An Algebraic Approach to Voting Theory

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Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

1^{st} :	A	A	B	B	C	C
2^{nd} :	B	C	A	C	A	B
3^{rd} :	C	B	C	A	B	A
	13	0	19	0	17	5

So who should be president?

Method 1: Vote for your first choice.

A	B	C
13	19	22

Carly wins!

Method 2: Tell us your full ranking, and we'll pair them off.

$A>B$	$B>C$	$A>C$
30	32	32
$A<B$	$B<C$	$A<C$
24	22	22

$A>B, \quad B>C, \quad A>C.$

Alice wins!

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2^{nd} :	B	C	A	C	A	B
3^{rd} :	C	B	C	A	B	A
	13	0	19	0	17	5

So who should be president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

A	B	C
13	19	22

Alice loses.

Round 2:

B	C
32	22

Becca wins!

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

A	B	C
$13 + 36t$	$19 + 18t$	$22 + 0t$

$(0 \leqslant t \leqslant 1)$

Suppose Alice, Becca, and Carly are running for president of their AWM chapter. The personal preferences of the 54 members are given by

1^{st} :	A	A	B	B	C	C
2^{nd} :	B	C	A	C	A	B
3^{rd} :	C	B	C	A	B	A
	13	0	19	0	17	5

So who should be president?

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

	A	B	C
$(0 \leqslant t \leqslant 1)$	$13 + 36t$	$19 + 18t$	$22 + 0t$
$t = 0$:	13	19	22
$t = 0.25$:	22	23.5	22
$t = 0.5$:	31	28	22
$t = 0.75$:	40	32.5	22
$t = 1$:	49	37	22

t: 0 — $C > B > A$ — $1/6$ — $B > C > A$ — $1/4$ — $B > A > C$ — $1/3$ — $A > B > C$ $\cdots$ — 1

In 2000, in Florida, 2,912,790 people voted for Bush, 2,912,253 voted for Gore, 97,488 voted for Nader, and 40,575 for other. Neglecting the "others", suppose we had asked the rest of the voters for their full rankings of the top three candidates.

1st :	B	G	G	N
2nd :	G	N	B	G
3rd :	N	B	N	B
	2.902mil	1.421mil	1.481mil	0.118mil

So who should have been president?

Method 1: Vote for your first choice.

B	G	N
2.90mil	2.90mil	0.11mil

Bush won.

Method 2: Tell us your full ranking, and we'll pair them off.

$B > G$	$G > N$	$B > N$
2.90mil	5.80mil	4.32mil

$B < G$	$G < N$	$B < N$
3.02mil	0.12mil	1.60mil

$G > B, \quad G > N, \quad B > N.$

Gore wins.

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2nd :	G	N	B	G
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	2.902mil	1.421mil	1.481mil	0.118mil

So who should have been president?

Method 3: Tell us your full ranking, and we'll run an instant run off.

Round 1:

B	G	N
2.90mil	2.90mil	0.12mil

Nader loses.

Round 2:

B	G
2.90mil	3.02mil

Gore wins.

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

B	G	N
$2.902 + 1.421\ t$	$2.902 + 3.020\ t$	$0.118 + 1.481\ t$

$(0 \leqslant t \leqslant 1)$

In 2000, in Florida, 2,912,790 people voted for Bush, 2,912,253 voted for Gore, 97,488 voted for Nader, and 40,575 for other. Neglecting the "others", suppose we had asked the rest of the voters for their full rankings of the top three candidates.

1st :	B	G	G	N
2nd :	G	N	B	G
3rd :	N	B	N	B
	2.902mil	1.421mil	1.481mil	0.118mil

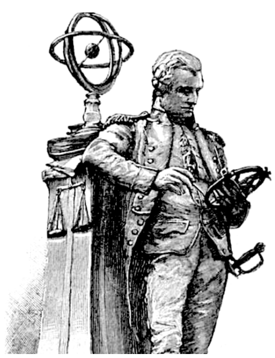
So who should have been president?

Method 4: Tell us your full ranking, and we'll give points according to the candidates' positions—1 for 1st, t for 2nd, 0 for 3rd.

	A	B	C
$(0 \leqslant t \leqslant 1)$	$2.902 + 1.421\ t$	$2.902 + 3.020\ t$	$0.118 + 1.481\ t$
$t = 0$:	2.90mil	2.90mil	0.19mil
$t = 0.25$:	3.26mil	3.66mil	0.49mil
$t = 0.5$:	3.61mil	4.41mil	0.86mil
$t = 0.75$:	3.97mil	5.17mil	1.23mil
$t = 1$:	4.32mil	5.92mil	1.60mil

t: $B > G > N$ on $[0, .000335]$; $G > B > N$ from $.000335$ to 1

0 .000335 1



Jean-Charles, Chevalier de Borda
1733–1799
Mariner and scientist.
1770: formulated a ranked voting system, the "Borda count". Used by the French Academy of Sciences, until Napolean.

Nicolas de Caritat, Marquis de Condorcet
1743–1794
Philosopher and mathematician.
In 1785, wrote an essay on probability of decisions made on a majority vote, describing likelihood of good jury outcomes; and Condorcet's paradox, which shows that majority preferences can become intransitive with three or more options

Dr. Donald G. Saari (1940–)
Professor of Mathematics and Economics.
1999: Used geometric methods to model voting data as vector spaces, and decompose them based on how they affect various tallying methods.

Kernel: Doesn't affect any fair voting system.
Borda: Influences both point-based and pairwise systems.
Condorcet: Introduces Condorcet paradox.
Reversal: Influences point-based systems, but not pairwise.

	Kernel	Borda			Condorcet	Reversal		
		$\mathbf{b_A}$	$\mathbf{b_B}$	$\mathbf{b_C}$		$\mathbf{r_A}$	$\mathbf{r_B}$	$\mathbf{r_C}$
ABC	1	1	0	-1	1	1	-2	1
ACB	1	1	-1	0	-1	1	1	-2
BAC	1	0	1	-1	-1	-2	1	1
BCA	1	-1	1	0	1	1	1	-2
CAB	1	0	-1	1	1	-2	1	1
CBA	1	-1	0	1	-1	1	-2	1

	Points $\mathbf{w} = (1, t, 0)$ A	B	C	$A > B$ vs. $A < B$	$B > C$ vs. $B < C$	$C > A$ vs. $C < A$
Ker	$2 + 2t$	$2 + 2t$	$2 + 2t$	0	0	0
$\mathbf{b}_A$	2	-1	-1	4	0	-4
$\mathbf{b}_B$	-1	2	-1	-4	4	0
$\mathbf{b}_C$	-1	-1	2	0	-4	4
Cond	0	0	0	2	2	2
$\mathbf{r}_A$	$2 - 4t$	$-1 + 2t$	$-1 + 2t$	0	0	0
$\mathbf{r}_B$	$-1 + 2t$	$2 - 4t$	$-1 + 2t$	0	0	0
$\mathbf{r}_C$	$-1 + 2t$	$-1 + 2t$	$2 - 4t$	0	0	0

Don't worry about negatives: negative votes are fixed by the kernel.

Note: The reversal space is trivial *precisely* when $t = 1/2$.

Theorem (Saari '00) Given a full ranking of n candidates, the reversal space is trivial precisely for weight

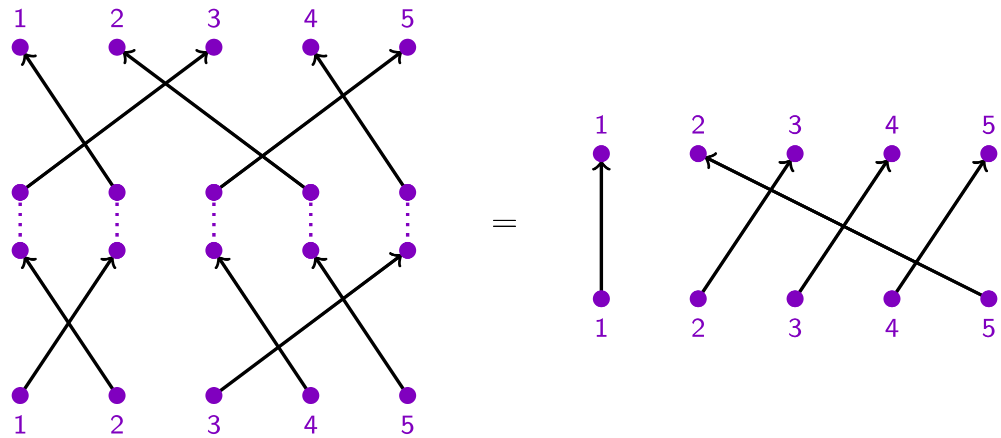
$$\mathbf{w} = \left(1,\ \frac{n-2}{n-1},\ \ldots,\ \frac{2}{n-1},\ \frac{1}{n-1},\ 0\right).$$

Permutations and the symmetric group

A permutation is a bijective (one-to-one and onto) function

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}.$$

Permutation diagrams:

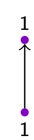


Permutations "multiply" by stacking and resolving.

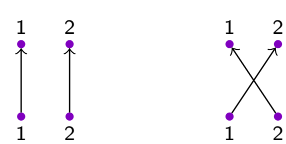
The symmetric group S_n is the group of permutations of $1, \ldots, n$ with multiplication given by function composition.

Some examples:

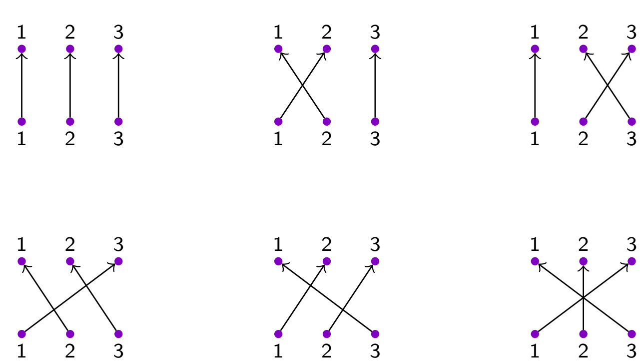
S_1 :



S_2 :

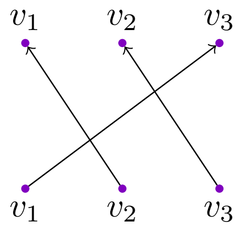


S_3 :



A representation of a group is a map from the group to a set of matrices that follows same multiplication rules.

Example: Permutation representation of the symmetric group.



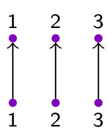
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Pick a basis for $\mathbb{Q}^3$:

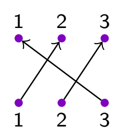
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Map each permutation to the matrix that permutes the basis vectors in the same way. (Recall: ith col. is image of ith basis vector)

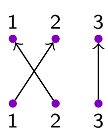
Aside: we actually have a representation of the group ring $\mathbb{Q}S_n = \left\{ \sum_{\sigma \in S_n} r_\sigma \sigma \mid r_\sigma \in \mathbb{Q} \right\}$, with multiplication like polynomials.



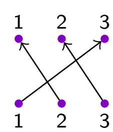
$$\mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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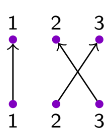

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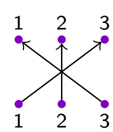


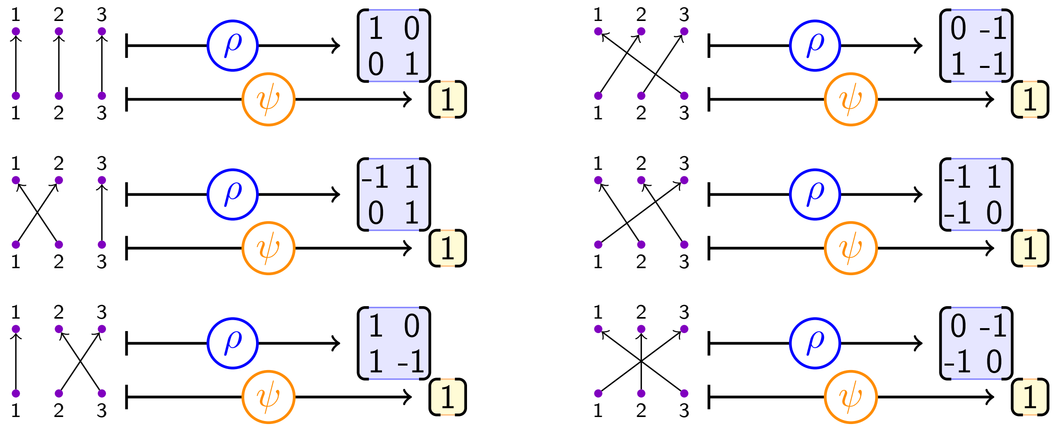
$$\mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that the permutation representation has an invariant subspace $\mathbb{Q}\{v_1 + v_2 + v_3\}$, since

$$M(v_1 + v_2 + v_3) = v_1 + v_2 + v_3$$

for all permutation matrices M.
Change to basis

$$w_1 = v_1 - v_2, \qquad w_2 = v_2 - v_3, \qquad w_3 = v_1 + v_2 + v_3$$



Start with the permutation representation P with basis $\{v_1, v_2, v_3\}$. Change to basis

$$w_1 = v_1 - v_2, \qquad w_2 = v_2 - v_3, \qquad w_3 = v_1 + v_2 + v_3$$

We say P is isomorphic to the sum of two smaller representations:

$$P \cong \rho \oplus \psi$$

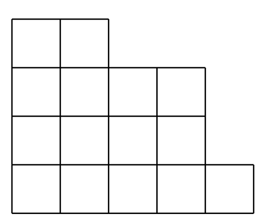
We say ρ and ψ are simple because neither has any invariant subspaces.

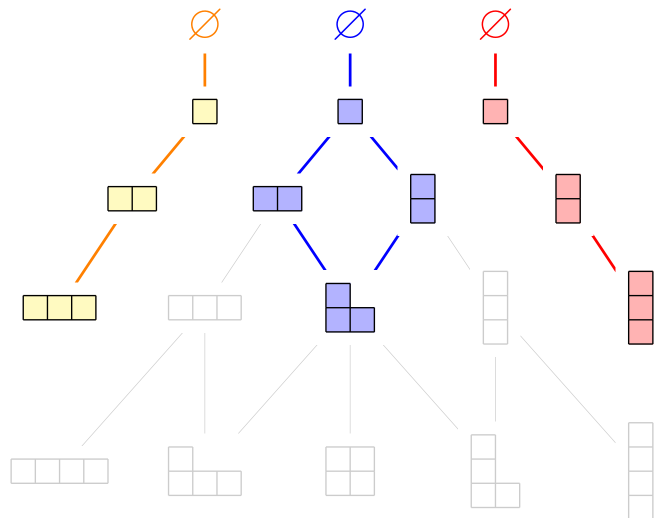
Some combinatorics.

Let n be a non-negative integer.
A partition λ of n is a non-ordered list of positive integers which sum to n.

Example: the partitions of 3 are (3), $(2, 1)$, and $(1, 1, 1)$.

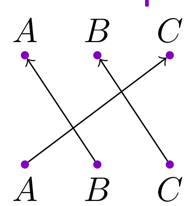
We draw partitions as n boxes left-justified, where the parts are the number of boxes in a row (reading from the bottom):

$\lambda = (5, 4, 4, 2) =$ 



λ-Tableau: a path from $\varnothing$ down to a partition λ.
Theorem 1: (Up to isomorphism) the simple S_n-representations are indexed by partitions of n.
Theorem 2: If λ is a partition of n, then the corresponding representation has basis indexed by λ-tableaux, and matrices determined by other combinatorial data about those paths.

Why do *we* care about representations of S_n?



$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Pick a basis for $\mathbb{Q}^3$:

$$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Map each permutation to the matrix that permutes the basis vectors in the same way.

We use the permutation representation to model the outcome space. For example, the vector

$$(13, 19, 22) = 13A + 19B + 22C$$

means A, B, and C got 13, 19, and 22 votes, respectively.

Theorem. The permutation representation decomposes into one simple 1-dimensional (trivial) representation and one simple $(n-1)$-dimensional (reflection) representation.

The permutation representation models the outcome space.
More: all voting data spaces are symmetric group "modules".

In other words:

Permutations naturally move around voting and outcome spaces.

(Permute the candidate's names, or their places on the ballot.)

Tally functions (how you add up the vote) are "S_n-module homomorphisms", i.e. maps from the voter data to the outcome space that preserve

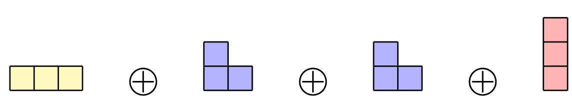
- addition,
  If individual precincts add up votes, and then combine results, that should be the same as if the tallying happened all in one place.
- scaling, and
  If everyone's vote counted 5 times, the outcome should be the same.
- permutations.
  Changing the order that candidates appear on the ballot ideally shouldn't change the outcome.

Big representation theory theorems

Maschke's Theorem: All S_n modules decompose uniquely(ish) into simple modules.

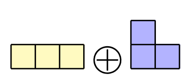
Schur's Lemma: If $\varphi : M \to N$ is a S_n-module homormophism, then on each simple piece of M, φ is either an isomorphism (is bijective) or trivial (sends everything to 0).

Example: The space of possible votes in a 3-way race with full-rankings is



Kernel $\oplus$ Borda $\oplus$ Reversal $\oplus$ Condorcet

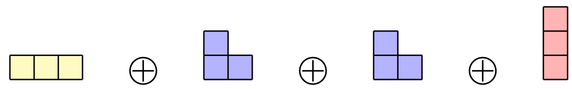
The space of possible outcomes in a 3-way race is



Ties $\oplus$ Relative positions

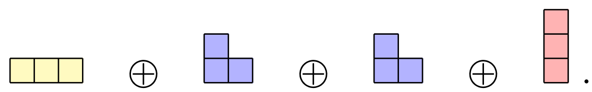
First result: If you decide a winner based on a points system, Condorcet cycles get lost in the tally.

Example: The space of possible votes in a 3-way race with full-rankings is

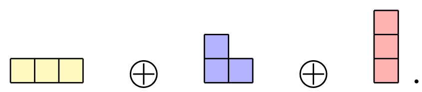


Kernel $\oplus$ Borda $\oplus$ Reversal $\oplus$ Condorcet

The space of possible outcomes of a pairwise comparison (e.g. how many times is $A > B$, how many times is $B > A$, ...) in a 3-way race is

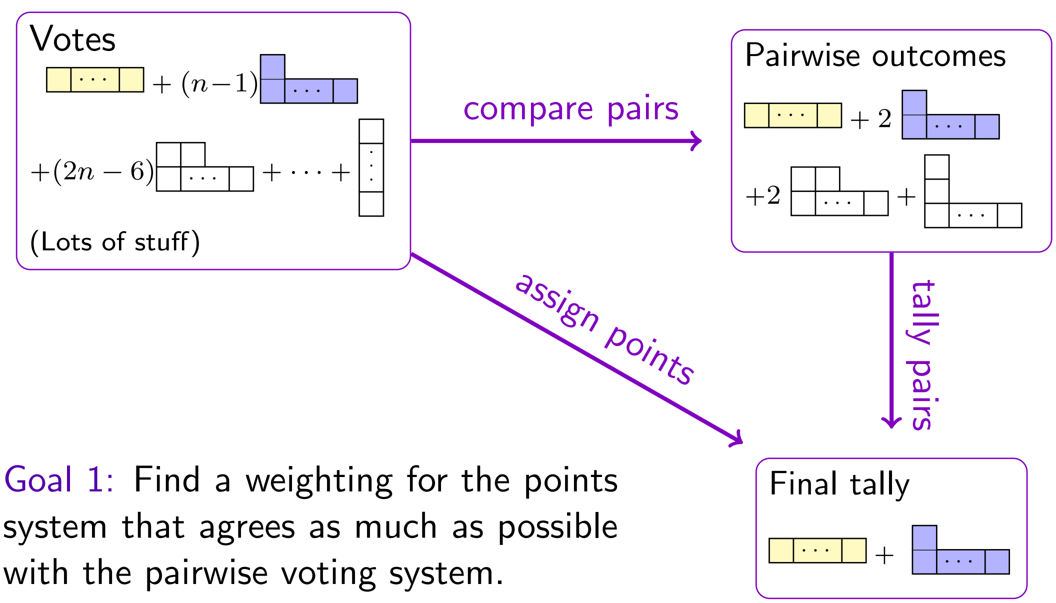


But by further analysis, one can compute that the image is at most



In fact, the information lost is precisely the "reversal" space.

Collect full rankings of preferences for n candidates...



Goal 1: Find a weighting for the points system that agrees as much as possible with the pairwise voting system.

Result 1 (Saari, **D.**): The point-based system that agrees most with the pairwise outcomes is the (modified) Borda count, with weight

$$\mathbf{w} = (n-1,\ n-2, \ldots,\ 1,\ 0)$$

(as expected).

Other results...

Partial rankings (**D.** '05): Ask voters to rank their top k choices (more practical if there are *many* choices). Then the points system that agrees most with the pairwise outcomes is the natural analogue to the Borda count, with weight

$$n - i \qquad \text{for the } i\text{th candidate } (1 \leqslant i \leqslant k), \text{ and}$$

$$\frac{1}{2}(n-k-1) \qquad \text{for the last } n-k \text{ candidates.}$$

(Average the remaining points amongst the last-place candidates.) For example, if you ask for the top 3 out of 20, the points are $19, 18, 17, 8, \ldots, 8$.

[**D.**, Eustis, Minton, Orrison; '07]: Notes on "Approval voting" (ask voters which candidates they approve of), and "Effective spaces" (what kind of voting profiles influence elections).

Committees (Barcelo, Bernstein, Bockting-Conrad, McNichols, Nyman, Viel; '18): How to tally votes for committees with representation from several departments (using representation theory of $S_m \wr S_n$).

What do we want, and why do we care?

- Mathematicians: More data preservation is better.
- Idealists: Everyone should have a fair say.
- Pragmatists: Simpler voting systems are easier to implement.
- Cynics: Stupid things happen when people, en mass, are forced to game their votes.
- (Over-educated) Conspiracy Theorists: Our voting system is provably about as bad as it can be without *everyone* noticing.
- Kenneth Arrow: No voting system is ideal, so...
- Mathematicians (again): Oh, come *on*!

Some references. . .

- Many many publications of Donal Saari, particularly around 1999–2000. Also, "Decisions and Elections; Explaining the Unexpected", Cambridge University Press, 2001.

- "Voting, the symmetric group, and representation theory", by **D.**, Eustis, Minton, and Orrison. American Mathematical Monthly 116 (2009), no. 8, 667–687.

- "Algebraic voting theory and representations of $S_m \wr S_n$", by Barcelo, Bernstein, Bockting-Conrad, McNichols, Nyman, Viel. (preprint: arXiv:1807.03743)

- "How not to be wrong", by Jordan Ellenberg. Chapter 17: "There is no such thing as public opinion."

And even though our system is non-ideal as is. . .

Go vote tomorrow!
http://vote.nyc.ny.us