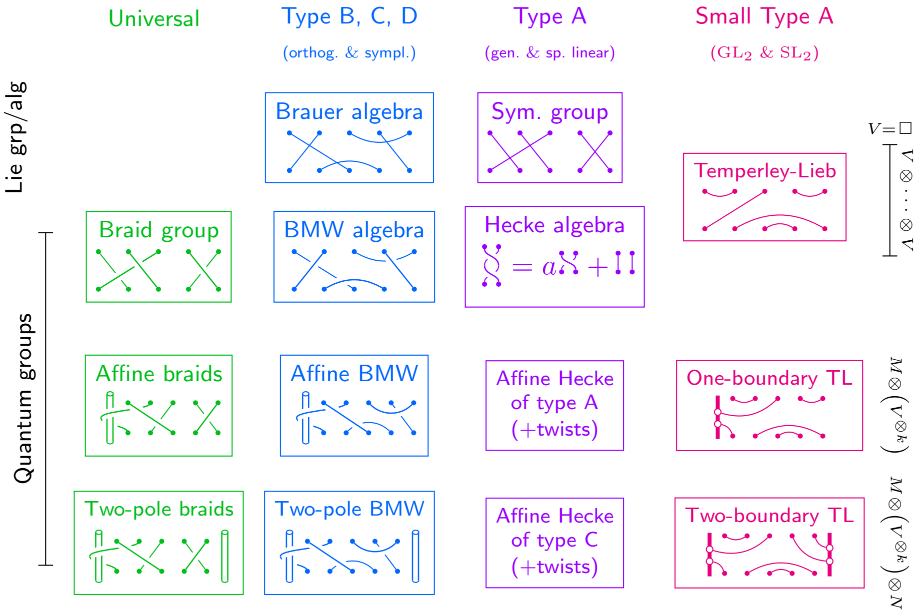
(3) *If* $\mathfrak{g} = \mathfrak{gl}_2$, *then the action of the two-boundary Temperley-Lieb algebra factors through the T.L. quotient of* H_k.

Some results:

(a) A diagrammatic intuition for H_k.

(b) A combinatorial classification and construction of irreducible representations of H_k (type C with distinct parameters) via central characters and generalizations of Young tableaux.

(c) A classification of the representations of TL_k in [dGN08] via central characters, including answers to open questions and conjectures regarding their irreducibility and isomorphism classes.


Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL_2 & SL_2)
Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
Temperley-Lieb
$V = \square$
$V \otimes \cdots \otimes V$
Braid group
BMW algebra
Hecke algebra
$= a \quad + $
Affine braids
Affine BMW
Affine Hecke of type A (+twists)
One-boundary TL
$(V^{\otimes k}) \otimes M$
Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two-boundary TL
$N \otimes (V^{\otimes k}) \otimes M$