Signed Brauer algebras and their translations

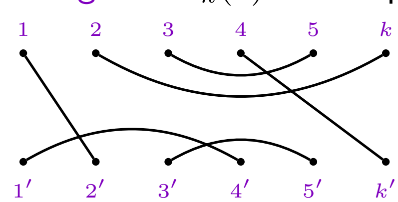
Zajj Daugherty
The City College of New York
& The CUNY Graduate Center

Joint with M. Balagovic, I. Entova-Aizenbud, I. Halacheva,
J. Hennig, M. S. Im, G. Letzter, E. Norton,
V. Serganova, and C. Stroppel

October 2, 2019

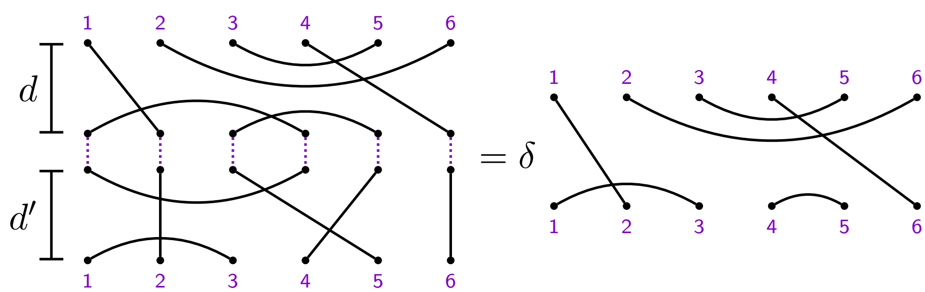
The classical Brauer algebra

The Brauer algebra $B_k(\delta)$ is the space spanned by Brauer diagrams

$d =$ 

perfect matchings of $\{1, \ldots, k, 1', \ldots, k'\}$

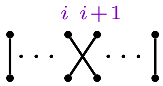
(equivalent under isotopy), with multiplication given by vertical concatenation, subject to the relation $\bigcirc = \delta$. For example,

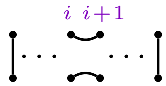
$dd' =$ 

.

Action on tensor space

The Brauer algebra $B_k(\delta)$ is generated by

$s_i =$ 

and $e_i =$ 

, $\quad i = 1, \ldots, k-1,$

with expected relations.
Let V be a finite dimensional vector space, with $\beta : V \otimes V \to \mathbb{C}$ a non-degenerate symmetric (resp. skew symmetric) bilinear form on V, and β^* its dual. Then the map $B_k(\delta) \to \mathrm{End}(V^{\otimes k})$ that sends

$$s_i \mapsto 1^{\otimes i-1} \otimes s \otimes 1^{k-i-1}, \qquad e_i \mapsto 1^{\otimes i-1} \otimes \beta^*\beta \otimes 1^{k-i-1},$$

where $s(u \otimes v) = v \otimes u$, is a map

$$B_k(\delta) \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes k})$$

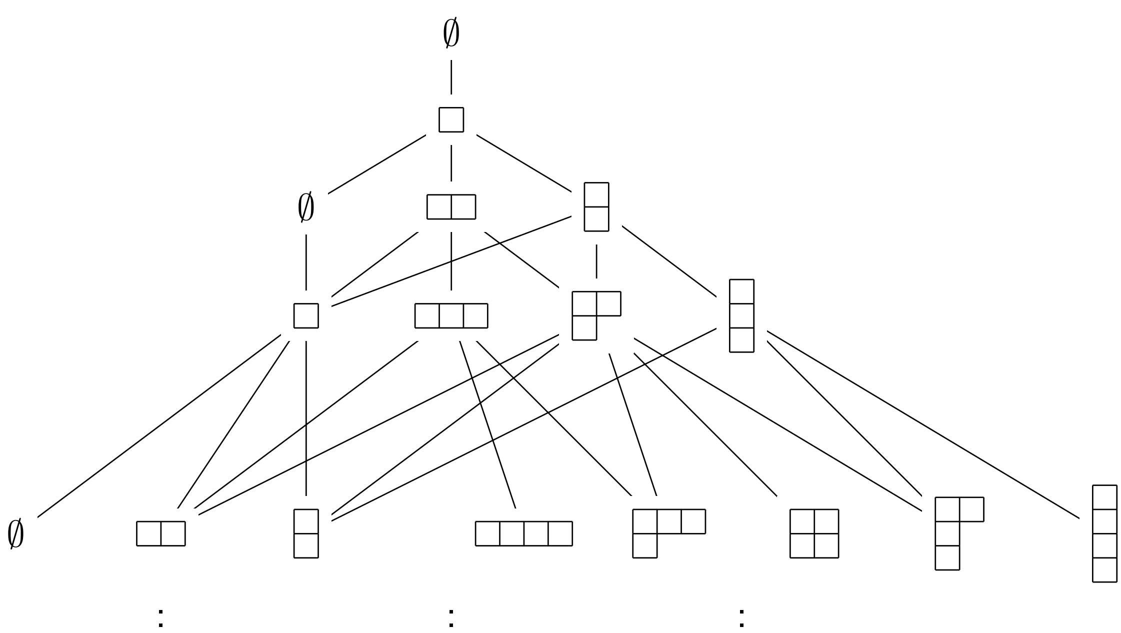
when $\mathfrak{g} = \mathfrak{so}(V)$ (resp. $\mathfrak{sp}(V)$), $\delta = \mathrm{dim}V$ (resp. $-\mathrm{dim}V$).

Consequence: Schur-Weyl duality between modules for $B_k(\delta)$ and $\mathfrak{g}$,

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k, k-2, \ldots} B_k^\lambda \otimes L(\lambda),$$

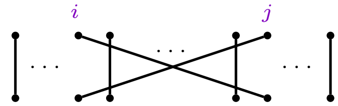
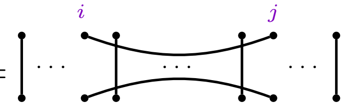
where B_k^λ are distinct simple Brauer modules, and $L(\lambda)$ are distinct simple (highest weight) $\mathfrak{g}$-modules.

Decompositions of $V^{\otimes k}$



Jucys-Murphy elements

For $i < j$, let

$s_{i,j} =$  and $e_{i,j} =$  .

Brauer algebra $B_k(\delta)$ has Jucys-Murphy elements

$$x_j = \text{constant} + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad j = 1, \ldots, k,$$

that (See Nazarov '96, D.-Ram-Virk '13 & '14.)

1. pairwise commute;
2. generate the center; and
3. have eigenvalues in $\mathrm{End}(V^{\otimes k})$ given by combinatorial data from the partitions lattice.

Jucys-Murphy elements

Brauer algebra $B_k(\delta)$ has Jucys-Murphy elements

$$x_j = \text{constant} + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad j = 1, \ldots, k,$$

that (See Nazarov '96, D.-Ram-Virk '13 & '14.)

1. pairwise commute;
2. generate the center; and
3. have eigenvalues in $\mathrm{End}(V^{\otimes k})$ given by combinatorial data from the partitions lattice.

Let Γ be a basis for $\mathfrak{g}$, and $\Gamma = \{b^* \mid b \in \Gamma\}$ be the dual basis with respect to a nice bilinear form. The split Casimir invariant $U\mathfrak{g} \otimes U\mathfrak{g}$ is

$$\gamma = \sum_{b \in \Gamma} b \otimes b^*.$$

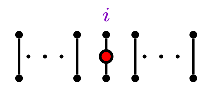
Then γ acts on $V \otimes V$ by $s_1 - e_1$, so the action of x_i is given by

$$x_i\Big|_{V^{\otimes k}} = \text{constant} + \sum_{j=1,\ldots,i-1} \gamma\Big|_{V^{(j)},V^{(i)}}.$$

Jucys-Murphy elements

Now let M be a simple $\mathfrak{g}$ module, and define an operator on $M \otimes V^{\otimes k}$ by

$$y_i = \text{constant} + \gamma\Big|_{M,V^{(i)}} + \sum_{j=1,\dots,i-1} \gamma\Big|_{V^{(j)},V^{(i)}} =$$



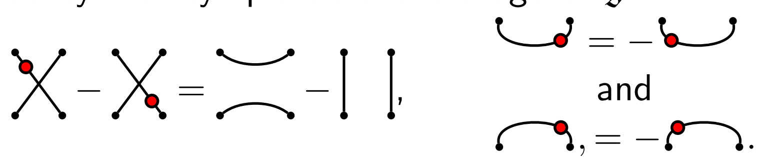
(When $M = L(0)$, this is the same as x_i from before.)

Nice facts: Still...

1. $y_1, \dots, y_k$ commute;
2. $y_i \in \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes k})$;
3. they generate the center of the action; and
4. have eigenvalues given by combinatorial data from the partitions lattice.

The graded Brauer algebra is the algebra generated by s_i and e_i for $i = 1, \dots k-1$, and $y_1, \dots, y_k$ (modulo relations), and is also in Schur-Weyl duality with symplectic and orthogonal $\mathfrak{g}$.

Relations:



Lie superalgebras

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a super Lie bracket

$$[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

satisfying

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$$

and

$$[x, [y, z]]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]],$$

where x, y, z are each homogeneous, and $\bar{x}$ means degree.

Three types: basic, Cartan type, and strange (two families: periplectic and queer).

Let $V = V_0 \oplus V_1 = \mathbb{C}^{m|n}$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$.

The general linear Lie superalgebra is

$$\mathfrak{gl}(m|n) = \mathrm{End}(V)$$

Lie superalgebras

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a super Lie bracket $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying a super symmetry and super Jacobi identity.

Three types: basic, Cartan type, and strange (two families: periplectic and queer).

Let $V = V_0 \oplus V_1 = \mathbb{C}^{m|n}$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$. For (homogeneous) $v \in V_i$, write $\bar{v} = i$ for its degree.

The general linear Lie superalgebra is

$$\mathfrak{gl}(m|n) = \mathrm{End}(V) = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\middle|\; A \in \mathrm{End}(V_0), D \in \mathrm{End}(V_1) \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \;\middle|\; B \in \mathrm{Hom}(V_1, V_0), C \in \mathrm{Hom}(V_0, V_1) \right\}.$$

Bracket: $[x, y] = xy - (-1)^{\bar{x}\bar{y}}yx$.

Lie superalgebras

Let $\beta : V \otimes V \to \mathbb{C}$ be a nondegenerate, homogeneous, bilinear form satisfying

$$\beta(v, w) = (-1)^{\bar{v}\bar{w}} \beta(w, v) \quad \text{(supersymmetric)}.$$

Then

$$\mathfrak{g} = \{x \in \mathrm{End}(V) \mid \beta(xu, v) + (-1)^{\bar{x}\bar{u}} \beta(v, xu)\}$$

is a Lie superalgebra ($\mathbb{Z}_2$-graded). For example, if β is even, $\mathfrak{g} = \mathfrak{osp}(V)$ the orthosymplectic Lie superalgebra (if $V_1 = 0$, $\mathfrak{g} = \mathfrak{so}(V)$; and if $V_0 = 0$, $\mathfrak{g} = \mathfrak{sp}(V)$).

Lie superalgebras

Let $\beta : V \otimes V \to \mathbb{C}$ be a nondegenerate, homogeneous, bilinear form satisfying

$$\beta(v, w) = (-1)^{\bar{v}\bar{w}} \beta(w, v) \quad \text{(supersymmetric)}.$$

If β is odd, then $\mathfrak{g}$ is the periplectic Lie superalgebra,

$$\mathfrak{p}(V) = \mathfrak{p}(n) = \{x \in \mathrm{End}(V) \mid \beta(xv, w) + (-1)^{\bar{x}\bar{v}} \beta(v, xw) = 0\}.$$

Specifically, we have

$$\mathfrak{p}(n) \cong \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n|n) \;\middle|\; B = B^t, C = -C^t \right\}.$$

Then, as vector spaces $\mathfrak{p}(n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$, where

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \right\} \cong \mathfrak{gl}(n)$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\}.$$

Goal: Study the representation theory of $\mathfrak{p}(n)$. In particular, study the category $\mathcal{F}_n$ of finite-dimensional integrable representations (a "highest weight category").

Translation functors

Key ingredients for other cases: a large center in $\mathcal{U}\mathfrak{g}$, and translation functors given by tensoring with the natural representation followed by the projection onto a block (given by eigenvalues of y_i's).

Namely, you study the action of $\mathcal{U}\mathfrak{g}$ on

$$M \otimes V \otimes V \otimes \cdots \otimes V = M \otimes V^{\otimes d},$$

where V is $\mathfrak{g}$'s favorite module, and M is another simple module, by constructing operators in $\mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$ that commute with the $\mathfrak{g}$-action. Many commuting operators are generated by taking coproducts of central elements (again, like y_i's).

Examples: If $\mathfrak{g} = \mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, then the commuting operators generate the graded Brauer algebra; when $\mathfrak{g} = \mathfrak{sl}(V)$, you get the "graded Hecke algebra of type A".

Obstruction: The center of $\mathcal{U}\mathfrak{p}(V)$ is trivial! But we'll figure it out anyway...

Example: $V \otimes V$

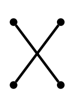
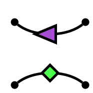
Recall: $V = V_0 \oplus V_1 = \mathbb{C}^{m|n}$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$.
For (homogeneous) $v \in V_i$, write $\bar{v} = i$ for its degree.

The algebra $\mathrm{End}_{\mathfrak{p}(V)}(V \otimes V)$ is 3-dimensional with basis 1,

$$s : v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v, \quad \text{and} \quad e = \beta^*\beta : v \otimes w \mapsto \beta(v, w)c,$$

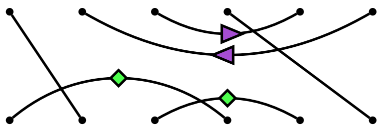
where c spans the (super) sign module.

Draw:

$s =$  and $e =$  . (signed Brauer)

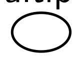
Relation: $e \circ s = e = -s \circ e$. Also, $e^2 = 0$. (non-semisimple case)

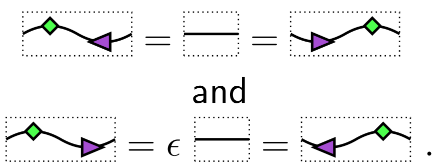
(Kujawa-Tharp 2014) The marked Brauer algebra $B_k(\delta, \epsilon)$, $\epsilon = \pm 1$, is the space spanned by marked Brauer diagrams

$d =$ 

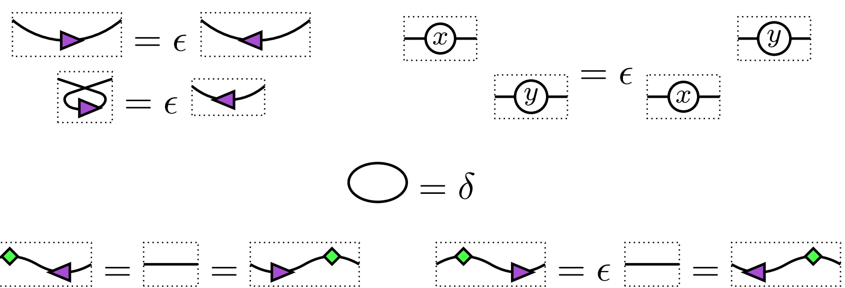
caps get one ◆ each,
cups get one ▶ or ◀ each,
no two markings at same height.

with equivalence up to isotopy except for the local relations

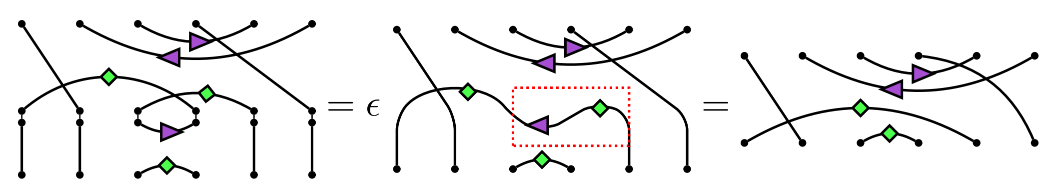
for any adjacent markings ⓧ and ⓨ (meaning no markings of height between these two). Again, multiplication is given by vertical concatenation, with relations  $= \delta$,



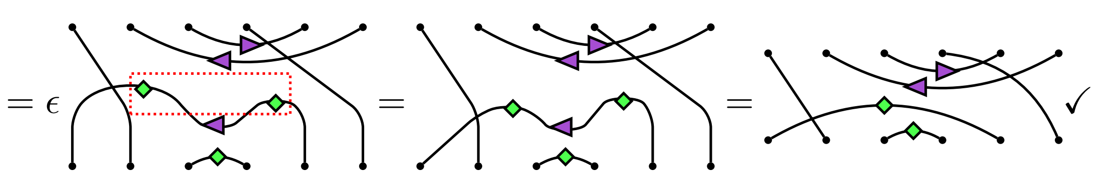
Note:
(1) $B_k(\delta, 1) = B_k(\delta)$.
(2) If $\epsilon = -1$, then multiplication is well-defined exactly when $\delta = 0$.



For example,



Alternatively,



The marked Brauer algebra $B_k(\delta, \epsilon)$ is generated by

for $i = 1, \ldots, k-1$, with relations exactly analogous to those for the Brauer algebra, with some ϵ's.

Back to Lie superalgebras: $V = V_0 \oplus V_1$, let $\beta : V \otimes V \to \mathbb{C}$ be a non-degenerate, homogeneous, bilinear form on V, and let $\mathfrak{g}$ be the corresponding β-invariant Lie superalgebra. Then with

$$\beta^* : \mathbb{C} \to V \otimes V \quad \text{and} \quad \begin{aligned} s : V \otimes V &\to V \otimes V \\ u \otimes v &\mapsto (-1)^{\bar{u}\bar{v}} v \otimes u, \end{aligned}$$

the map

$$e_i \mapsto 1^{\otimes i-1} \otimes \beta^*\beta \otimes 1^{k-i-1}, \quad s_i \mapsto 1^{\otimes i-1} \otimes s \otimes 1^{k-i-1},$$

for $i = 1, \ldots, k-1$, gives

$$B_k(\delta, \epsilon) \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes k})$$

when $\delta = \mathrm{dim} V_0 - \mathrm{dim} V_1$ and $\epsilon = (-1)^{\bar{\beta}}$ [KT14].

Jucys-Murphy elements for $B_k(\delta, \epsilon)$

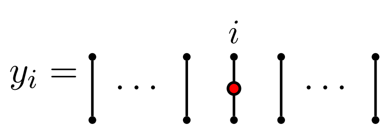
For the marked Brauer algebra,

$$x_j = \text{constant} + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad c \in \mathbb{C}, \ j = 1, \ldots, k,$$

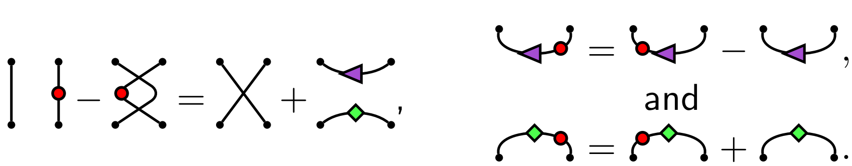
are still the Jucys-Murphy elements. So we define the graded version similarly, with ϵ's where needed,

$$\mathcal{B}_k(\delta, \epsilon) = \mathbb{C}[y_1, \ldots, y_k] \otimes B_k(\delta, \epsilon) / \langle y_i\text{-relations} \rangle$$

Namely, if we draw



then

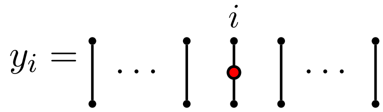


Jucys-Murphy elements for $B_k(\delta, \epsilon)$

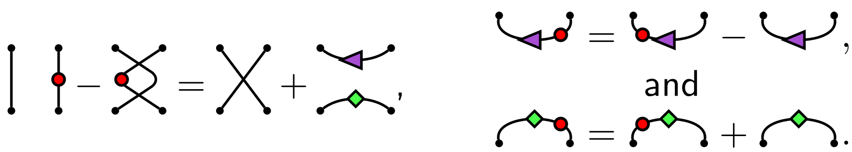
We define the graded version similarly, with ϵ's where needed,

$$\mathcal{B}_k(\delta, \epsilon) = \mathbb{C}[y_1, \ldots, y_k] \otimes B_k(\delta, \epsilon)/\langle y_i\text{-relations}\rangle$$

Namely, if we draw



then



Questions: For $\mathcal{B}_k(0, -1)$,
(1) what tensor space do we want analogous to $M \otimes V^{\otimes k}$?
(2) what's the action of the y_i's?

Start with (2): $\mathfrak{p}(V)$ has trivial center! Namely, if Γ is a basis of $\mathfrak{p}(V)$, then $\mathfrak{p}(V)$ does not contain a dual basis with respect to β.

Sneaky split Casimir

$$\mathcal{B}_k(\delta, \epsilon) = \mathbb{C}[y_1, \ldots, y_k] \otimes B_k(\delta, \epsilon)/\langle y_i\text{-relations}\rangle.$$

Questions: For $B_k(0, -1)$,
(1) what tensor space do we want analogous to $M \otimes V^{\otimes k}$?
(2) what's the action of the y_i's?

Start with (2): $\mathfrak{p}(V)$ has trivial center! Namely, if Γ is a basis of $\mathfrak{p}(V)$, then $\mathfrak{p}(V)$ does not contain a dual basis with respect to β.

In particular, considering $\mathfrak{p}(V) \subseteq \mathfrak{gl}(V)$, then $\{b^* \mid b \in \Gamma\}$ is a basis for $\mathfrak{p}(V)^\perp \subseteq \mathfrak{gl}(V)$. So

$$\gamma = \sum_{b \in \Gamma} b \otimes b^* \in U\mathfrak{p}(V) \otimes U\mathfrak{p}(V)^\perp.$$

However, γ *does* have a natural action on $M \otimes V$, since V is also a $\mathfrak{gl}(V)$-module. And since it commutes with the action of $\mathfrak{gl}(V)$, it commutes with $\mathfrak{p}(V)$. In particular, as before,

$$\gamma_{i,j} \text{ acts on } V^{\otimes k} \text{ as } s_{i,j} - e_{i,j}.$$

What should M be in $M \otimes V^{\otimes k}$?

Try 1: For the partition λ of size ℓ, take the indecomposable $M(\lambda)$ indexed by λ (the one paired with B^{λ} by Moon, Kujawa-Tharp) in $V^{\otimes \ell}$. Write the action of $\mathcal{B}_k(0,-1)$ on $M(\lambda) \otimes V^{\otimes k}$ in terms of the the action of $B_k(0,-1)$ on $V^{\otimes \ell+k}$; make an inductive argument.

Issues:

(a) Not big enough. In $V \otimes V$, the minimal polynomial for γ is $(\gamma-1)(\gamma+1)$. So the image of $\mathcal{B}_1(0,-1)$ in $\mathrm{End}(V \otimes V)$ (think $M = V$, $k=1$) is at most

$$\mathcal{B}_1(0,-1)/\langle (y_1-1)(y_1+1)\rangle \quad (\text{dim} = 2).$$

But $\mathrm{End}_{\mathfrak{p}(V)}(V \otimes V) \cong B_2(0,-1)$ (dim $=3$).

(b) Non-semisimple actions. In $V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V$,

$$e_1 : \mathrm{Sym}^2 V \xrightarrow{\beta} \mathbb{C} \xrightarrow{\beta^*} \bigwedge\nolimits^2(V)$$

has non-trivial image. So, for example, the action of $B_3(0,-1)$ on $V^{\otimes 3}$ does not restrict to a closed action on $(\mathrm{Sym}^2 V) \otimes V$.

What should M be in $M \otimes V^{\otimes k}$?

Try 1: $M(\lambda) \otimes V^{\otimes k} \subseteq V^{\otimes |\lambda|+k}$ (nope)

Try 2: Induce $\mathfrak{gl}(V) = \mathfrak{g}_0$ modules $L(\lambda)$ up to $\mathfrak{p}(V)$. Again, the dimensions to not match.

Try 3: Kac modules of two types: $K(\lambda)$ (thin) and $\tilde{K}(\lambda)$ (thick) . Let $\phi = \sum_{\text{neg. roots } \alpha} \alpha$ (like the staircase partition) and let $V(\lambda)$ be the simple $\mathfrak{g}_0$-module of highest weight λ. Define

$$K(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V(\lambda - \phi) \qquad \tilde{K}(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}}^{\mathfrak{g}} V(\lambda).$$

Then $K(\lambda) \otimes V \cong M_1 \oplus \cdots \oplus M_n$ where

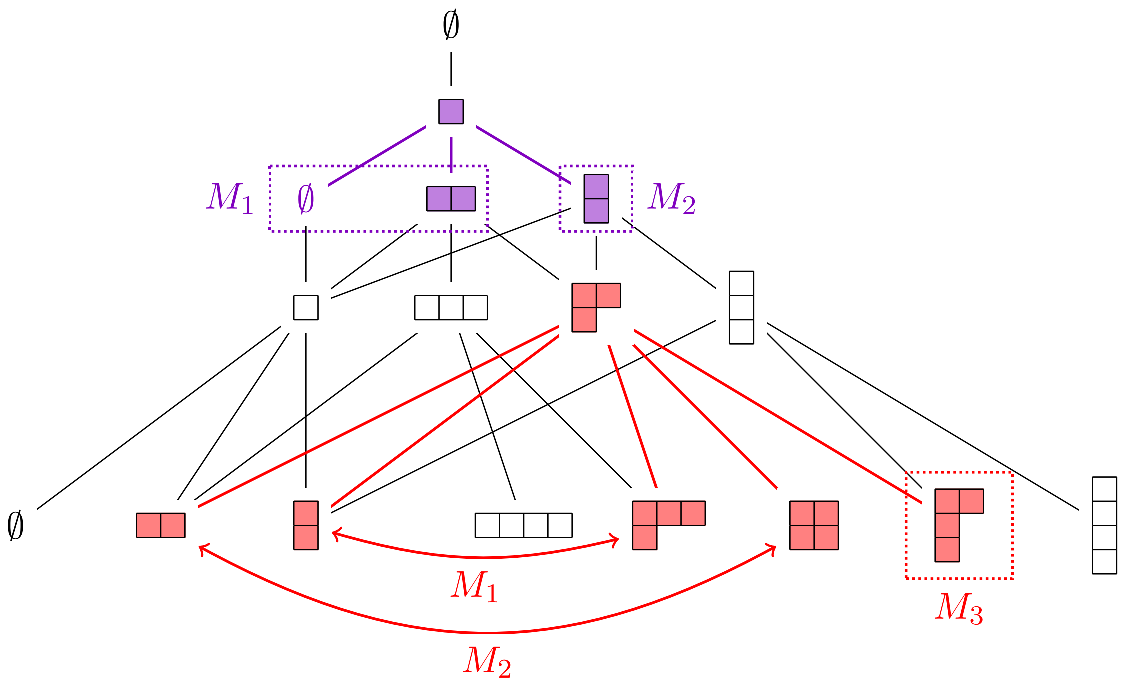
$$0 \to K(\lambda + \varepsilon_i) \to M_i \to K(\lambda - \varepsilon_i) \to 0,$$

whenever $\lambda \pm \varepsilon_i$ are dominant (add or remove a box to λ), or replace $K(*)$ with 0 whenever they're not (similar statement for $\tilde{K}$). (Proof uses eigenvalues of γ on $K(\lambda) \otimes V$ and $\tilde{K}(\lambda) \otimes V$, which are combinatorial data in terms of boxes added/removed.)

$K(\lambda) \otimes V \cong M_1 \oplus \cdots \oplus M_n$ where

$$0 \to K(\lambda + \varepsilon_i) \to M_i \to K(\lambda - \varepsilon_i) \to 0,$$

whenever $\lambda \pm \varepsilon_i$ are dominant (add or remove a box to λ), or replace $K(*)$ with 0 whenever they're not (similar statement for $\tilde{K}$).



Some more results: ([BDEHHILNSS-1&2])

- Presentation of the graded signed Brauer algebra and related algebras/categories.
  - Basis and spanning sets in terms of decorated diagrams.
  - Center given by a certain class of symmetric functions.
  - Filtrations and specializations similar to the classical cases.
- Action on tensor space and translation functors.
  - Translation functors given by actions on "weight diagrams" (akin to spin chain diagrams).
  - Algebraic structure: 0-Temperley-Lieb algebra.

With Martina Balagovic, Inna Entova-Aizenbud, Iva Halacheva, Johanna Hennig, Mee Seong Im, Gail Letzter, Emily Norton, Vera Serganova, and Catharina Stroppel:

[1] "Translation functors and Kazhdan-Lusztig multiplicities for the Lie superalgebra $\mathfrak{p}(n)$, Mathematical Research Letters Vol.26, no.3.

[2] "The affine VW supercategory", to appear in Selecta. arXiv:1801.04178