

Combinatorics and representation theory of Temperley-Lieb algebras

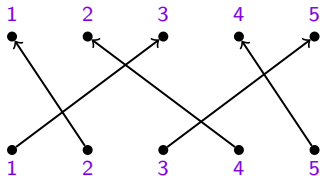
Zajj Daugherty

The City College of New York
& CUNY Graduate Center

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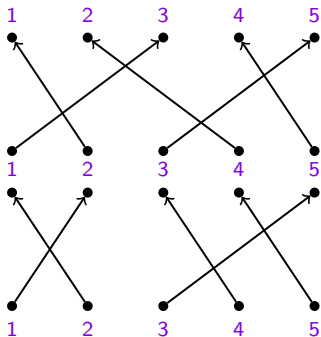
Motivating example: Schur-Weyl Duality

The symmetric group S_k (permutations) as diagrams:



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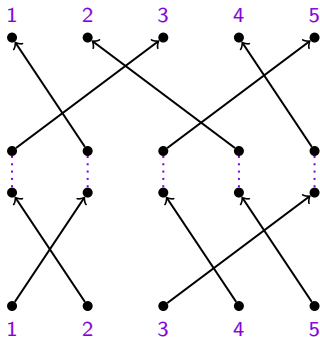
The symmetric group S_k (permutations) as diagrams:



(with multiplication given by concatenation)

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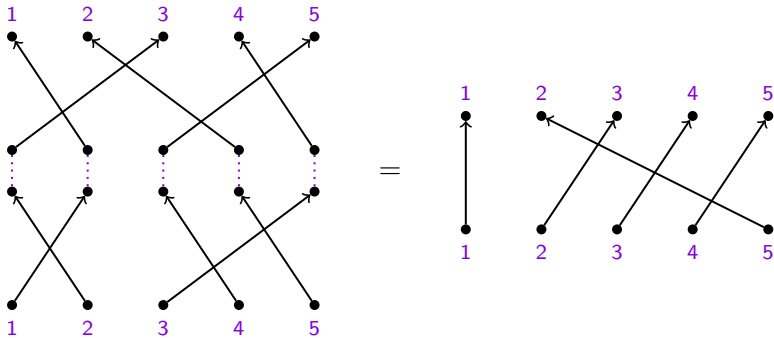
The symmetric group S_k (permutations) as diagrams:



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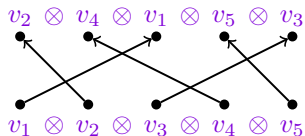
$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.

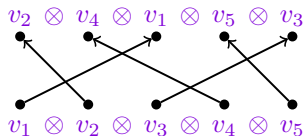


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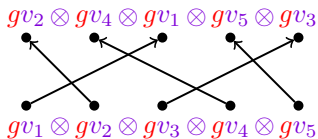
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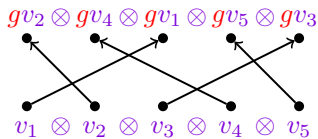
S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



These actions commute!



vs.



Motivating example: Schur-Weyl Duality

Consider the representations induced by these commuting actions,

$$\pi : \mathbb{C}S_k \rightarrow \text{End}((\mathbb{C}^n)^{\otimes k}) \quad \text{and} \quad \rho : \mathbb{C}GL_n \rightarrow \text{End}((\mathbb{C}^n)^{\otimes k}).$$

Thm. (Schur 1901)

$$\underbrace{\text{End}_{GL_n} \left((\mathbb{C}^n)^{\otimes k} \right)}_{\substack{\text{(all linear maps that} \\ \text{commute with } GL_n)}} = \underbrace{\pi(\mathbb{C}S_k)}_{\substack{\text{(img of } S_k \\ \text{action)}}} \quad \text{and} \quad \text{End}_{S_k} \left((\mathbb{C}^n)^{\otimes k} \right) = \underbrace{\rho(\mathbb{C}GL_n)}_{\substack{\text{(img of } GL_n \\ \text{action)}}}.$$

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Powerful consequence: a duality between representations

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} G^\lambda \otimes S^\lambda \quad \text{as a } GL_n\text{-}S_k \text{ bimodule,}$$

where G^λ are distinct irreducible GL_n -modules,
 S^λ are distinct irreducible S_k -modules.

Temperley-Lieb algebras

Caution! The representation

$$\pi : \mathbb{C}S_k \rightarrow \text{End} \left((\mathbb{C}^n)^{\otimes k} \right)$$

is not always injective!

Thm. $\ker(\pi) \neq 0$ when $n < k$.

Temperley-Lieb algebras

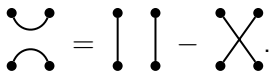
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Case $n = 2$: Define


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Temperley-Lieb algebras

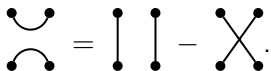
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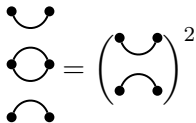
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Then in $\mathbb{C}S_k$ (for general k),


$$\begin{array}{c} \bullet \quad \bullet \\ \smile \\ \bullet \quad \bullet \\ \bullet \quad \bullet \\ \smile \\ \bullet \quad \bullet \end{array} = \left(\begin{array}{c} \bullet \quad \bullet \\ \smile \\ \bullet \quad \bullet \\ \bullet \quad \bullet \\ \smile \\ \bullet \quad \bullet \end{array} \right)^2$$

Temperley-Lieb algebras

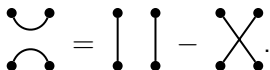
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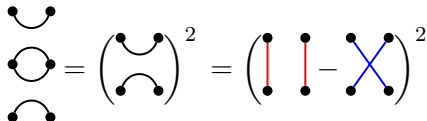
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Case $n = 2$: Define

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$$= 2 \left(\begin{array}{c} \circ \quad \circ \\ | \quad | \\ \circ \quad \circ \\ | \quad | \\ \circ \quad \circ \end{array} - \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \circ \quad \circ \end{array} \right) = 2 \left(\begin{array}{c} \circ \quad \circ \\ \text{---} \\ \circ \quad \circ \\ \text{---} \\ \circ \quad \circ \end{array} \right).$$

Because $\begin{array}{c} \circ \quad \circ \\ \text{---} \\ \circ \quad \circ \end{array}$ is $(2 \times)$ the projection onto the sign representation for S_2 .

Temperley-Lieb algebras

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Thm. $\ker(\pi) \neq 0$ when $n < k$.

Case $n = 2$: Define

$$\text{cup} = \text{parallel lines} - \text{crossing}$$

Only true for $n \leq 2$:

$$\text{Diagram 1} - \text{Diagram 2} \in \ker(\pi)$$

$$\text{Diagram 3} - \text{Diagram 4} \in \ker(\pi)$$

Temperley-Lieb algebras

Fix $\delta \in \mathbb{C}$. The *Temperley-Lieb algebra* TL_k is a diagram algebra generated over \mathbb{C} by diagrams

$$e_i = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdots \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ \cup \\ \bullet \\ \cap \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdots \begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \quad \text{for } i = 1, \dots, k-1,$$

with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$e_i e_{i \pm 1} e_i = e_i$$

for $1 \leq i \leq k-1$,

$$\boxed{\begin{array}{c} \cup \\ | \\ \cap \\ | \\ \cup \end{array}} = \begin{array}{c} \cup \\ | \\ \cup \end{array} \quad \text{or} \quad \boxed{\begin{array}{c} \cup \\ | \\ \cup \\ | \\ \cup \end{array}} = \begin{array}{c} | \\ \cup \end{array}$$

$$e_i^2 = \delta e_i.$$

$$\boxed{\begin{array}{c} \cup \\ \cap \\ \cup \end{array}} = \delta \begin{array}{c} \cup \\ \cup \end{array}$$

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Basis: all non-crossing diagrams

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$$\boxed{\begin{array}{c} \curvearrowright \\ | \\ \curvearrowleft \\ | \\ \curvearrowright \\ | \\ \curvearrowleft \end{array} = \begin{array}{c} \curvearrowright \\ | \\ \curvearrowleft \end{array}}$$

or

$$\boxed{\begin{array}{c} \curvearrowleft \\ | \\ \curvearrowright \\ | \\ \curvearrowleft \\ | \\ \curvearrowright \end{array} = \begin{array}{c} \curvearrowleft \\ | \\ \curvearrowright \end{array}}$$

(*)

$$e_i^2 = \delta e_i.$$

$$\boxed{\begin{array}{c} \curvearrowright \\ | \\ \circ \\ | \\ \curvearrowleft \end{array} = \delta \begin{array}{c} \curvearrowright \\ | \\ \curvearrowleft \end{array}}$$

Thm. The quotient of $\mathbb{C}S_k$ by relations (*) factors through the representation

$$\pi : \mathbb{C}S_k \rightarrow \text{End} \left((\mathbb{C}^2)^{\otimes k} \right)$$


(i.e. when $\delta = 2$, TL_k centralizes the action of GL_2 on $(\mathbb{C}^2)^{\otimes k}$).

Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra \mathfrak{g} (deform the Lie algebra by a parameter q).

Quantum groups and braids

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$$\check{\mathcal{R}}_{VW}: V \otimes W \longrightarrow W \otimes V$$



that

- (1) satisfies braid relations, and
- (2) commutes with the \mathcal{U} -action on $V \otimes W$

for any \mathcal{U} -module V .

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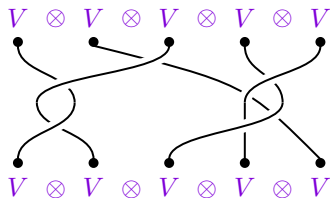
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The braid group \mathcal{B}_k shares a commuting action with \mathcal{U} on $V^{\otimes k}$:



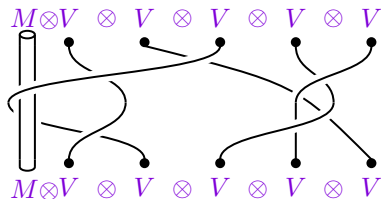
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The **one-pole/affine** braid group $\mathcal{B}_k^{(1)}$ shares a commuting action with \mathcal{U} on $M \otimes V^{\otimes k}$:




Around the pole:



$$= \check{R}_{MV} \check{R}_{VM}$$

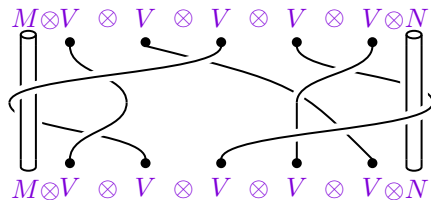
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$$\check{\mathcal{R}}_{VW}: V \otimes W \longrightarrow W \otimes V$$


that (1) satisfies braid relations, and
 (2) commutes with the \mathcal{U} -action on $V \otimes W$
 for any \mathcal{U} -module V .

The **two-pole** braid group $\mathcal{B}_k^{(2)}$ shares a commuting action with \mathcal{U} on $M \otimes V^{\otimes k} \otimes N$:



Around the pole:



$$= \check{R}_{MV} \check{R}_{VM}$$

The **type-A Hecke algebra** is the quotient of the group algebra of the **braid group** \mathcal{B}_k by relations

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}. \quad (*)$$

Thm. The action of $\mathbb{C}\mathcal{B}_k$ on $V^{\otimes k}$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

The **type-A Hecke algebra** is the quotient of the group algebra of the **braid group** \mathcal{B}_k by relations

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array}. \quad (*)$$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

The **affine type- GL_k Hecke algebra** is the quotient of the group algebra of the **one-pole braid group** $\mathcal{B}_k^{(1)}$ by relations (*).

Thm. The action of $\mathbb{C}B_k^{(1)}$ on $M \otimes V^{\otimes k}$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

The **type-A Hecke algebra** is the quotient of the group algebra of the **braid group** \mathcal{B}_k by relations

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}. \quad (*)$$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

The **affine type- GL_k Hecke algebra** is the quotient of the group algebra of the **one-pole braid group** $\mathcal{B}_k^{(1)}$ by relations (*).

Thm. The action of $\mathbb{C}B_k^{(1)}$ on $M \otimes V^{\otimes k}$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

The **affine type-C Hecke algebra** is the quotient of the group algebra of the **two-pole braid group** $\mathcal{B}_k^{(2)}$ by relations (*),

$$\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} = a \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = b \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \quad (**).$$

Thm. The action of $\mathbb{C}B_k^{(2)}$ on $M \otimes V^{\otimes k} \otimes N$ factors through the quotient by (*) and (**), when $V = \mathbb{C}^n$, M and N are “rectangular”, and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

The **type-A Hecke algebra** is the quotient of the group algebra of the **braid group \mathcal{B}_k** by relations

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}. \quad (*)$$

The **affine type- GL_k Hecke algebra** is the quotient of the group algebra of the **one-pole braid group $\mathcal{B}_k^{(1)}$** by relations (*).

The **affine type-C Hecke algebra** is the quotient of the group algebra of the **two-pole braid group $\mathcal{B}_k^{(2)}$** by relations (*),

$$\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} = a \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} = b \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \quad (**).$$

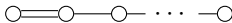
“Type what-now?”

Dynkin diagrams:

Type A



Affine Type GL



Affine Type C



The **type-A Hecke algebra** is the quotient of the group algebra of the **braid group \mathcal{B}_k** by relations

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \quad (*)$$

The **affine type- GL_k Hecke algebra** is the quotient of the group algebra of the **one-pole braid group $\mathcal{B}_k^{(1)}$** by relations (*).

The **affine type-C Hecke algebra** is the quotient of the group algebra of the **two-pole braid group $\mathcal{B}_k^{(2)}$** by relations (*),

$$\begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = a \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = b \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \quad (**)$$

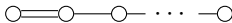
“Type what-now?”

Dynkin diagrams:

Type A



Affine Type GL



Affine Type C



The **two-pole/affine type-C braid group** is the group $\mathcal{B}_k^{(2)}$ generated by T_0, T_1, \dots, T_k , with relations



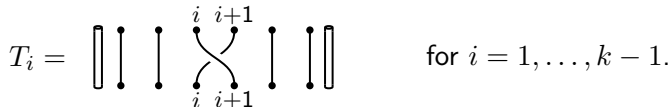
The **two-pole/affine type-C braid group** is the group $\mathcal{B}_k^{(2)}$ generated by T_0, T_1, \dots, T_k , with relations



Pictorially, the generators of $\mathcal{B}_k^{(2)}$ are identified with the diagrams



and



The **two-pole/affine type-C braid group** is the group $\mathcal{B}_k^{(2)}$ generated by T_0, T_1, \dots, T_k , with relations



Pictorially,

$$T_i T_{i+1} T_i = \begin{array}{c} \bullet & \bullet & \bullet \\ & \diagdown & \diagup \\ & \bullet & \bullet \\ & \diagup & \diagdown \\ \bullet & \bullet & \bullet \end{array} = \begin{array}{c} \bullet & \bullet & \bullet \\ \diagdown & & \diagup \\ \bullet & \bullet & \bullet \\ \diagup & & \diagdown \\ \bullet & \bullet & \bullet \end{array} = T_{i+1} T_i T_{i+1}$$

The **two-pole/affine type-C braid group** is the group $\mathcal{B}_k^{(2)}$ generated by T_0, T_1, \dots, T_k , with relations



Pictorially,

$$T_i T_{i+1} T_i = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \end{array} = T_{i+1} T_i T_{i+1}$$

$$T_1 T_0 T_1 T_0 = \begin{array}{c} \bullet \quad \bullet \\ \parallel \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \parallel \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \parallel \\ \bullet \quad \bullet \\ \parallel \quad \diagup \\ \bullet \quad \bullet \end{array} = T_0 T_1 T_0 T_1$$

(similar picture for $T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k$)

Back to Temperley-Lieb algebras

The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group \mathcal{B}_k by relations

$$\begin{array}{c} \bullet & & \bullet \\ & \searrow & \nearrow \\ \bullet & & \bullet \\ & \nearrow & \searrow \\ \bullet & & \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet & & \bullet \\ & \searrow & \nearrow \\ \bullet & & \bullet \\ & \nearrow & \searrow \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array}. \quad (*)$$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

Back to Temperley-Lieb algebras

The type-A Hecke algebra HA_k is the quotient of the group algebra of the braid group \mathcal{B}_k by relations

$$\begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ \bullet & & \bullet \\ & / & \searrow \\ \bullet & & \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & & \\ & / & \searrow \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array}. \quad (*)$$

Thm. The action of $\mathbb{C}\mathcal{B}_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

Case $n = 2$: Define

$$\begin{array}{c} \bullet & & \bullet \\ \cup & & \cup \\ \bullet & & \bullet \end{array} = q \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array} - \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & & \\ & / & \searrow \\ \bullet & & \bullet \end{array}. \quad (\diamond)$$

Back to Temperley-Lieb algebras

The **type-A Hecke algebra** HA_k is the quotient of the group algebra of the **braid group** \mathcal{B}_k by relations

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}. \quad (*)$$

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Then in HA_k (for general k),

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = \left(\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \right)^2$$

Back to Temperley-Lieb algebras

The **type-A Hecke algebra** HA_k is the quotient of the group algebra of the **braid group** \mathcal{B}_k by relations

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}. \quad (*)$$

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$$\begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} = q \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array}. \quad (\diamond)$$

Then in HA_k (for general k),

$$\begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = \left(\begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} \right)^2 = \left(q \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \right)^2$$

Back to Temperley-Lieb algebras

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$$\begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & \bullet & \\ & / & \searrow \\ \bullet & & \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & & \bullet \\ & / & \searrow \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array}. \quad (*)$$

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Case $n = 2$: Define

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$$\begin{array}{c} \bullet & & \bullet \\ \cup & & \cup \\ \bullet & & \bullet \\ \cup & & \cup \\ \bullet & & \bullet \end{array} = \left(\begin{array}{c} \bullet & & \bullet \\ \cup & & \cup \\ \bullet & & \bullet \end{array} \right)^2 = \left(q \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array} - \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & \bullet & \\ & / & \searrow \\ \bullet & & \bullet \end{array} \right)^2 = q^2 \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array} - q \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & \bullet & \\ & / & \searrow \\ \bullet & & \bullet \end{array} - q \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & \bullet & \\ & / & \searrow \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & \bullet & \\ & / & \searrow \\ \bullet & & \bullet \end{array}$$

Back to Temperley-Lieb algebras

The **type-A Hecke algebra** HA_k is the quotient of the group algebra of the **braid group** \mathcal{B}_k by relations

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}. \quad (*)$$

Thm. The action of $\mathbb{C}\mathcal{B}_k$ on $V^{\otimes k}$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

Case $n = 2$: Define

$$\begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} = q \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array}. \quad (\diamond)$$

Then in HA_k (for general k),

$$\begin{aligned} \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} &= \left(\begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} \right)^2 = \left(q \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \right)^2 \\ &= q^2 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - q \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - q \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \\ &= q^2 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - q \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \left((q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right) \end{aligned}$$

Back to Temperley-Lieb algebras

The *type-A Hecke algebra* HA_k is the quotient of the group algebra of the *braid group* B_k by relations

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \quad (*)$$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

Case $n = 2$: Define

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} = q \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \quad (\diamond)$$

Then in HA_k (for general k),

$$\begin{aligned} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} &= \left(\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \right)^2 = \left(q \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \right)^2 = q^2 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - q \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} - q \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \\ &= q^2 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - q \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \left((q - q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right) = (q + q^{-1}) \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \end{aligned}$$

Because $\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array}$ is $(q + q^{-1}) \times (\text{proj. onto sign representation for } HA_2)$.

Back to Temperley-Lieb algebras

The **type-A Hecke algebra** HA_k is the quotient of the group algebra of the **braid group** \mathcal{B}_k by relations

$$\begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ \bullet & & \bullet \\ & / & \searrow \\ \bullet & & \bullet \end{array} = (q - q^{-1}) \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & & \\ & / & \searrow \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array}. \quad (*)$$

Thm. The action of $\mathbb{C}B_k$ on $V^{\otimes k}$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or \mathfrak{sl}_n .

Case $n = 2$: Define


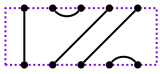

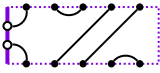
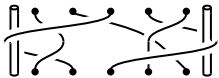
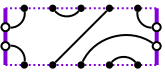
$$\begin{array}{c} \bullet & & \bullet \\ \cup & & \cup \\ \bullet & & \bullet \end{array} = q \begin{array}{c} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{array} - \begin{array}{c} \bullet & & \bullet \\ & \searrow & / \\ & & \\ & / & \searrow \\ \bullet & & \bullet \end{array}. \quad (\diamond)$$

Thm. Using the identification in (\diamond) , the action of HA_k on $(\mathbb{C}^2)^{\otimes k}$ factors through the Temperley-Lieb quotient when $\delta = q + q^{-1} = [2]_q$, i.e. TL_k centralizes $\mathcal{U}_q \mathfrak{gl}_2$ and $\mathcal{U}_q \mathfrak{sl}_2$ in $\text{End}((\mathbb{C}^2)^{\otimes k})$ when $\bigcirc = [2]_q$.

$$\text{cup} = q \text{parallel} - \text{cross}$$

$$\text{cup with pole} = q_0 \text{parallel} - \text{cross}$$

$$\text{cup with two poles} = q_k \text{parallel} - \text{cross}$$

tensor space	centralizer of $\mathcal{U}_q \mathfrak{g}$	centralizer of $\mathcal{U}_q \mathfrak{gl}_n$	centralizer of $\mathcal{U}_q \mathfrak{gl}_2$
$V^{\otimes k}$	Braids on k strands 	Type-A Hecke (twist relations)	Temperley-Lieb 
$M \otimes V^{\otimes k}$	One-pole braids 	Affine type-GL Hecke (twist relations)	1-boundary TL 
$M \otimes V^{\otimes k} \otimes N$	Two-pole braids 	Affine type-C Hecke (twist & wrap relations)	2-boundary TL 

Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $\delta, \delta_0, \delta_k \in \mathbb{C}$. The *two-boundary Temperley-Lieb algebra* $TL_k^{(2)}$ is a diagram algebra generated over \mathbb{C} by diagrams

$$e_0 = \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right], \quad e_k = \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right], \quad \text{and} \quad e_i = \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right]$$

The diagrams are as follows:

- e_0 : A rectangular diagram with 8 vertical strands. The left two strands are connected by two arcs (top and bottom) that cross each other. The right two strands are also connected by two arcs (top and bottom) that cross each other. The top and bottom boundaries are labeled '1'.
- e_k : A rectangular diagram with 8 vertical strands. The left two strands are connected by two arcs (top and bottom) that cross each other. The right two strands are also connected by two arcs (top and bottom) that cross each other. The top and bottom boundaries are labeled 'k'.
- e_i : A rectangular diagram with 8 vertical strands. The left two strands are connected by two arcs (top and bottom) that cross each other. The right two strands are also connected by two arcs (top and bottom) that cross each other. The top and bottom boundaries are labeled 'i'.

for $i = 1, \dots, k - 1$

Two-boundary Temperley-Lieb algebras

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$$e_0 = \left[\begin{array}{c} 1 \\ \text{Diagram 1} \\ 1 \end{array} \right], \quad e_k = \left[\begin{array}{c} \text{Diagram 2} \\ k \\ k \end{array} \right], \quad \text{and} \quad e_i = \left[\begin{array}{c} \text{Diagram 3} \\ i \\ i \end{array} \right]$$

for $i = 1, \dots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$e_i e_{i \pm 1} e_i = e_i$$

for $1 \leq i \leq k - 1$,

$$e_i^2 = \delta_i e_i.$$

Two-boundary Temperley-Lieb algebras

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$$e_i e_{i\pm 1} e_i = e_i$$

for $1 \leq i \leq k-1$,

Diagrammatic relations for the two-boundary Temperley-Lieb algebra:

- Box 1: A vertical line with a loop on the left side is equal to a vertical line with a loop on the right side.
- Box 2: A vertical line with a loop on the left side is equal to a vertical line with a loop on the right side.
- Box 3: A vertical line with a loop on the left side is equal to a vertical line with a loop on the right side.

$$e_i^2 = \delta_i e_i.$$

Two-boundary Temperley-Lieb algebras

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$$e_0 = \begin{array}{c} 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 1 \end{array}, \quad e_k = \begin{array}{c} k \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ k \end{array}, \quad \text{and} \quad e_i = \begin{array}{c} i \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ i \end{array}$$

for $i = 1, \dots, k-1$, with relations $e_i e_j = e_j e_i$ for $|i-j| > 1$,

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Two-boundary Temperley-Lieb algebras

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for $i = 1, \dots, k-1$, with relations $e_i e_j = e_j e_i$ for $|i-j| > 1$,

$$e_i e_{i\pm 1} e_i = e_i \quad \text{for } 1 \leq i \leq k-1,$$

$$e_i^2 = \delta_i e_i.$$

(Side loops are resolved with a 1 or a δ_i depending on whether there are an even or odd number of connections below their lowest point.)

Diagram multiplication:



Diagram multiplication:

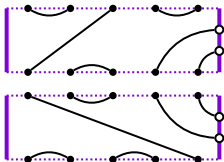


Diagram multiplication:

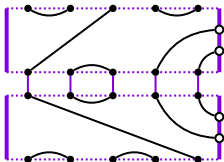


Diagram multiplication:

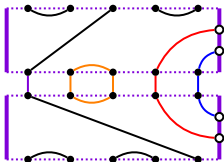


Diagram multiplication:

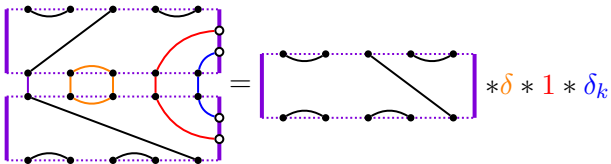
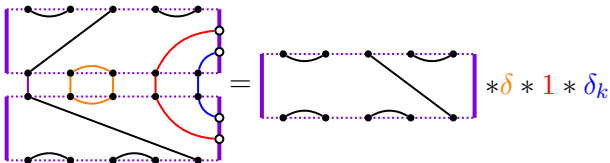


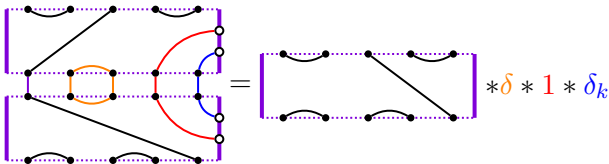
Diagram multiplication:



In short, $TL_k^{(2)}$ has basis given by non-crossing diagrams with

- (1) k connections to the top and to the bottom,
- (2) an even number of connections to the right and to the left, and
- (3) no edges with both ends on the left or both ends on the right.

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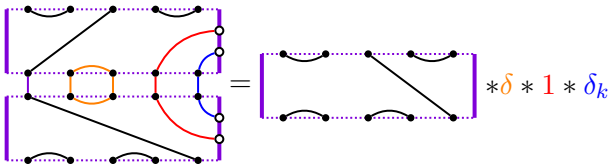
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However,



So unlike the classical T-L algebras, $TL_k^{(2)}$ is not finite dimensional!

Diagram multiplication:



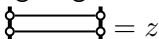
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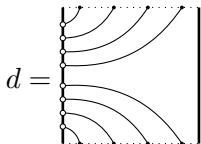
However,



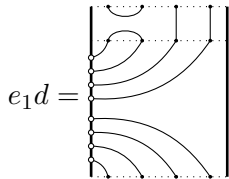
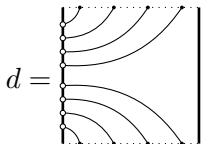
So unlike the classical T-L algebras, $TL_k^{(2)}$ is not finite dimensional! Take quotient giving



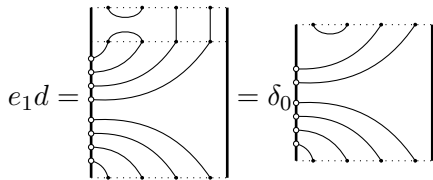
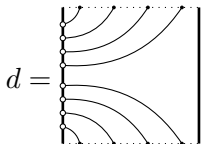
Representation theory of $TL_k^{(2)}$: action on diagrams



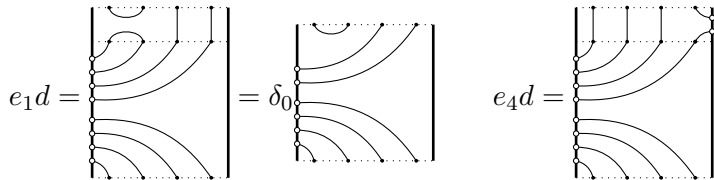
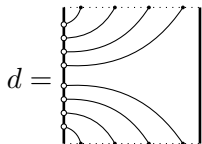
Representation theory of $TL_k^{(2)}$: action on diagrams



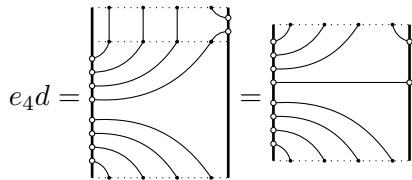
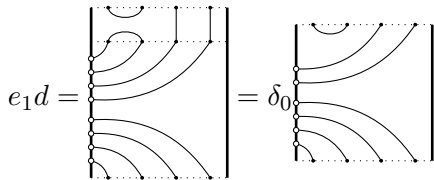
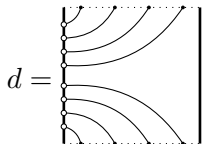
Representation theory of $TL_k^{(2)}$: action on diagrams



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Representation theory of $TL_k^{(2)}$: action on diagrams



Representation theory of $TL_k^{(2)}$: action on diagrams

$$d = \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a dot. Curved lines connect the top two strands on the left to the top two on the right, and the bottom two strands on the left to the bottom two on the right. The top and bottom strands have small arcs at their ends.$$

$$e_1 d = \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a dot. Curved lines connect the top two strands on the left to the top two on the right, and the bottom two strands on the left to the bottom two on the right. The top two strands have small arcs at their ends. A vertical line is added on the right side, connecting the top two dots to the bottom two dots. This is equal to δ_0 times the diagram d .$$

$$e_4 d = \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a dot. Curved lines connect the top two strands on the left to the top two on the right, and the bottom two strands on the left to the bottom two on the right. The top two strands have small arcs at their ends. A vertical line is added on the right side, connecting the top two dots to the bottom two dots. This is equal to the diagram d with a horizontal line connecting the two dots on the right side of the top two strands.$$

$$e_3 e_4 d = \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a dot. Curved lines connect the top two strands on the left to the top two on the right, and the bottom two strands on the left to the bottom two on the right. The top two strands have small arcs at their ends. A vertical line is added on the right side, connecting the top two dots to the bottom two dots. A horizontal line is added on the right side, connecting the two dots on the right side of the top two strands. A vertical line is added on the left side, connecting the top two dots to the bottom two dots.$$

Representation theory of $TL_k^{(2)}$: action on diagrams

$$d = \text{Diagram with 4 strands on the left and 4 on the right, with arcs connecting them. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with dots at the top and bottom.$$

$$e_1 d = \text{Diagram with a crossing on the left strand} = \delta_0 \text{Diagram with a crossing on the right strand} \quad e_4 d = \text{Diagram with a crossing on the right strand} = \text{Diagram with a crossing on the left strand}$$

$$e_3 e_4 d = \text{Diagram with a crossing on the right strand and a crossing on the left strand} = \text{Diagram with two horizontal strands in the middle and arcs connecting them to the left and right strands}$$

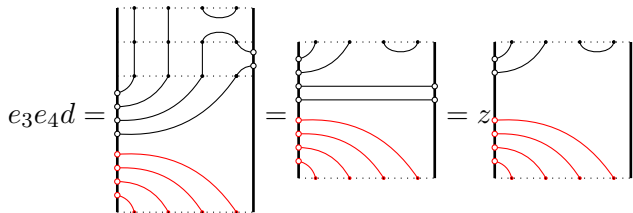
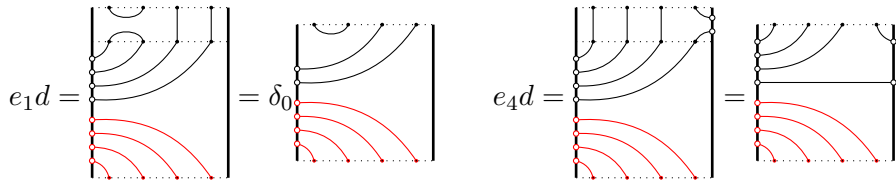
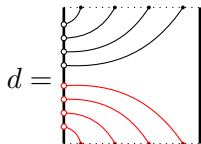
Representation theory of $TL_k^{(2)}$: action on diagrams

$$d = \text{Diagram with 4 strands on the left and 4 on the right, with arcs connecting them. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.$$

$$e_1 d = \text{Diagram with a vertical line on the left and arcs connecting strands. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.} = \delta_0 \text{Diagram with 4 strands on the left and 4 on the right, with arcs connecting them. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.} \\ e_4 d = \text{Diagram with a vertical line on the left and arcs connecting strands. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.} = \text{Diagram with a vertical line on the left and arcs connecting strands. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.}$$

$$e_3 e_4 d = \text{Diagram with a vertical line on the left and arcs connecting strands. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.} = \text{Diagram with a vertical line on the left and arcs connecting strands. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.} = z \text{Diagram with 4 strands on the left and 4 on the right, with arcs connecting them. The top two strands on the left are connected to the top two on the right, and the bottom two on the left are connected to the bottom two on the right. The strands are labeled with small circles at the top and bottom.}$$

Representation theory of $TL_k^{(2)}$: action on diagrams



Representation theory of $TL_k^{(2)}$: half diagrams

$$d = \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

$$e_1 d = \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = \delta_0 \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

$$e_4 d = \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

$$e_3 e_4 d = \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = z \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

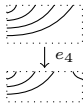
Generic module:

(act by e_i , don't make loops)



Generic module:

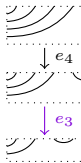
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Generic module:

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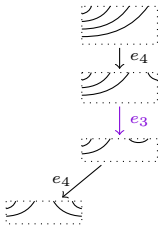
Red arrows indicate coef of z .



Generic module:

(act by e_i , don't make loops)

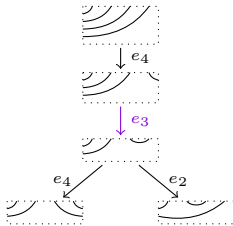
Red arrows indicate coef of z .



Generic module:

(act by e_i , don't make loops)

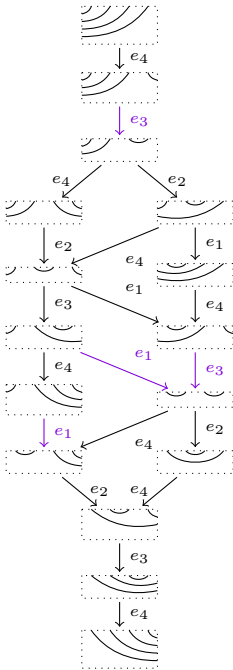
Red arrows indicate coef of z .



Generic module:

(act by e_i , don't make loops)

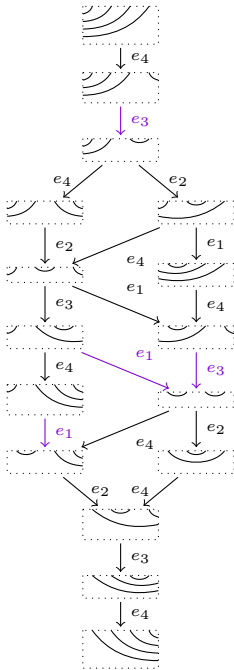
Red arrows indicate coef of z .



Generic module:

(act by e_i , don't make loops)

Red arrows indicate coef of z .



For what z does this module split?

(1) The two-boundary (two-pole) braid group \mathcal{B}_k is generated by

$$T_k = \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array}, \quad T_0 = \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ i \quad i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations $\begin{array}{c} T_0 \\ \circ \end{array} = \begin{array}{c} T_1 \\ \circ \end{array} - \begin{array}{c} T_2 \\ \circ \end{array} \dots \dots \begin{array}{c} T_{k-2} \\ \circ \end{array} - \begin{array}{c} T_{k-1} \\ \circ \end{array} = \begin{array}{c} T_k \\ \circ \end{array}.$

(1) The two-boundary (two-pole) braid group \mathcal{B}_k is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \overset{i}{\bullet} \quad \overset{i+1}{\bullet} \\ \diagdown \quad \diagup \\ \underset{i}{\bullet} \quad \underset{i+1}{\bullet} \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations $\begin{array}{c} T_0 \\ \circ \end{array} = \begin{array}{c} T_1 \\ \circ \end{array} = \begin{array}{c} T_2 \\ \circ \end{array} \text{---} \text{---} \text{---} \begin{array}{c} T_{k-2} \\ \circ \end{array} = \begin{array}{c} T_{k-1} \\ \circ \end{array} = \begin{array}{c} T_k \\ \circ \end{array}.$

(2) Fix constants $q_0, q_k, q \in \mathbb{C}$.

The affine type C Hecke algebra \mathcal{H}_k is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - q_0)(T_0 + q_0^{-1}) = 0, \quad (T_k - q_k)(T_k + q_k^{-1}) = 0$$

and $(T_i - q)(T_i + q^{-1}) = 0 \quad \text{for } i = 1, \dots, k-1.$

(1) The two-boundary (two-pole) braid group \mathcal{B}_k is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} i \quad i+1 \\ \diagdown \quad \diagup \\ i \quad i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations $T_0 \text{---} T_1 \text{---} T_2 \text{---} \dots \text{---} T_{k-2} \text{---} T_{k-1} \text{---} T_k$.

(2) Fix constants $q_0, q_k, q = q_1 = q_2 = \dots = q_{k-1} \in \mathbb{C}$.

The affine type C Hecke algebra \mathcal{H}_k is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - q_i^{1/2})(T_i + q_i^{-1/2}) = 0$.

(3) Set

$$\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = q_0 \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \end{array} \quad (e_0 = q_0 - T_0)$$

$$\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = q_k \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \end{array} \quad (e_k = q_k - T_k)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = q \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \end{array} \quad (e_i = q - T_i)$$

so that $e_j^2 = z_j e_j$ (for good z_j).

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subject to relations $T_0 \text{---} T_1 \text{---} T_2 \text{---} \dots \text{---} T_{k-2} \text{---} T_{k-1} \text{---} T_k$.

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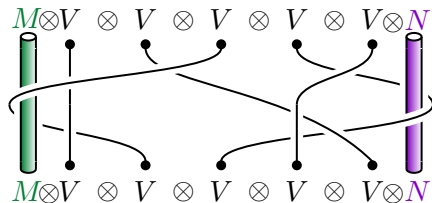
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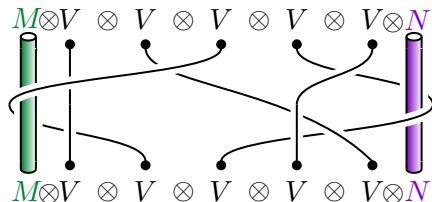
so that $e_j^2 = z_j e_j$ (for good z_j).

The **two-boundary Temperley-Lieb algebra** is the quotient of \mathcal{H}_k by the relations $e_i e_{i\pm 1} e_i = e_i$ for $i = 1, \dots, k-1$.

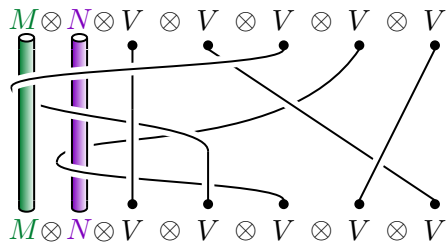
One advantage of using braids:



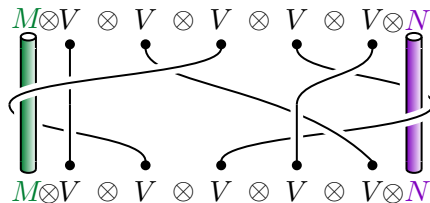
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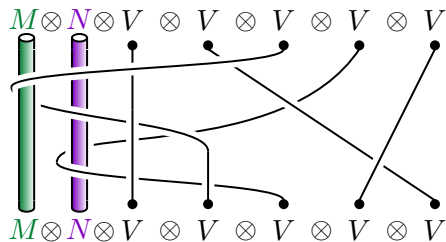
Move both poles
to the left ↓



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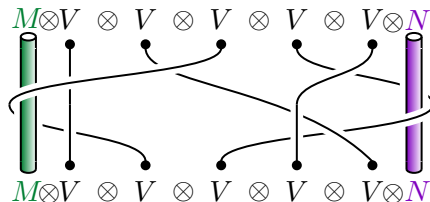
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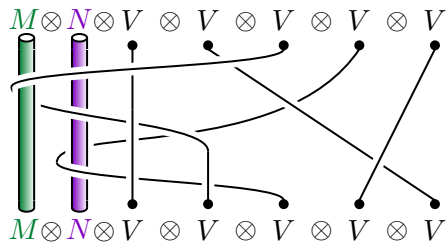
Jucys-Murphy elements:

$$Z_i = \text{Diagram with } i \text{ strands, where the } i\text{-th strand from the left is connected to the } i\text{-th strand from the right, and the other } i-1 \text{ strands are straight vertical lines.}$$

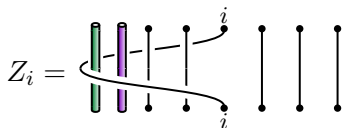
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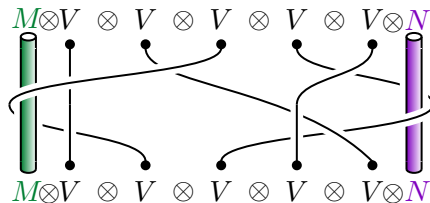


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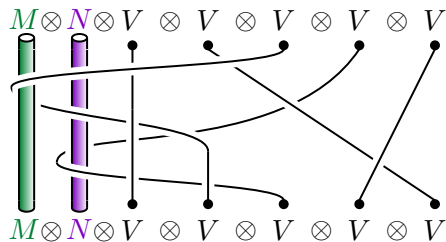


► Pairwise commute

One advantage of using braids:



Move both poles
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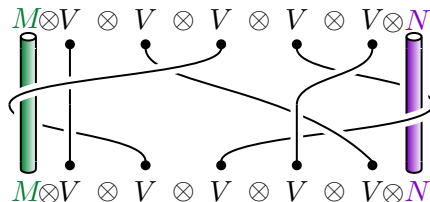


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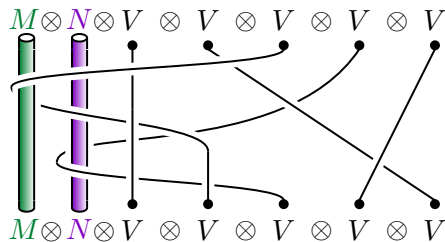
$$Z_i = \text{Diagram showing } i \text{ strands, with a crossing between the } i \text{th and } (i+1)\text{th strands. The } i \text{th strand is green and the } (i+1)\text{th strand is purple. The crossing is labeled } i \text{ at the top and } i \text{ at the bottom. To the right are } k-i \text{ vertical strands.}$$

- ▶ Pairwise commute
- ▶ $Z(\mathcal{H}_k)$ is (type-C) symmetric Laurent polynomials in Z_i 's

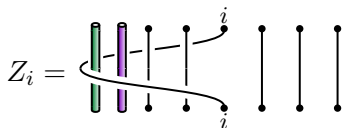
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Jucys-Murphy elements:



- ▶ Pairwise commute
- ▶ $Z(\mathcal{H}_k)$ is (type-C) symmetric Laurent polynomials in Z_i 's
- ▶ Central characters indexed by $\mathfrak{c} \in \mathbb{C}^k$ (modulo signed permutations)

Representation theory of \mathcal{H}_k

The representations of \mathcal{H}_k are indexed by pairs (\mathbf{c}, J) , where

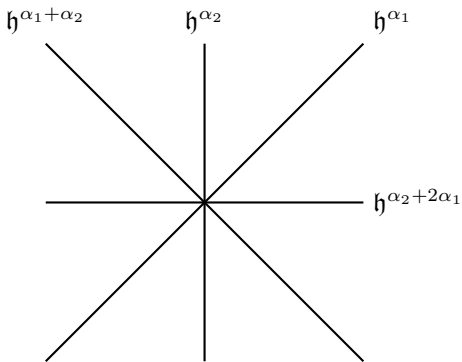
\mathbf{c} is a point in the fundamental chamber of
the (finite) type C hyperplane system, and
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Example: $k = 2$

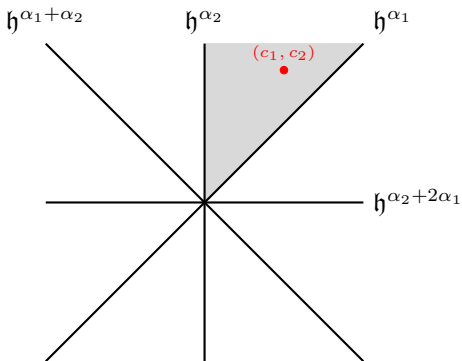


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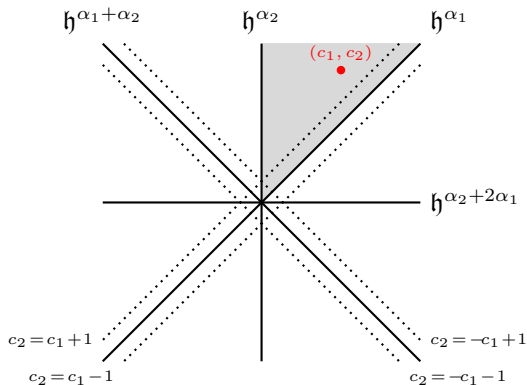


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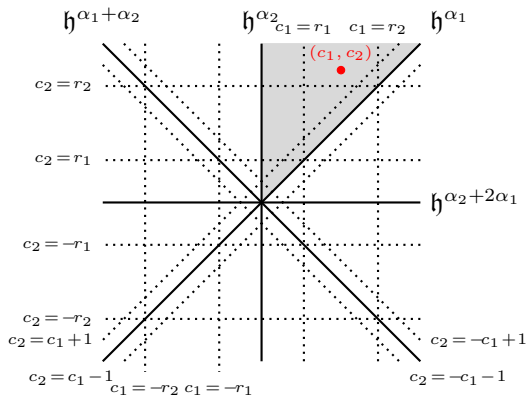


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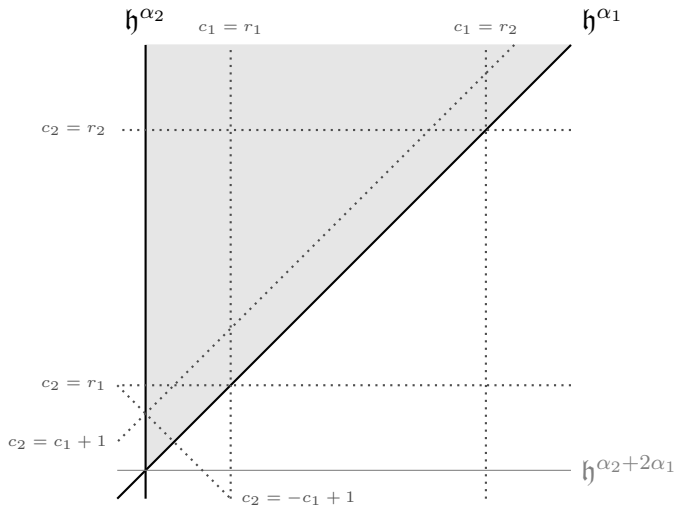
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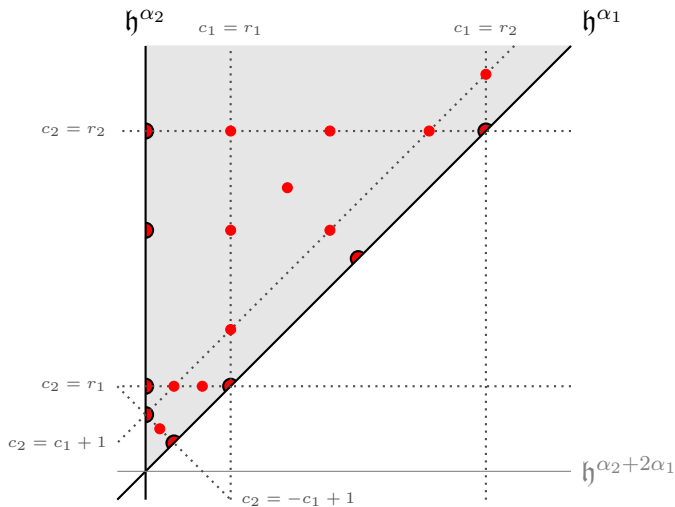
The r_i s depend on \mathcal{H}_k 's parameters q_0 and q_k : $r_1 = \log_q(q_0/q_k)$, $r_2 = \log_q(q_0q_k)$.

Representation theory of \mathcal{H}_k



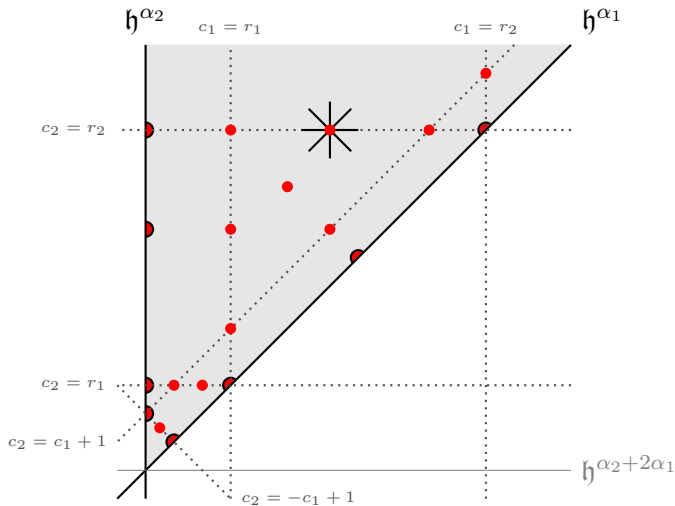
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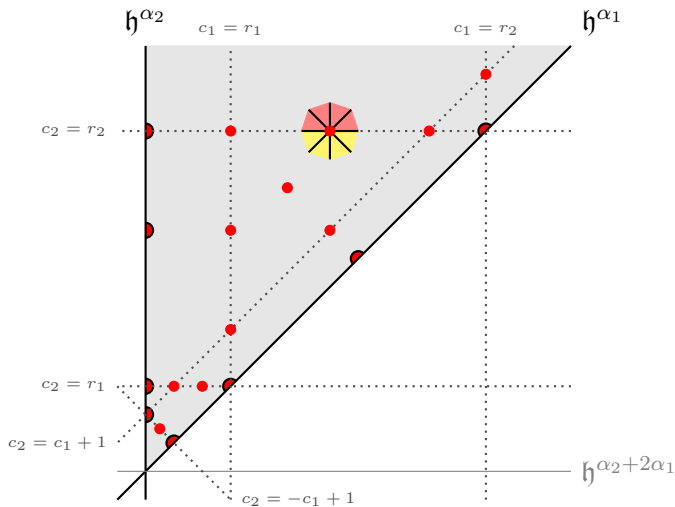
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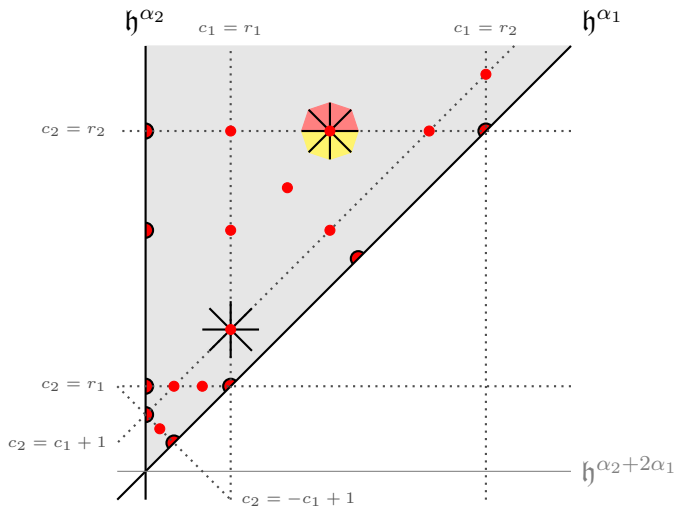
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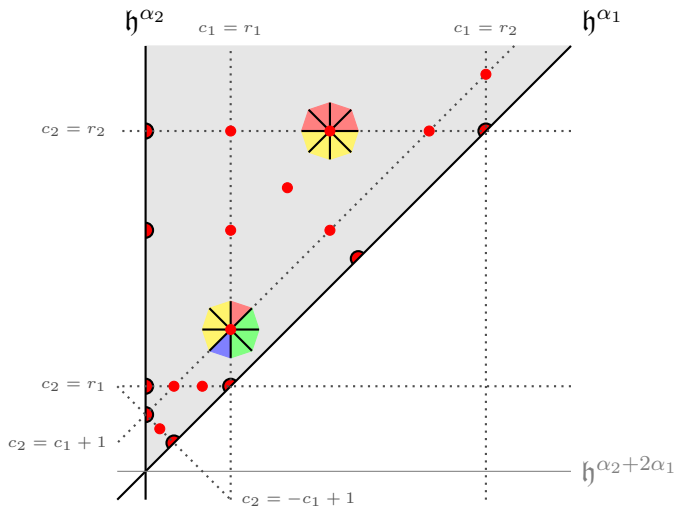
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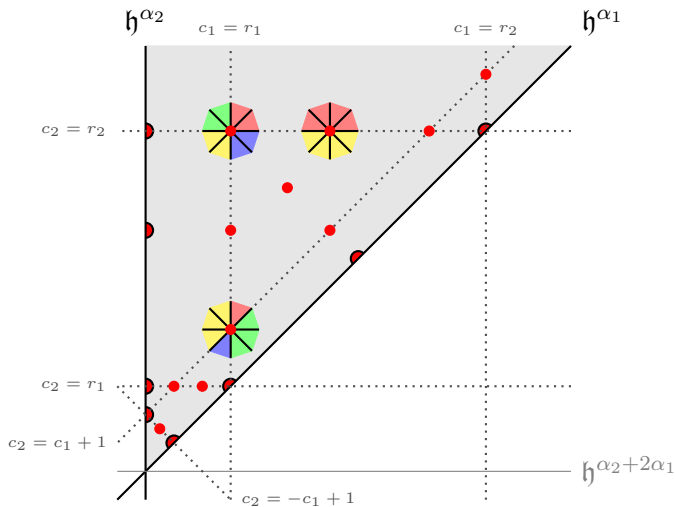
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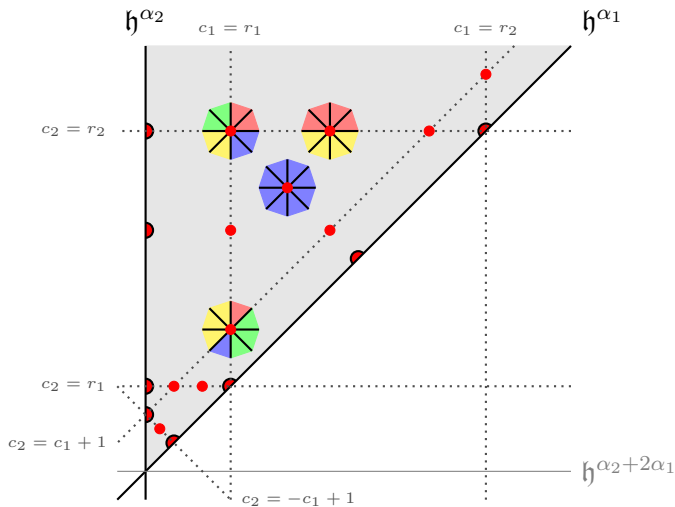
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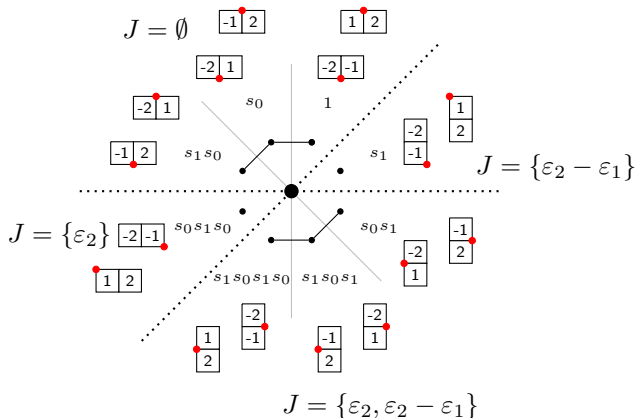
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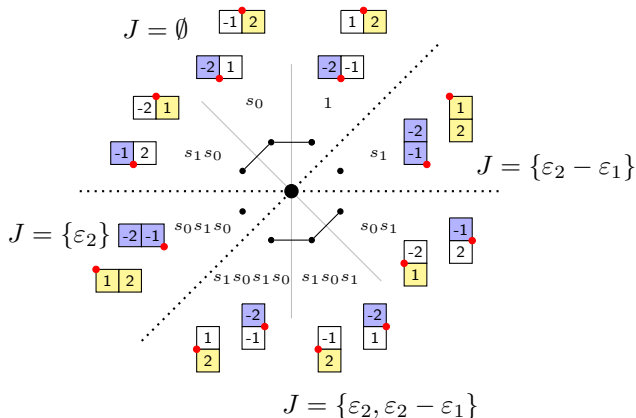
A little more detail

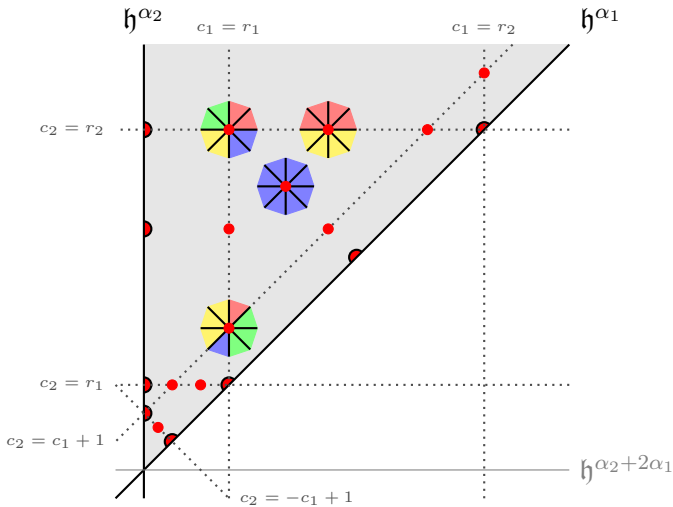
- J is determined by a set of positive roots (corresp. to hyperplanes).
- For “nice” characters, there is a bijection between alcoves and marked type-C generalized Young tableaux.
- “Intertwining operators” τ_i move between alcoves; dotted lines correspond to $\tau_i = 0$.



A little more detail

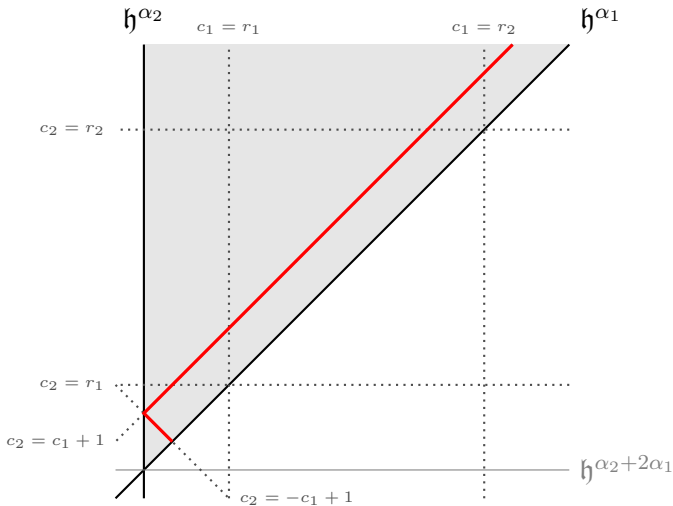
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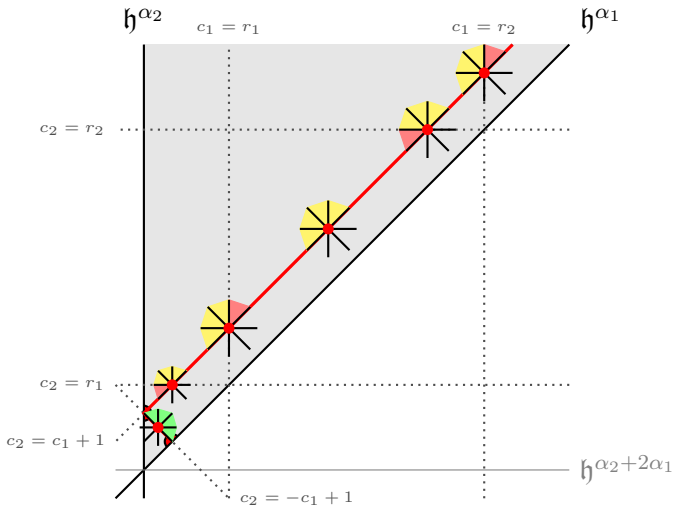
Thm. (D.-Ram)

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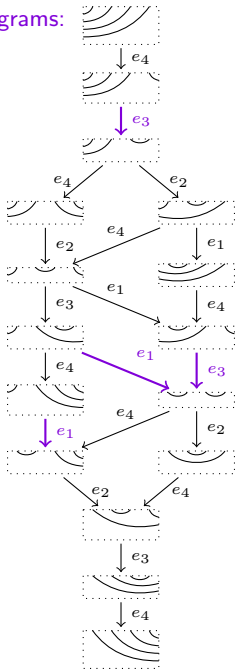
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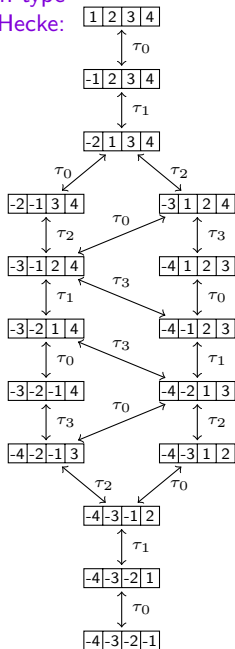
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Diagrams:

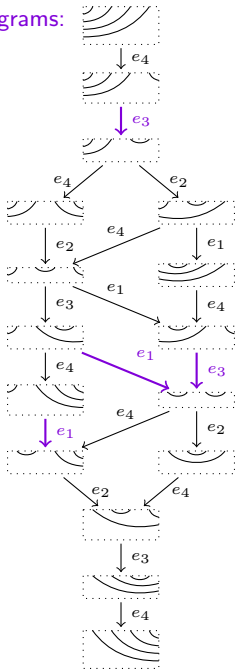


[GN] [DR]

Aff. type
C Hecke:

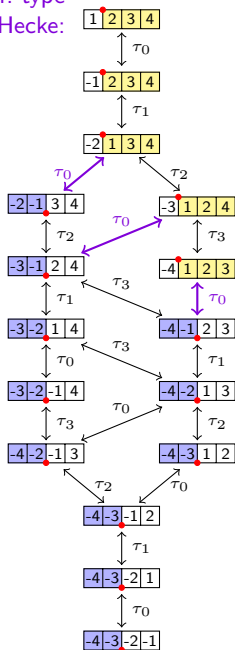


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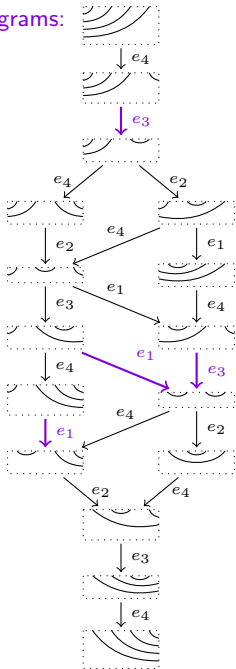


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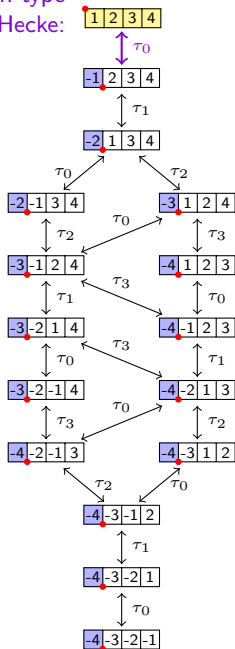


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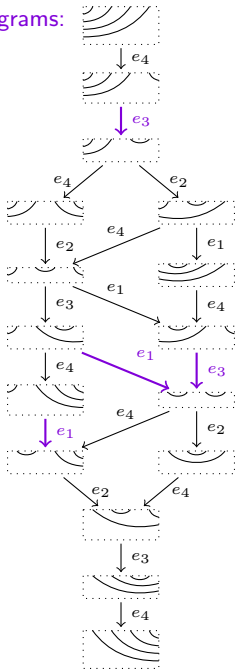


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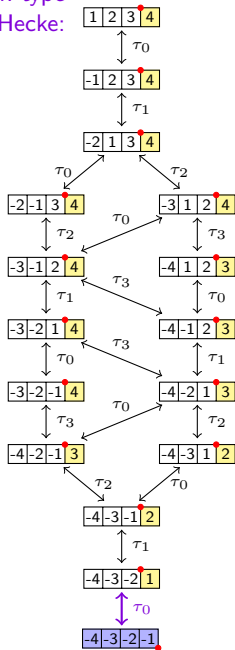


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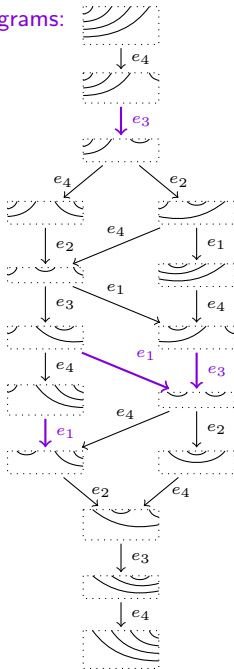


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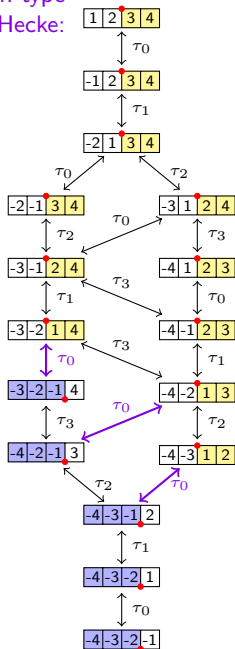


Diagrams:



[GN] [DR]

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See also...

Specific:

- [DR] Daugherty and Ram, “Two boundary Hecke Algebras and combinatorics of type C”, arXiv:1804.10296.
- [GN] de Gier and Nichols, “The two-boundary Temperley-Lieb algebra”, J. Algebra 321 (2009), no. 4, 1132–1167.

General:

- [BR] Barcelo and Ram, “Combinatorial representation theory,” New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), 23–90, Math. Sci. Res. Inst. Publ., 38, Cambridge Univ. Press, Cambridge, 1999.

<https://zdaugherty.ccny.cuny.edu>

Thanks!