Combinatorics and representation theory of Temperley-Lieb algebras

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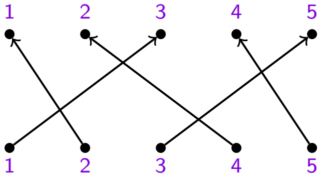
Motivating example: Schur-Weyl Duality

The symmetric group S_k (permutations) as diagrams:



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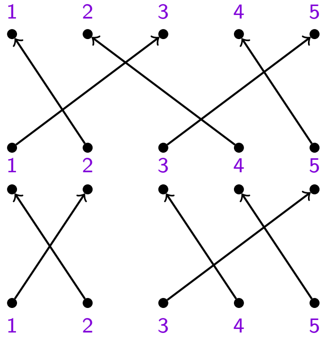

(with multiplication given by concatenation)

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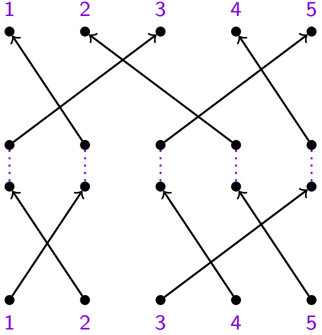

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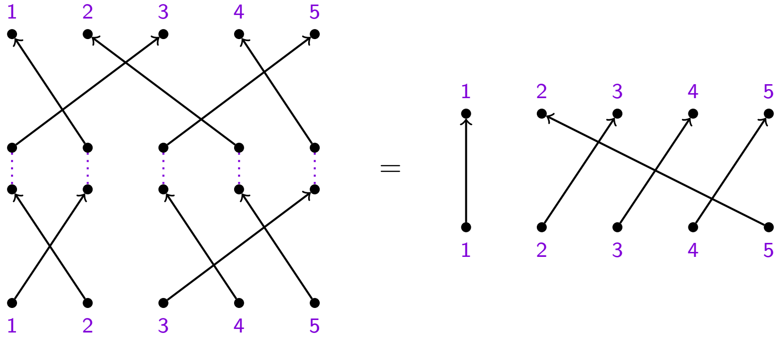

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Motivating example: Schur-Weyl Duality

$\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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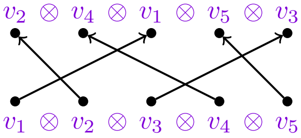
S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



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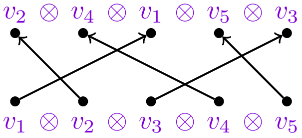
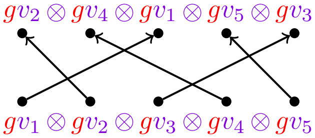
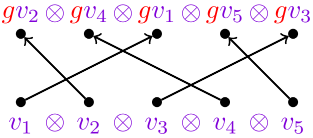


These actions commute!



vs.



Motivating example: Schur-Weyl Duality

Consider the representations induced by these commuting actions,

$$\pi : \mathbb{C}S_k \to \mathrm{End}((\mathbb{C}^n)^{\otimes k}) \quad \text{and} \quad \rho : \mathbb{C}\mathrm{GL}_n \to \mathrm{End}((\mathbb{C}^n)^{\otimes k}).$$

Thm. (Schur 1901)

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\text{(all linear maps that commute with } \mathrm{GL}_n\text{)}} = \underbrace{\pi(\mathbb{C}S_k)}_{\text{(img of } S_k \text{ action)}} \text{ and } \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\text{(img of } \mathrm{GL}_n \text{ action)}}.$$

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Powerful consequence: a duality between representations

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

where G^λ are distinct irreducible GL_n-modules,
S^λ are distinct irreducible S_k-modules.

Temperley-Lieb algebras

Caution! The representation

$$\pi : \mathbb{C}S_k \to \mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)$$

is not always injective!

Thm. $\ker(\pi) \neq 0$ when $n < k$.

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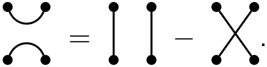

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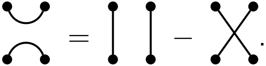
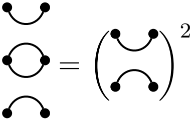


Then in $\mathbb{C}S_k$ (for general k),



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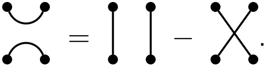

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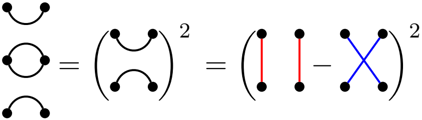

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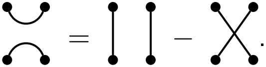

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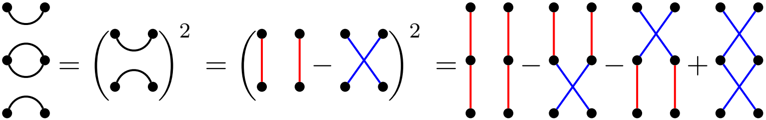

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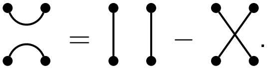

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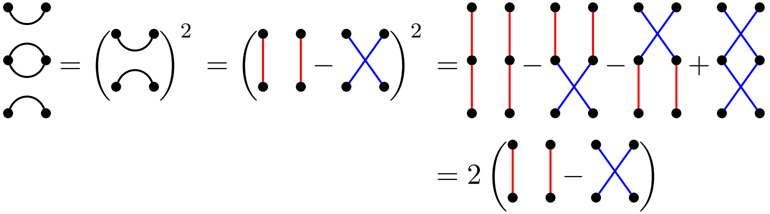

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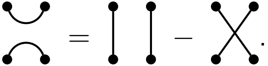

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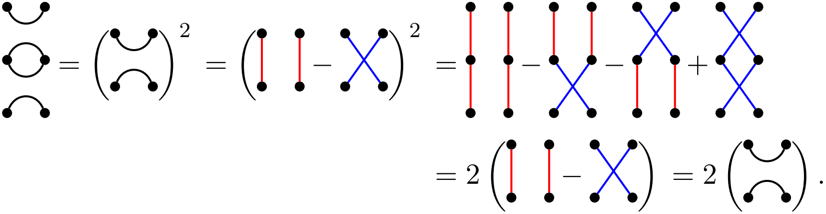

Because  *is* ($2\times$) *the projection onto the sign representation for* S_2*.*

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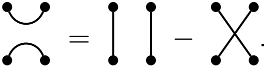
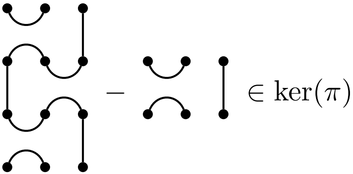
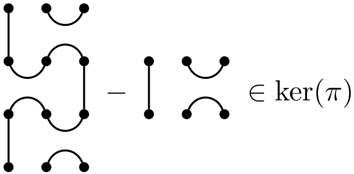


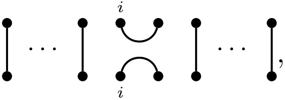
Only true for $n \leq 2$:





Temperley-Lieb algebras

Fix $\delta \in \mathbb{C}$. The *Temperley-Lieb algebra* TL_k is a diagram algebra generated over $\mathbb{C}$ by diagrams

$e_i =$ 
for $i = 1, \ldots, k-1$,

with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$e_i e_{i\pm 1} e_i = e_i$$
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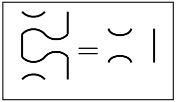
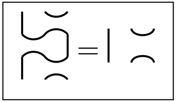
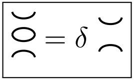

or


$$e_i^2 = \delta_i e_i.$$



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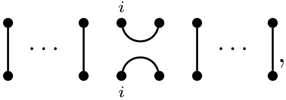
Basis: all non-crossing diagrams

$e_i =$  , for $i = 1, \ldots, k-1$,

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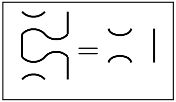

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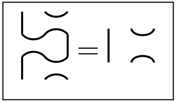

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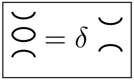

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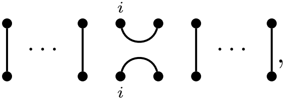
$e_i =$  for $i = 1, \ldots, k-1$,

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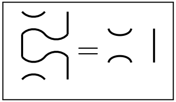
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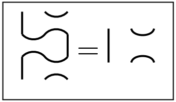
$(\star)$

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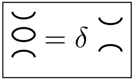

Thm. The quotient of $\mathbb{C}S_k$ by relations $(\star)$ factors through the representation

$$\pi : \mathbb{C}S_k \to \mathrm{End}\left(\left(\mathbb{C}^2\right)^{\otimes k}\right)$$

(i.e. when $\delta = 2$, TL_k centralizes the action of GL_2 on $\left(\mathbb{C}^2\right)^{\otimes k}$).

Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q\mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$ (deform the Lie algebra by a parameter q).

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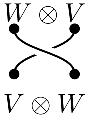
$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R}$ called an R-matrix that yields a map

$$\check{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$



that (1) satisfies braid relations, and
(2) commutes with the $\mathcal{U}$-action on $V \otimes W$
for any $\mathcal{U}$-module V.

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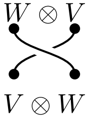

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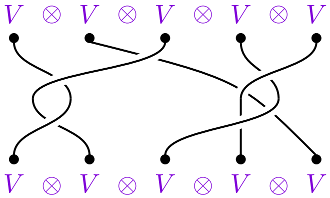
The braid group $\mathcal{B}_k$ shares a commuting action with $\mathcal{U}$ on $V^{\otimes k}$:



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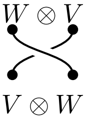

that (1) satisfies braid relations, and
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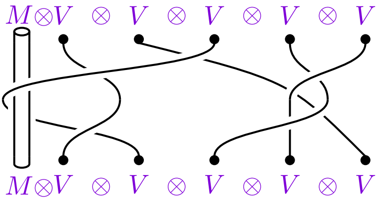
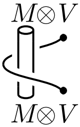
The one-pole/affine braid group $\mathcal{B}_k^{(1)}$ shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k}$:



Around the pole:



$= \check{R}_{MV}\check{R}_{VM}$

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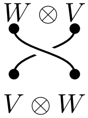

that (1) satisfies braid relations, and
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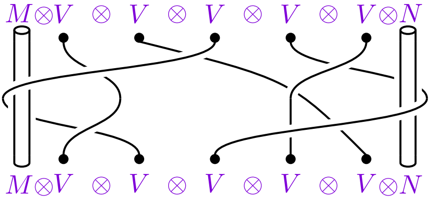
The two-pole braid group $\mathcal{B}_k^{(2)}$ shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k} \otimes N$:



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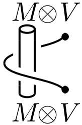
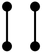


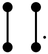
$$= \check{R}_{MV}\check{R}_{VM}$$

The type-A Hecke algebra is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations

 $= (q - q^{-1})$  $+$ . $\quad (*)$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The type-A Hecke algebra is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations

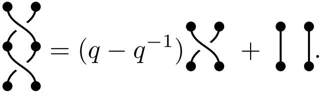
 $= (q - q^{-1})$  $+$ . $(*)$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The affine type-GL_k Hecke algebra is the quotient of the group algebra of the one-pole braid group $\mathcal{B}_k^{(1)}$ by relations $(*)$.

Thm. The action of $\mathbb{C}B_k^{(1)}$ on $M \otimes V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The type-A Hecke algebra is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



$(*)$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The affine type-GL_k Hecke algebra is the quotient of the group algebra of the one-pole braid group $\mathcal{B}_k^{(1)}$ by relations $(*)$.

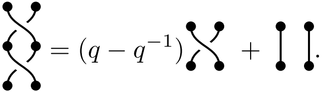
Thm. The action of $\mathbb{C}B_k^{(1)}$ on $M \otimes V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The affine type-C Hecke algebra is the quotient of the group algebra of the two-pole braid group $\mathcal{B}_k^{(2)}$ by relations $(*)$,

$(**)$

Thm. The action of $\mathbb{C}B_k^{(2)}$ on $M \otimes V^{\otimes k} \otimes N$ factors through the quotient by $(*)$ and $(**)$ when $V = \mathbb{C}^n$, M and N are "rectangular", and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

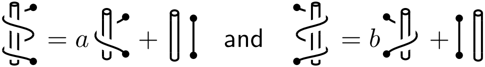
The type-A Hecke algebra is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



(∗)

The affine type-GL_k Hecke algebra is the quotient of the group algebra of the one-pole braid group $\mathcal{B}_k^{(1)}$ by relations (∗).

The affine type-C Hecke algebra is the quotient of the group algebra of the two-pole braid group $\mathcal{B}_k^{(2)}$ by relations (∗),

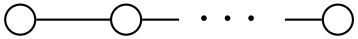


(∗∗)

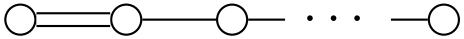
"Type what-now?"

Dynkin diagrams:

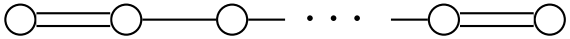
Type A



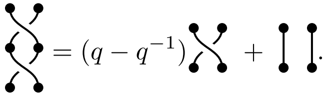
Affine Type GL



Affine Type C



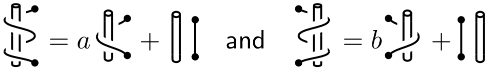
The type-A Hecke algebra is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



$(*)$

The affine type-GL_k Hecke algebra is the quotient of the group algebra of the one-pole braid group $\mathcal{B}_k^{(1)}$ by relations $(*)$.

The affine type-C Hecke algebra is the quotient of the group algebra of the two-pole braid group $\mathcal{B}_k^{(2)}$ by relations $(*)$,

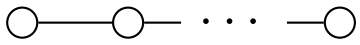


$(**)$

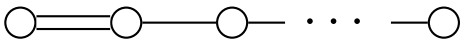
"Type what-now?"

Dynkin diagrams:

Type A

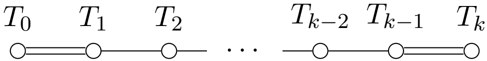


Affine Type GL

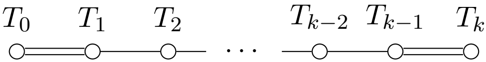


Affine Type C

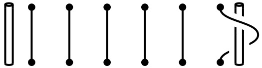
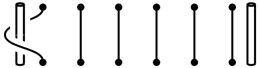
The two-pole/affine type-C braid group is the group $\mathcal{B}_k^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations



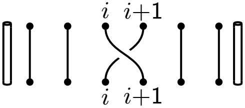
The two-pole/affine type-C braid group is the group $\mathcal{B}_k^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations



Pictorially, the generators of $\mathcal{B}_k^{(2)}$ are identified with the diagrams

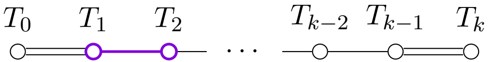
$T_k =$  , $T_0 =$  ,

and

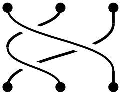
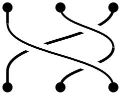
$T_i =$ 

for $i = 1, \ldots, k-1$.

The two-pole/affine type-C braid group is the group $\mathcal{B}_k^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations

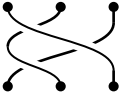
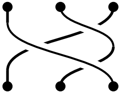


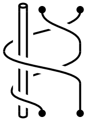
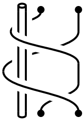
Pictorially,

$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$

The two-pole/affine type-C braid group is the group $\mathcal{B}_k^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations

Pictorially,

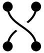
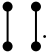
$T_iT_{i+1}T_i$ =  =  = $T_{i+1}T_iT_{i+1}$

$T_1T_0T_1T_0$ =  =  = $T_0T_1T_0T_1$

(similar picture for $T_kT_{k-1}T_kT_{k-1} = T_{k-1}T_kT_{k-1}T_k$)

Back to Temperley-Lieb algebras

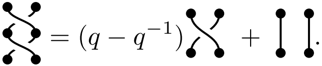
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations

 $= (q - q^{-1})$  $+$ . $(*)$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Back to Temperley-Lieb algebras

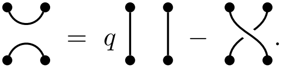
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



(∗)

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (∗) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

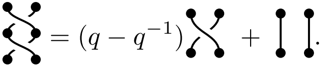
Case $n = 2$: Define



(◇)

Back to Temperley-Lieb algebras

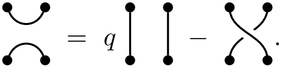
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



(∗)

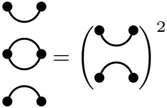
Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (∗) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define



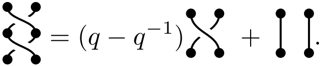
(◇)

Then in HA_k (for general k),



Back to Temperley-Lieb algebras

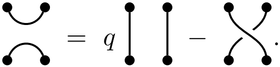
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



$(*)$

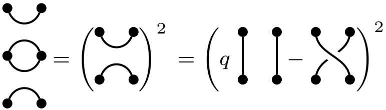
Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define



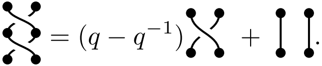
$(\diamond)$

Then in HA_k (for general k),



Back to Temperley-Lieb algebras

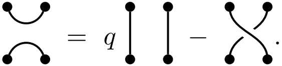
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



$(*)$

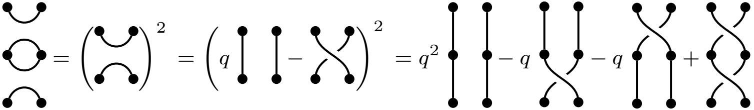
Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define



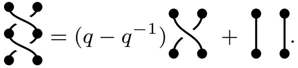
$(\diamond)$

Then in HA_k (for general k),



Back to Temperley-Lieb algebras

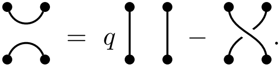
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



(∗)

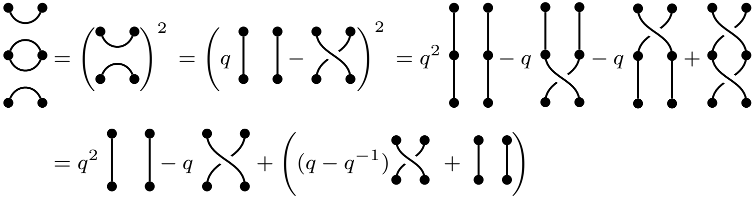
Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (∗) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define



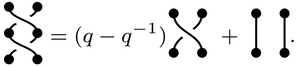
(◇)

Then in HA_k (for general k),



Back to Temperley-Lieb algebras

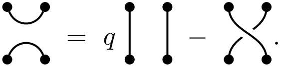
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



(∗)

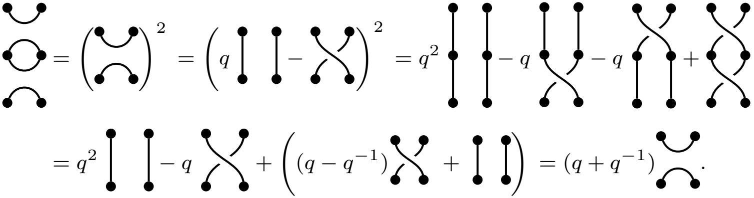
Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (∗) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define



(◇)

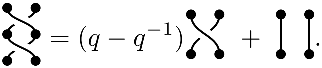
Then in HA_k (for general k),



Because  *is* $(q + q^{-1})\times$*(proj. onto sign representation for* HA_2*).*

Back to Temperley-Lieb algebras

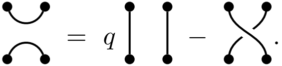
The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations



(∗)

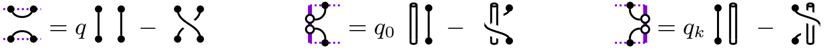
Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (∗) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

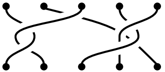
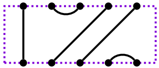
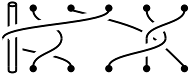
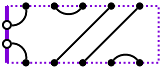
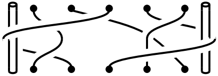
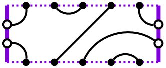
Case $n = 2$: Define



(◇)

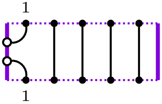
Thm. Using the identification in (◇), the action of HA_k on $(\mathbb{C}^2)^{\otimes k}$ factors through the Temperley-Lieb quotient when $\delta = q + q^{-1} = [2]_q$, i.e. TL_k centralizes $\mathcal{U}_q\mathfrak{gl}_2$ and $\mathcal{U}_q\mathfrak{sl}_2$ in $\mathrm{End}((\mathbb{C}^2)^{\otimes k})$ when $\bigcirc = [2]_q$.

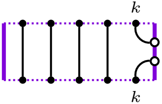


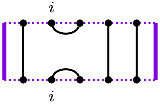
tensor space	centralizer of $\mathcal{U}_q\mathfrak{g}$	centralizer of $\mathcal{U}_q\mathfrak{gl}_n$	centralizer of $\mathcal{U}_q\mathfrak{gl}_2$
$V^{\otimes k}$	Braids on k strands 	Type-A Hecke (twist relations)	Temperley-Lieb 
$M \otimes V^{\otimes k}$	One-pole braids 	Affine type-GL Hecke (twist relations)	1-boundary TL 
$M \otimes V^{\otimes k} \otimes N$	Two-pole braids 	Affine type-C Hecke (twist & wrap relations)	2-boundary TL 

Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $\delta, \delta_0, \delta_k \in \mathbb{C}$. The *two-boundary Temperley-Lieb algebra* $TL_k^{(2)}$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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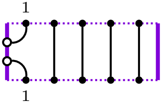
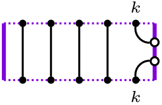
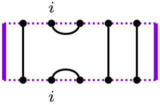
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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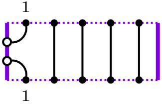
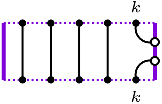
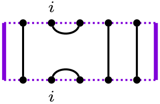
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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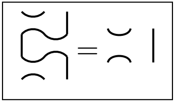
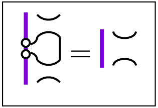
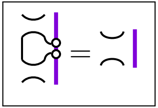
 or  or 

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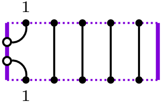
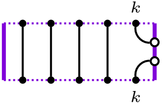
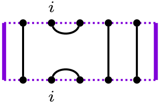
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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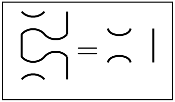
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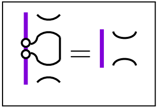
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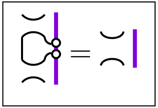

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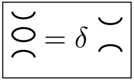

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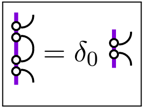

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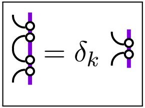

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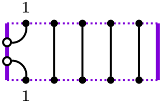
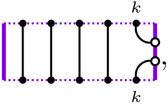
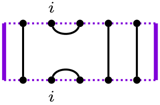
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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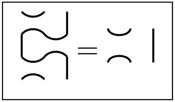
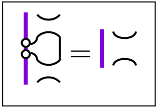
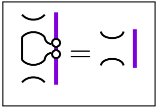
 or  or 

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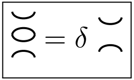
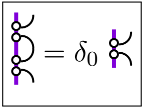
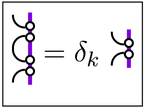
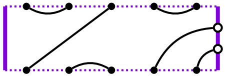


or 

or 

(Side loops are resolved with a 1 or a δ_i depending on whether there are an even or odd number of connections below their lowest point.)

Diagram multiplication:



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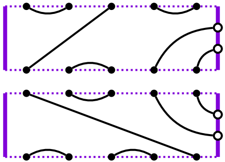

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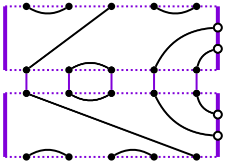

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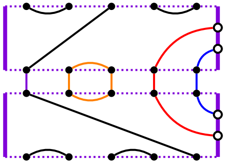

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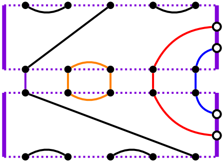

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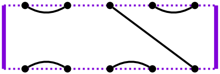

$* \delta * 1 * \delta_k$

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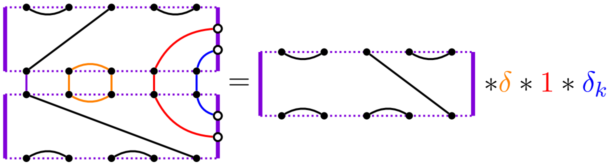

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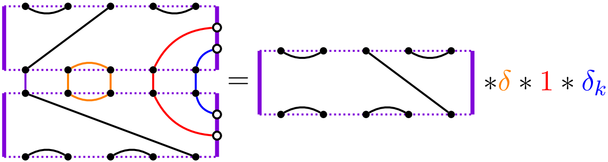

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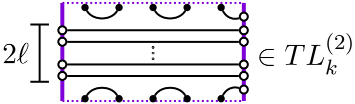

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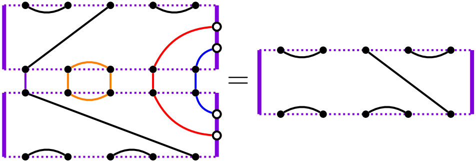

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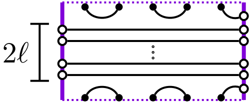

$\in TL_k^{(2)}$

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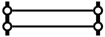
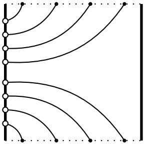


$= z$

Representation theory of $TL_k^{(2)}$: action on diagrams

$d =$ 

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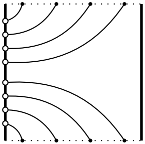
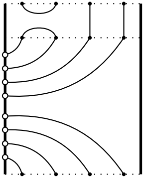


$e_1 d =$



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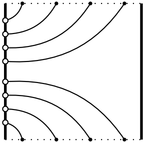
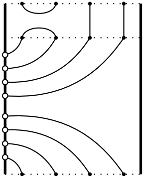
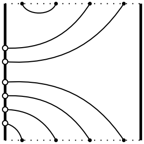
$d =$ 

$e_1 d =$  $= \delta_0$ 

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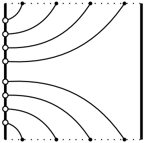
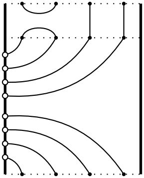
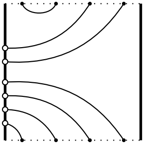
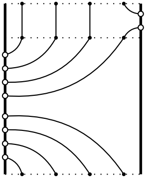
$d =$ 

$e_1 d =$  $= \delta_0$ 

$e_4 d =$ 

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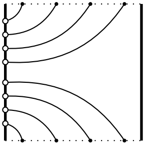
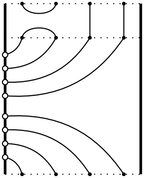
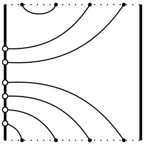
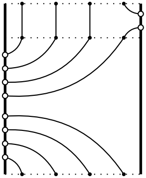
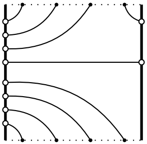
$d =$ 

$e_1 d =$  $= \delta_0$ 

$e_4 d =$  $=$ 

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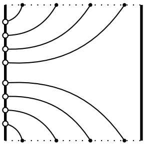

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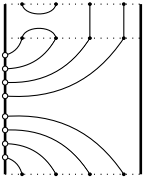

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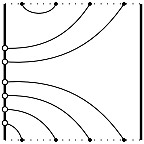

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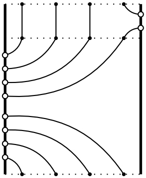
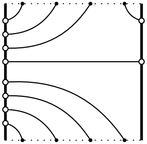


$=$



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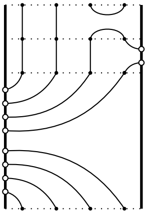
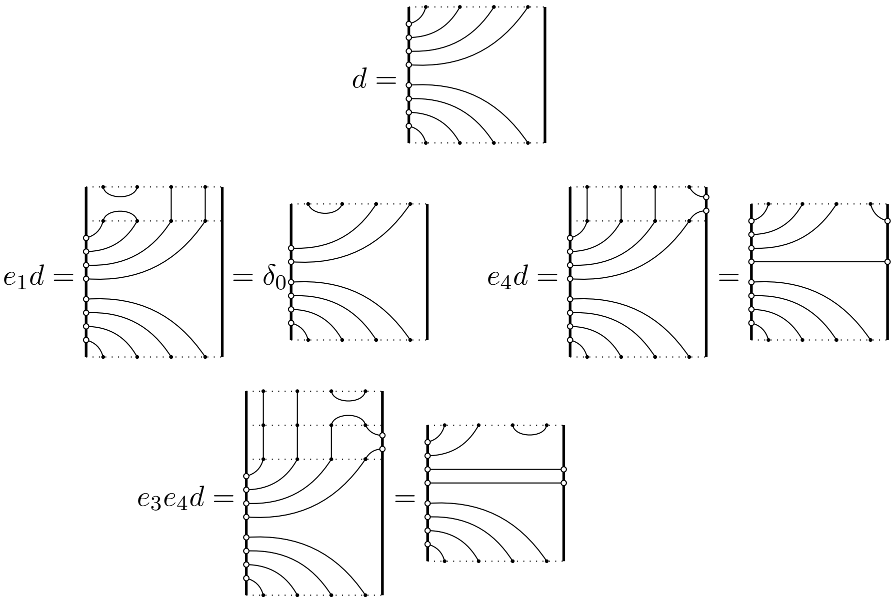
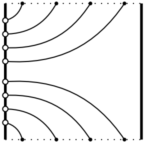
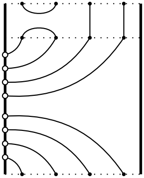
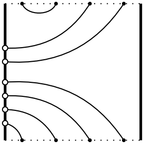


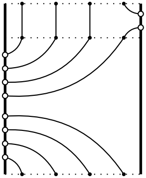
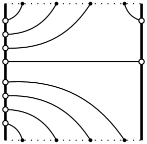
Representation theory of $TL_k^{(2)}$: action on diagrams

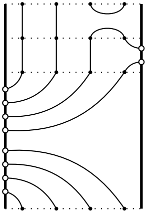
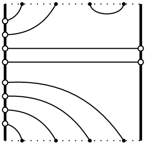
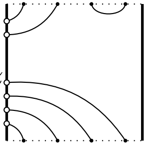


Representation theory of $TL_k^{(2)}$: action on diagrams

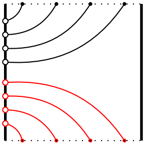
$d =$ 

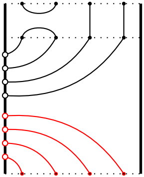
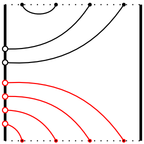
$e_1 d =$  $= \delta_0$ 

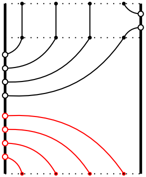
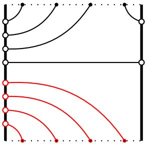
$e_4 d =$  $=$ 

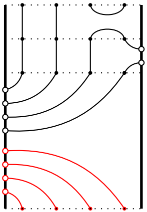
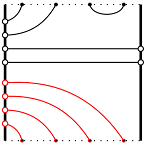
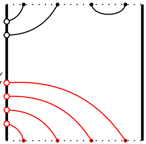
$e_3 e_4 d =$  $=$  $= z$ 

Representation theory of $TL_k^{(2)}$: action on diagrams

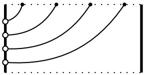
$d =$ 

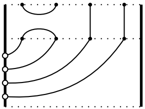
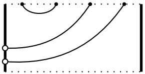
$e_1 d =$  $= \delta_0$ 

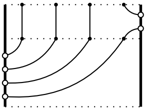
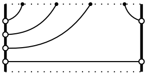
$e_4 d =$  $=$ 

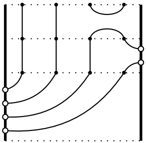
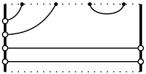
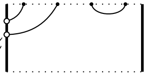
$e_3 e_4 d =$  $=$  $= z$ 

Representation theory of $TL_k^{(2)}$: half diagrams

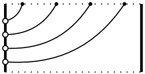
$d =$ 

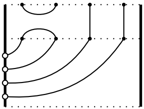
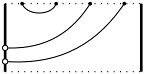
$e_1 d =$  $= \delta_0$ 

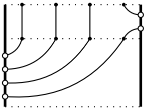
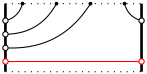
$e_4 d =$  $=$ 

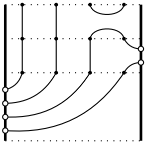
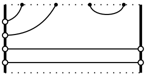
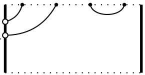
$e_3 e_4 d =$  $=$  $= z$ 

Representation theory of $TL_k^{(2)}$: half diagrams

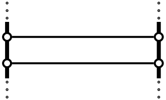
$d =$ 

$e_1 d =$  $= \delta_0$ 

$e_4 d =$  $=$ 

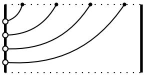
$e_3 e_4 d =$  $=$  $= z$ 

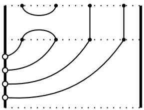
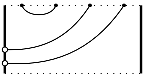
You can tell when to use

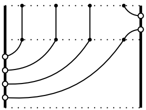
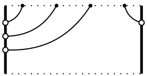
 $= z$

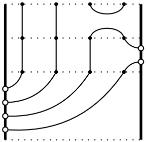
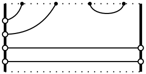
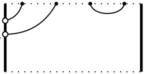
or not by the parity of connections to the left/right walls.

Representation theory of $TL_k^{(2)}$: half diagrams

$d =$ 

$e_1 d =$  $= \delta_0$ 

$e_4 d =$  $=$ 

$e_3 e_4 d =$  $=$  $= z$ 

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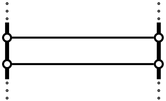
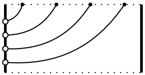
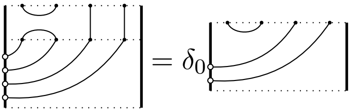
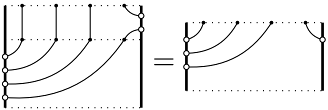
 $= z$

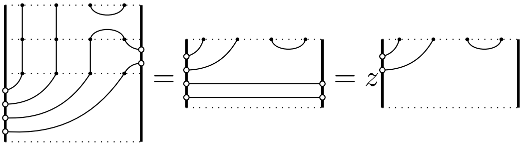
or not by the parity of connections to the left/right walls.

Representation theory of $TL_k^{(2)}$: half diagrams

$d =$ 

$e_1 d =$ 

$e_4 d =$ 

$e_3 e_4 d =$ 

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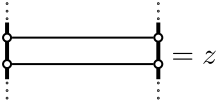

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Generic module:

(act by e_i, don't make loops)

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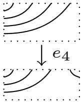

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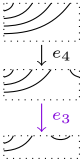
Red arrows indicate coef of z.



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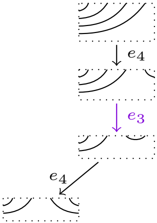

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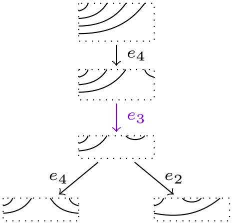

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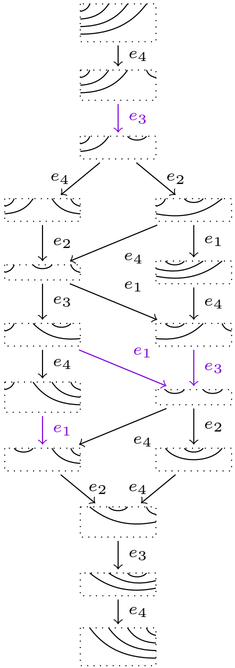

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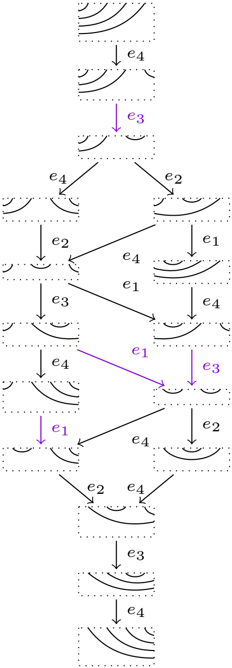
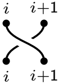


For what z does this module split?

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

subject to relations 

.

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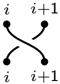
$T_k =$ , $T_0 =$  and $T_i =$ 

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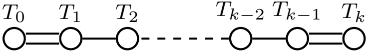
subject to relations 

.

(2) Fix constants $q_0, q_k, q \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - q_0)(T_0 + q_0^{-1}) = 0, \quad (T_k - q_k)(T_k + q_k^{-1}) = 0$$

$$\text{and} \quad (T_i - q)(T_i + q^{-1}) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

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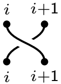
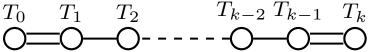
$T_k =$ , $T_0 =$  and $T_i =$ 

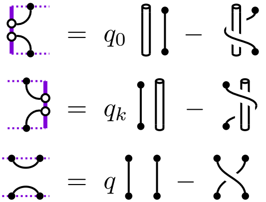
for $1 \le i \le k-1$,

subject to relations 

.

(2) Fix constants $q_0, q_k, q = q_1 = q_2 = \cdots = q_{k-1} \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - q_i^{1/2})(T_i + q_i^{-1/2}) = 0$.

(3) Set



$(e_0 = q_0 - T_0)$

$(e_k = q_k - T_k)$

$(e_i = q - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).

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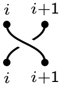
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

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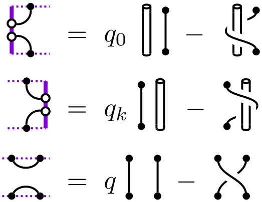
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - q_i^{1/2})(T_i + q_i^{-1/2}) = 0$.

(3) Set



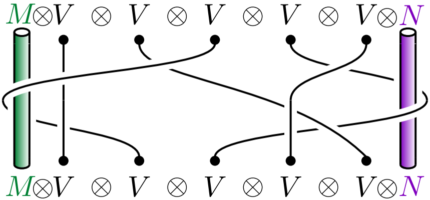
$(e_0 = q_0 - T_0)$

$(e_k = q_k - T_k)$

$(e_i = q - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).
The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm 1} e_i = e_i$ for $i = 1, \ldots, k-1$.

One advantage of using braids:



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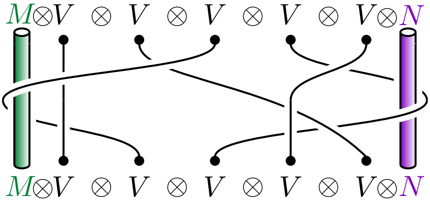
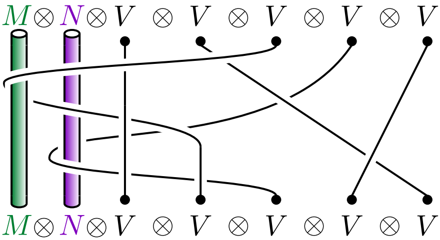


Move both poles to the left $\downarrow$



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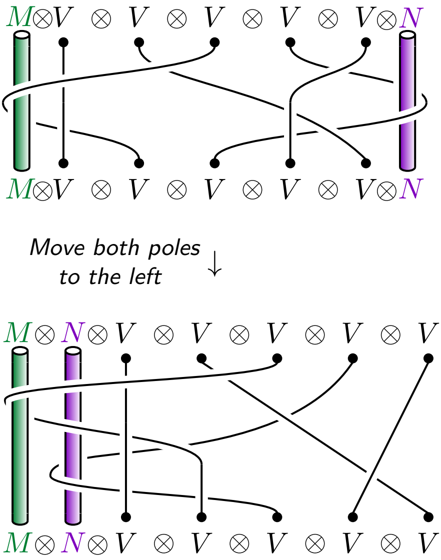
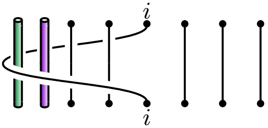


Jucys-Murphy elements:

$Z_i =$ 

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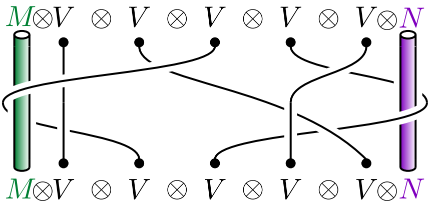
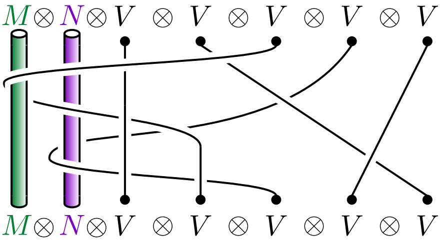


Move both poles to the left ↓



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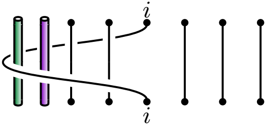
$Z_i =$ 

- Pairwise commute

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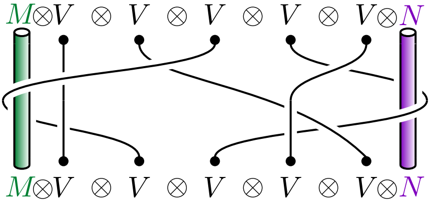

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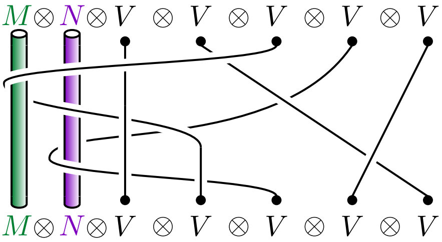

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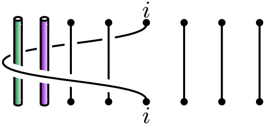

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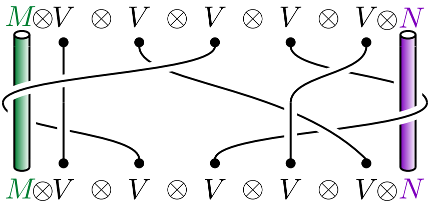

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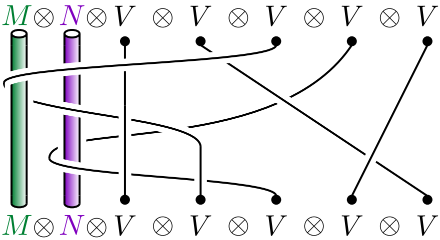

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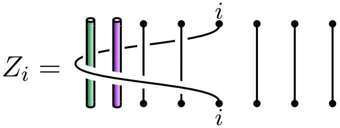

- Pairwise commute
- $Z(\mathcal{H}_k)$ is (type-C) symmetric Laurent polynomials in Z_i's
- Central characters indexed by $\mathbf{c} \in \mathbb{C}^k$ (modulo signed permutations)

Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$, where

$\mathbf{c}$ is a point in the fundamental chamber of the (finite) type C hyperplane system, and

J is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting $\mathbf{c}$

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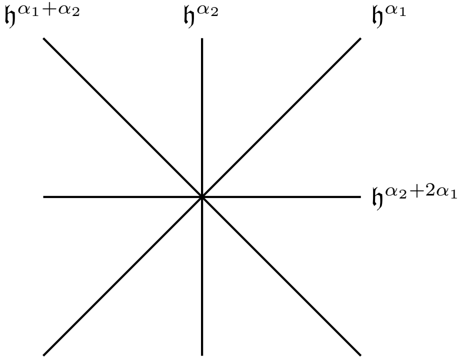

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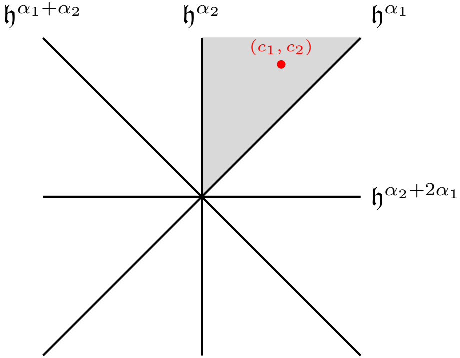

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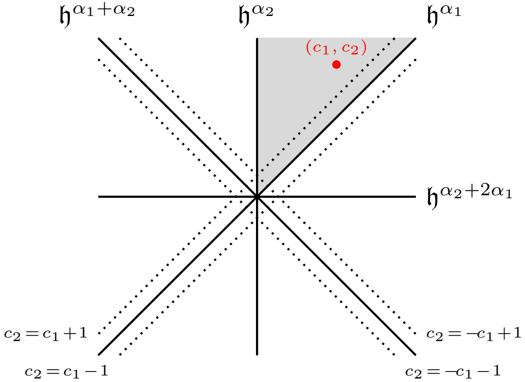

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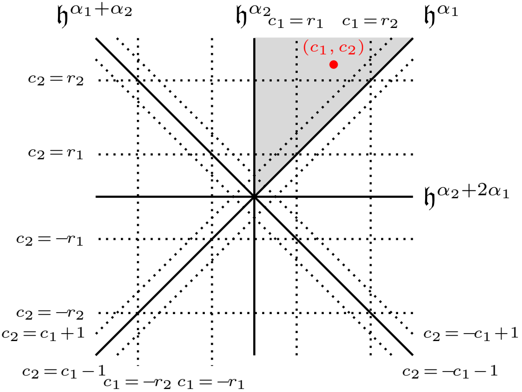

The r_is depend on $\mathcal{H}_k$'s parameters q_0 and q_k: $r_1 = \log_q(q_0/q_k)$, $r_2 = \log_q(q_0 q_k)$.

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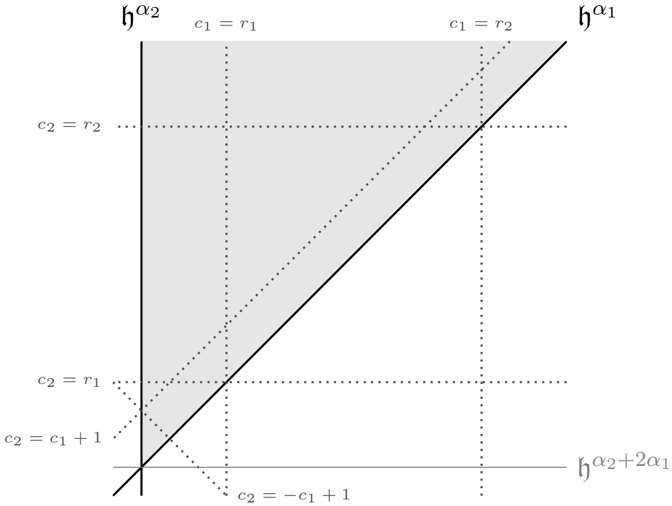

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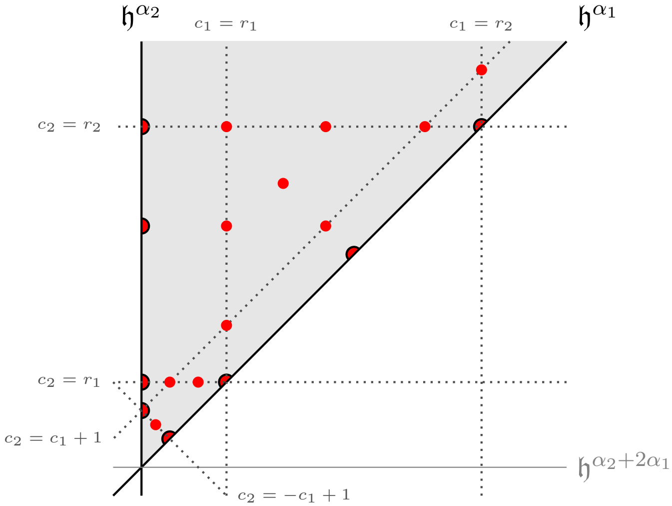

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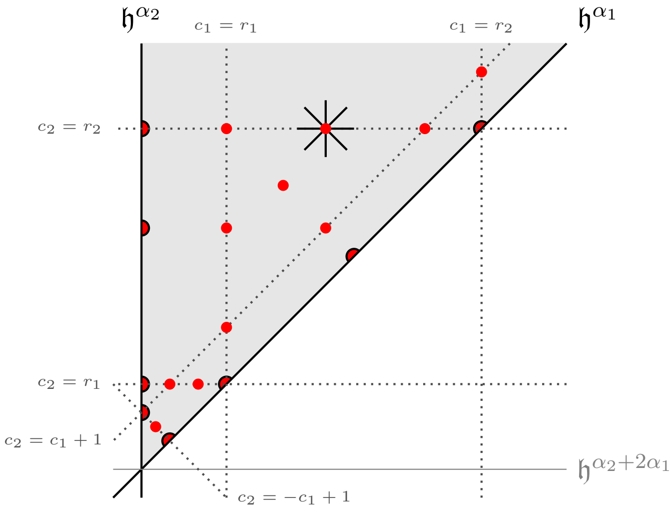

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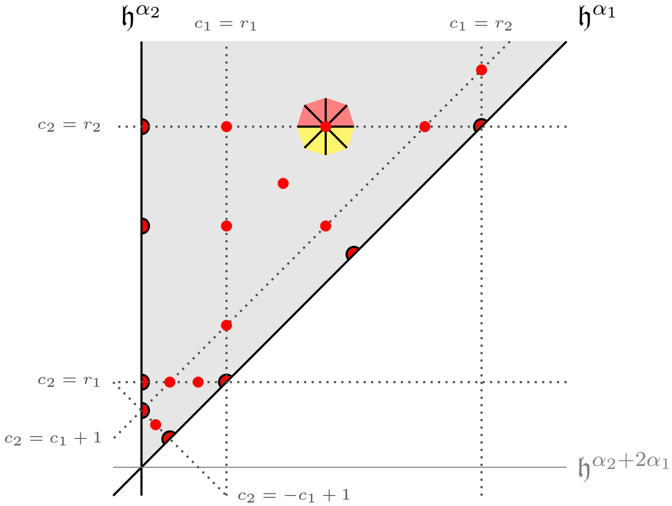

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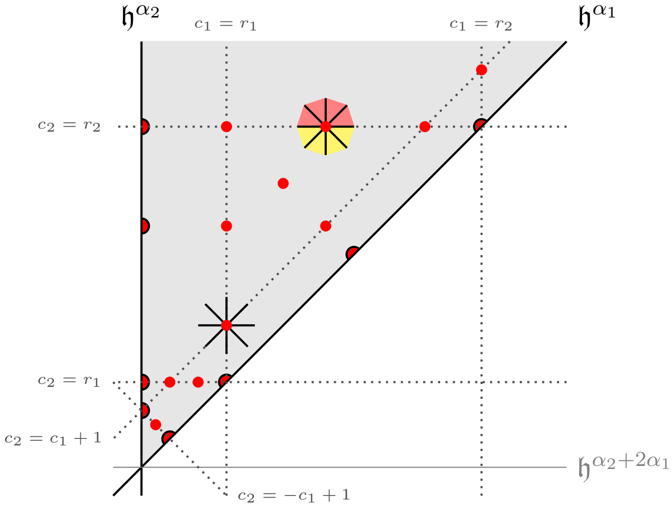

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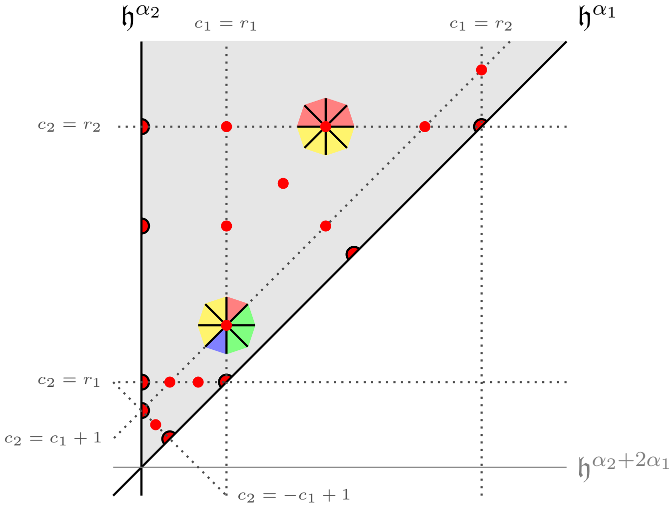

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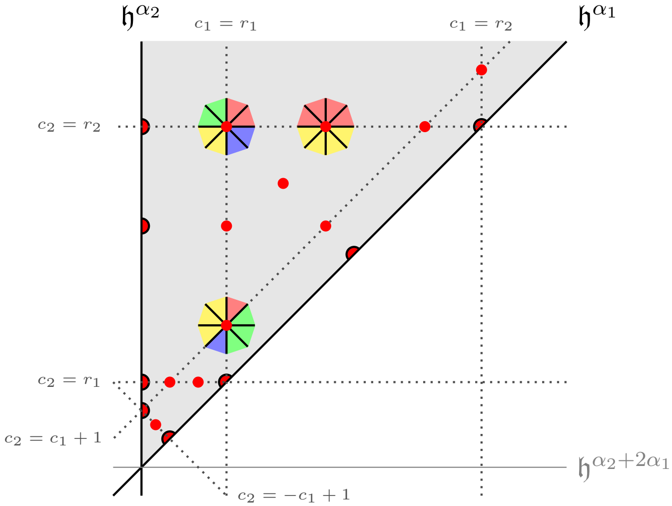

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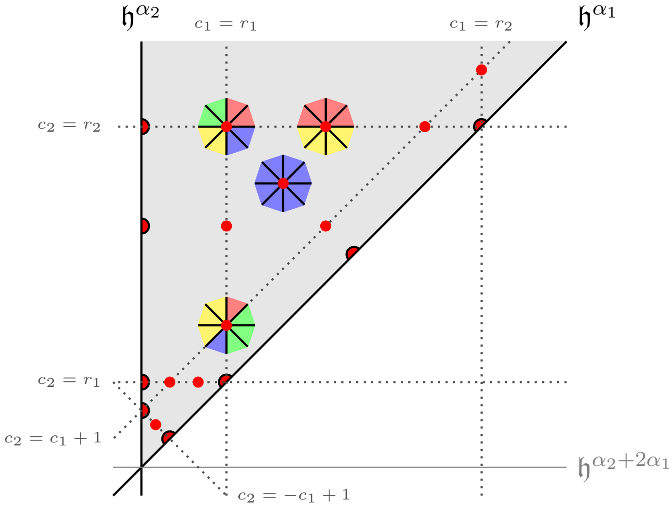

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A little more detail

- J is determined by a set of positive roots (corresp. to hyperplanes).
- For "nice" characters, there is a bijection between alcoves and marked type-C generalized Young tableaux.
- "Intertwining operators" τ_i move between alcoves;
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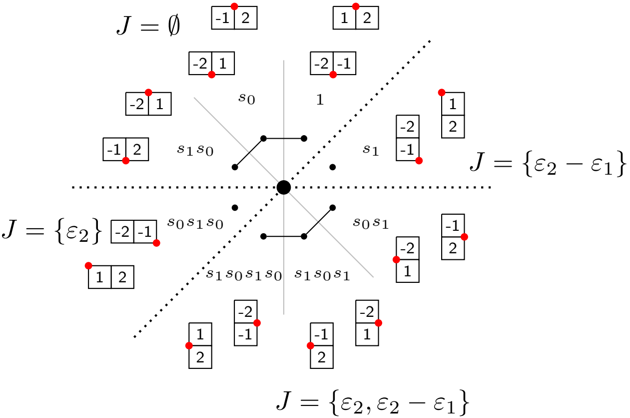

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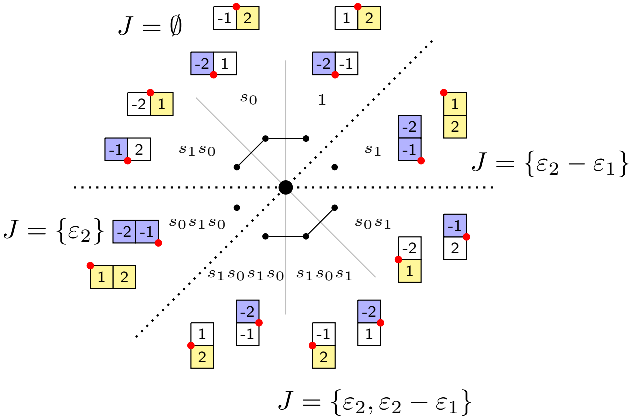

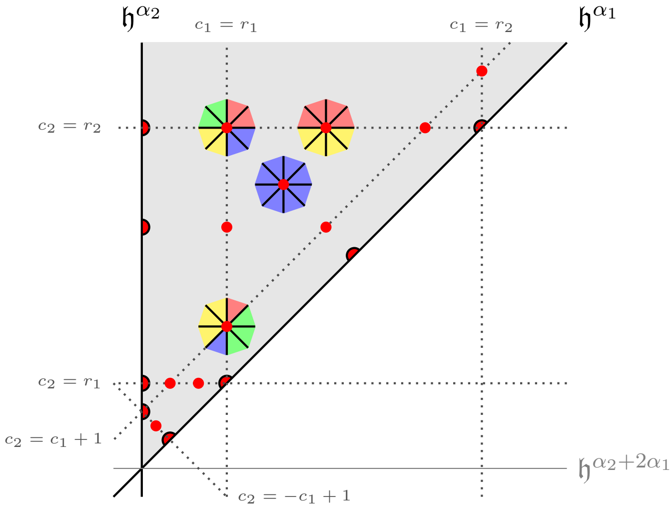

Thm. (D.-Ram)

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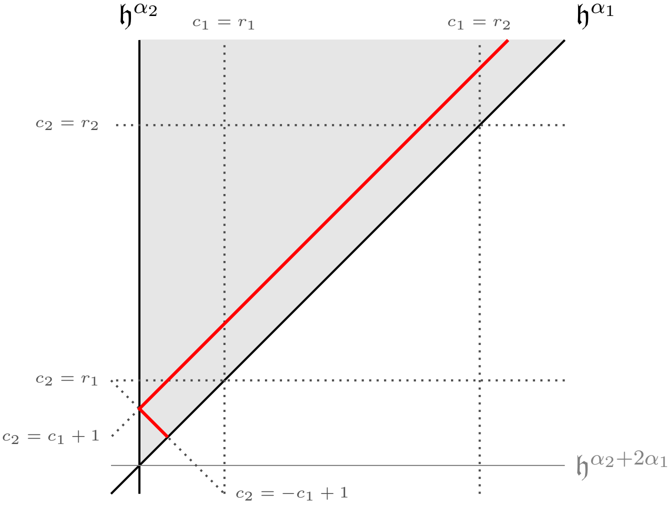

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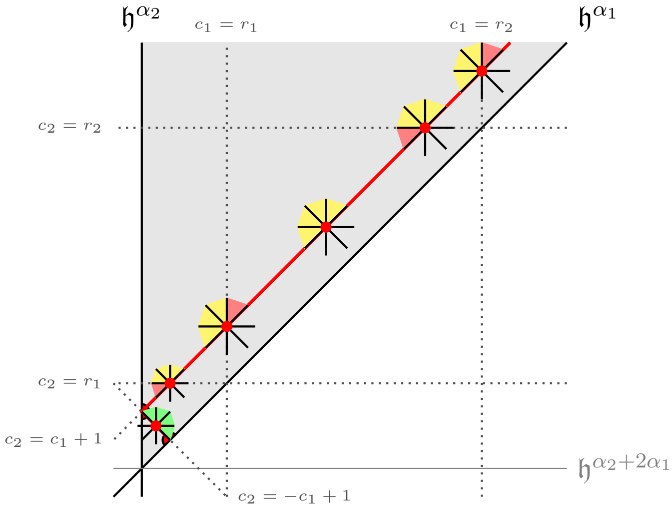

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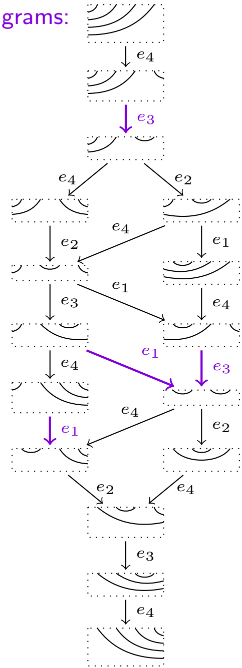
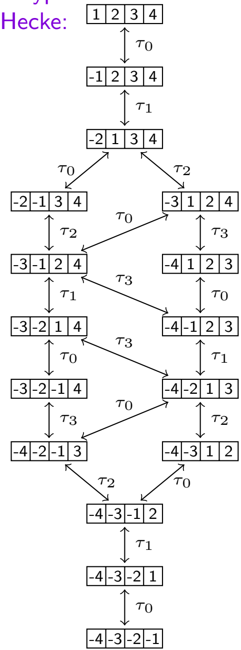
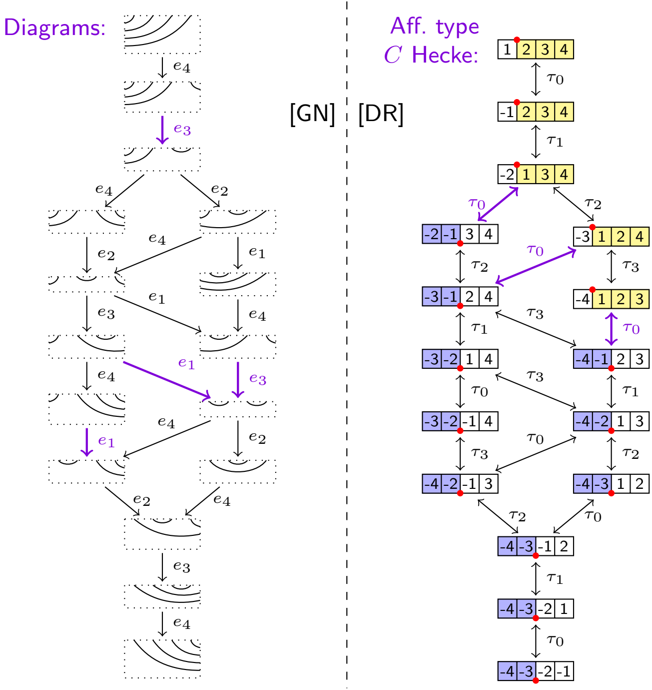


[GN]

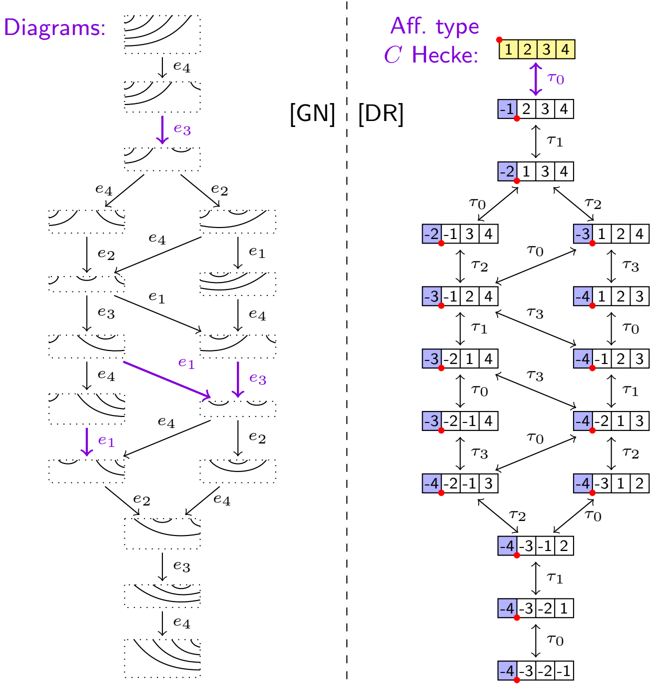
[DR]

Aff. type C Hecke:





Diagrams:
e_4
e_3
e_4 e_2
e_2 e_4 e_1
e_3 e_1 e_4
e_4 e_1 e_3
e_1 e_4 e_2
e_2 e_4
e_3
e_4
[GN]
[DR]
Aff. type
C Hecke:
1 2 3 4
τ_0
-1 2 3 4
τ_1
-2 1 3 4
τ_0 τ_2
-2 -1 3 4
-3 1 2 4
τ_2 τ_0 τ_3
-3 -1 2 4
-4 1 2 3
τ_1 τ_3 τ_0
-3 -2 1 4
-4 -1 2 3
τ_0 τ_3 τ_1
-3 -2 -1 4
-4 -2 1 3
τ_3 τ_0 τ_2
-4 -2 -1 3
-4 -3 1 2
τ_2 τ_0
-4 -3 -1 2
τ_1
-4 -3 -2 1
τ_0
-4 -3 -2 -1



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[GN]
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e_4 e_3 e_4 e_2 e_2 e_4 e_1 e_3 e_1 e_4 e_4 e_1 e_3 e_4 e_1 e_2 e_2 e_4 e_3 e_4
τ_0 τ_1 τ_0 τ_2 τ_2 τ_0 τ_3 τ_1 τ_3 τ_0 τ_0 τ_3 τ_1 τ_3 τ_0 τ_2 τ_2 τ_0 τ_1 τ_0
1 2 3 4
-1 2 3 4
-2 1 3 4
-2 -1 3 4
-3 1 2 4
-3 -1 2 4
-4 1 2 3
-3 -2 1 4
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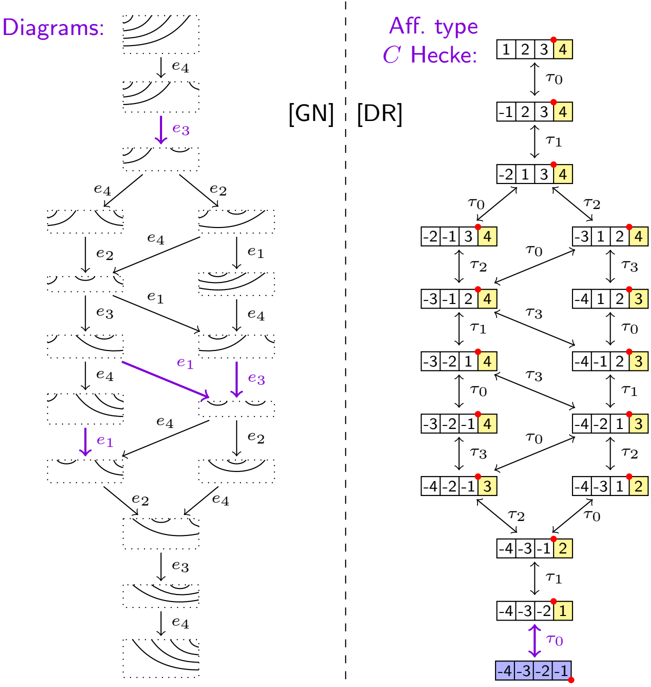
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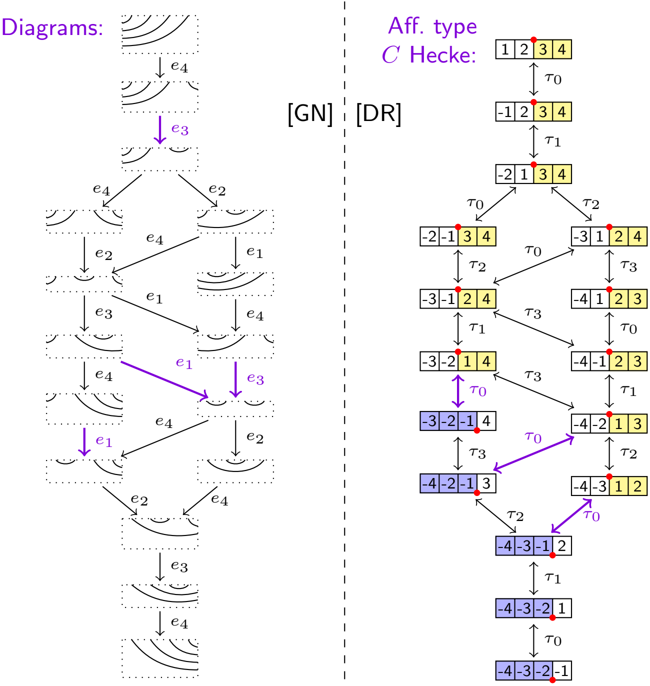
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See also...

Specific:

[DR] Daugherty and Ram, "Two boundary Hecke Algebras and combinatorics of type C", arXiv:1804.10296.

[GN] de Gier and Nichols, "The two-boundary Temperley-Lieb algebra", J. Algebra 321 (2009), no. 4, 1132–1167.

General:

[BR] Barcelo and Ram, "Combinatorial representation theory," New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), 23–90, Math. Sci. Res. Inst. Publ., 38, Cambridge Univ. Press, Cambridge, 1999.

https://zdaugherty.ccnysites.cuny.edu

Thanks!