Quasisymmetric power sums

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Joint work with
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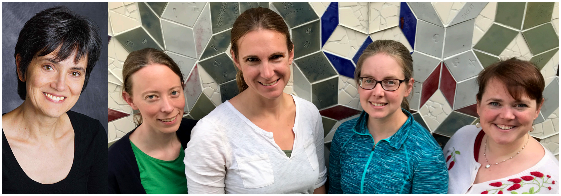
Sarah Mason, and Elizabeth Niese



Some combinatorics

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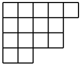 $= (5, 4, 4, 2) = \lambda$

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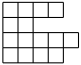 $= (4, 2, 5, 4) = \alpha$

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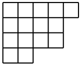 $= (5, 4, 4, 2) = \lambda$

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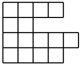 $= (4, 2, 5, 4) = \alpha$

For a composition α,

let $|\alpha|$ be the size (# boxes) of α;

let $\ell(\alpha)$ be the length (# parts) of α; and

let $\tilde{\alpha}$ be the rearrangement of the parts of α into decreasing order.

For example, $|\alpha| = 15$, $\ell(\alpha) = 4$, and $\tilde{\alpha} = \lambda$.

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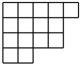 $= (5, 4, 4, 2) = \lambda$

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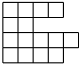 $= (4, 2, 5, 4) = \alpha$

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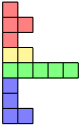
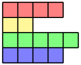
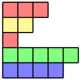
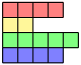
For compositions α and β, we say α refines β, written $\alpha \preccurlyeq \beta$, if β can be built by combining adjacent parts of α. For example,

 $\preccurlyeq$ ,

 $\not\preccurlyeq$ 

Let Sym be the ring of symmetric functions.

Many favorite bases indexed by integer partitions $\lambda \vdash n$:

- Monomial symmetric functions

$$m_\lambda = \sum_{\substack{\tilde{\alpha}=\lambda \\ i_1<i_2<\cdots<i_\ell}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

- Complete homogeneous symmetric functions

$$h_r = \sum_{\substack{|\alpha|=r \\ i_1<i_2<\cdots<i_\ell}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{|\lambda|=r} m_\lambda, \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots .$$

- Elementary symmetric functions

$$e_r = \sum_{1\le i_1<i_2<\cdots<i_r} x_{i_1} \cdots x_{i_r} = m_{(1,1,\ldots,1)} \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots .$$

⋆ Power sum symmetric functions

$$p_r = x_1^r + x_2^r + \cdots = m_{(r)}, \qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

We have a scalar product $\langle , \rangle : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathbb{C}$ defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu},$$

so that the homogeneous and monomial functions are dual.

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$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu},$$

where z_λ is the size of the stabilizer of a permutation of cycle type λ:

$$z_\lambda = \prod_k a_k! k^{a_k}, \quad a_k = \#\{\text{pts of length } k\} \qquad \text{Ex: } z_{\yng(3,3)} = 2!\ 3^2.$$

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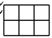
Ex: z  $= 2!\ 3^2$.

Generating functions:

$$H(t) = \sum_{k \geq 0} h_k t^k = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

$$E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i \geq 1} (1 + x_i t)$$

Note $H(t) = 1/E(-t)$.

$$P(t) = \sum_{k \geq 0} p_k t^k = \frac{d}{dt} \ln(H(t)) = \frac{d}{dt} \ln(1/E(-t))$$

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$$\text{if} \quad \mathbf{E}(t) = \sum_{k \geq 0} \mathbf{e}_k t^k \quad \text{and} \quad \mathbf{H}(t) = \sum_{k \geq 0} \mathbf{h}_k t^k,$$

then $\mathbf{H}(t) = 1/\mathbf{E}(-t)$. (Recall: $H(t) = 1/E(-t)$ in Sym).

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⋆ Noncommutative power sums: two choices, ψ and ϕ!

	In Sym:	In NSym:
Type 1:	$P(t) = \frac{d}{dt} \ln(H(t))$	$\frac{d}{dt}\mathbf{H}(t) = \mathbf{H}(t)\mathbf{\Psi}(t)$
Type 2:	$H(t) = \exp\left(\int P(t)dt\right)$	$\mathbf{H}(t) = \exp\left(\int \mathbf{\Phi}(t)dt\right)$

Not the same! (No unique notion of log derivative for power series with noncommutative coefficients.) But

$$\mathcal{A}b(\psi_\alpha) = p_{\tilde{\alpha}} = \mathcal{A}b(\phi_\alpha)$$

The ring of quasisymmetric functions QSym is a subring of $\mathbb{C}[\![x_1, x_2, \dots]\!]$ consisting of series where the coefficients on the monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell} \quad \text{and} \quad x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

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For example,

$$\sum_{i<j} x_i x_j^2 = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \cdots$$

is quasisymmetric but not symmetric (the coef. on $x_1^2 x_2$ is 0).

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$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_{\ell(\alpha)}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \quad \text{so that} \quad m_\lambda = \sum_{\tilde{\alpha} = \lambda} M_\alpha.$$

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Question: What is dual to ψ in QSym? to ϕ?

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Define

$$\Psi_\alpha = z_{\tilde{\alpha}} \psi^*_\alpha, \qquad \text{so that} \qquad \langle \psi_\alpha, \Psi_\beta \rangle = z_{\tilde{\alpha}} \delta_{\alpha\beta}.$$

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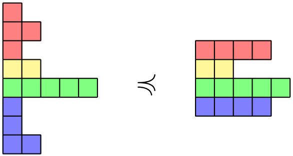

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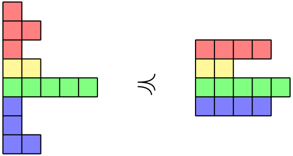
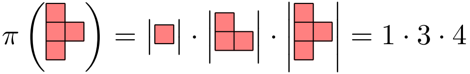


First, for each block, we compute the product of the partial sums:



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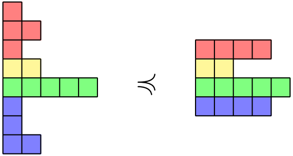

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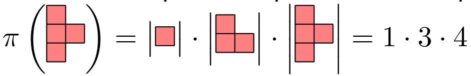

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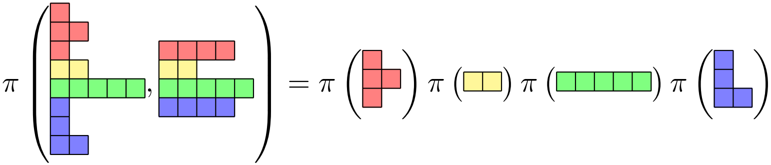

$$= (1 \cdot 3 \cdot 4)(2)(5)(1 \cdot 2 \cdot 4)$$

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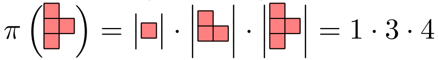

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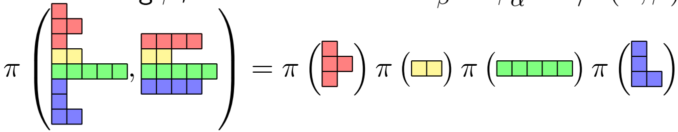
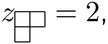
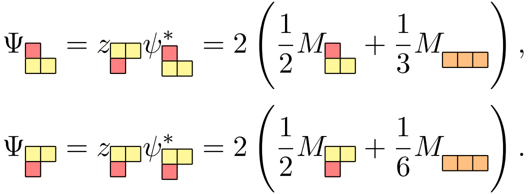


As another example, 



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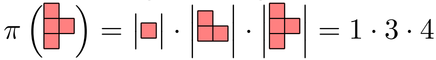

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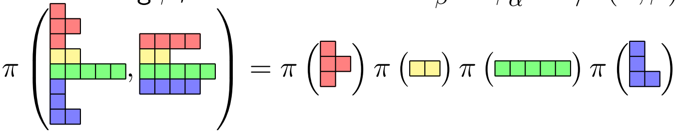
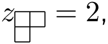
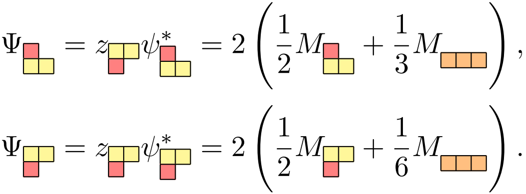


As another example, 



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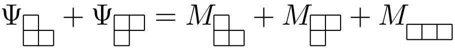

Computing coefficients

As another example, $z_{\ydiagram{2,1}} = 2$,

$$\Psi_{\ydiagram{1,2}} = z_{\ydiagram{2,1}} \psi^*_{\ydiagram{1,2}} = 2\left(\frac{1}{2} M_{\ydiagram{1,2}} + \frac{1}{3} M_{\ydiagram{3}}\right),$$

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$$\Psi_{\ydiagram{1,2}} + \Psi_{\ydiagram{2,1}} = M_{\ydiagram{1,2}} + M_{\ydiagram{2,1}} + M_{\ydiagram{3}}$$

$$= m_{\ydiagram{2,1}} + m_{\ydiagram{3}} = m_{\ydiagram{2}} m_{\ydiagram{1}} = p_2 p_1 = p_{\ydiagram{2,1}}.$$

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Theorem (BDHMN)

Type 1 QSym *power sums refine* Sym *power sums:*

$$p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha.$$

Theorem: $p_\lambda = \sum_{\tilde{\alpha}=\lambda} \Psi_\alpha$, where $\Psi_\alpha = z_{\tilde{\alpha}} \sum_{\alpha \preccurlyeq \beta} \frac{1}{\pi(\alpha,\beta)} M_\beta$.

Proof outline: For compositions α and β, define $\mathcal{O}_{\alpha,\beta}$ be the set of ordered set partitions $(B_1, \cdots, B_{\ell(\beta)})$ of $\{1, \cdots, \ell(\alpha)\}$ satisfying

$$\beta_j = \sum_{i \in B_j} \alpha_i \text{ for } 1 \leq j \leq \ell(\beta).$$

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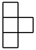
$\alpha =$  and $\beta =$ ,

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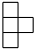
$\alpha =$  and $\beta =$ ,

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It has been shown that

$$p_\lambda = \sum_{\text{part'n } \mu} |\mathcal{O}_{\lambda,\mu}| m_\mu, \quad \text{so that} \quad p_\lambda = \sum_{\text{comp } \beta} |\mathcal{O}_{\lambda,\beta}| M_\beta.$$

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We combinatorially prove, for a fixed partition λ with size n, and a fixed composition β, that

$$|\mathcal{O}_{\lambda\beta}| \frac{n!}{z_\lambda} = \sum_{\substack{\alpha \preccurlyeq \beta \\ \tilde{\alpha}=\lambda}} \frac{n!}{\pi(\alpha,\beta)},$$

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where $S_n^\lambda = \{\sigma \in S_n \text{ of cycle type } \lambda\}$.

Two ways of thinking about permutations:

- In one-line notation:

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is the permutation sending

$$1 \mapsto 5,\ 2 \mapsto 7,\ 3 \mapsto 1,\ \text{and so on...}$$

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largest element of each cycle is last, and

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  Non-example: $571428369 \quad \rightarrow \quad (5)(7)\|(14)\|(2)(836)(9)$

$$\mathrm{Cons}_{(1,2,1)\preccurlyeq(1,2,1)} = \{1234, 1243, 1342, 2134, 2143, 2341, 3124, 3142, 3241, 4123, 4132, 4231\},$$

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Lemma

Fix $\alpha \preccurlyeq \beta$ *of size* n *Then*

$$n! = |\mathrm{Cons}_{\alpha\preccurlyeq\beta}| \cdot \pi(\alpha, \beta).$$

$$\begin{aligned}
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\pi((1,2,1),(1,2,1)) &= 2 \\
\mathrm{Cons}_{(1,2,1)\preccurlyeq(1,3)} &= \{1234, 2134, 3124, 4123\}, \\
\pi((1,2,1),(1,3)) &= 2 \cdot 3 \\
\mathrm{Cons}_{(1,2,1)\preccurlyeq(3,1)} &= \{1234, 1243, 1342, 2134, 2143, 2341, 3142, 3241\}, \\
\pi((1,2,1),(3,1)) &= 1 \cdot 3 \\
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so that $|A_{\alpha\preccurlyeq\beta}| = \pi(\alpha, \beta)$. Then there is a bijection

$$S_n \to \mathrm{Cons}_{\alpha\preccurlyeq\beta} \times A_{\alpha\preccurlyeq\beta}.$$

(For each permutation written in one-line notation, “cycle” into consistency with $\alpha \preccurlyeq \beta$, and record cycles with elements of $A_{\alpha\preccurlyeq\beta}$. See slides from “Discrete Math Day” October 2017.)

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so that

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as desired.

Type 2

In Sym the power sum basis is (essentially) self-dual:

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}.$$

In NSym, the type 2 power sum basis is defined by the generating function relation

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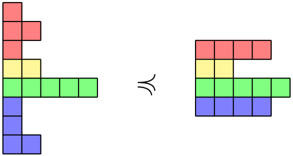
Define

$$\Phi_\alpha = z_{\tilde{\alpha}} \phi^*_\alpha, \quad \text{so that} \quad \langle \phi_\alpha, \Phi_\beta \rangle = z_\alpha \delta_{\alpha\beta}.$$

Computing coefficients

$$\Phi_\alpha = z_{\tilde{\alpha}} \sum_{\beta \succcurlyeq \alpha} \frac{1}{\mathrm{sp}(\alpha,\beta)} M_\beta .$$

For example, we saw that



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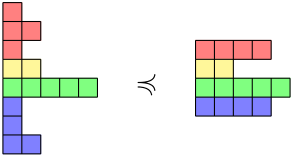
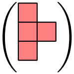


First, for each block, we compute $\mathrm{sp}(\gamma) = \ell(\gamma)! \prod_k \gamma_j$:

sp  $= 3!(1 \cdot 2 \cdot 1)$

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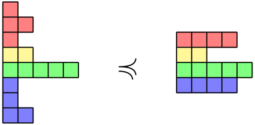

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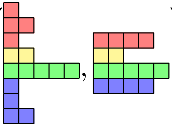
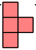
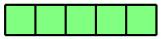
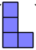
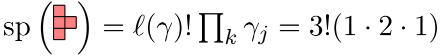
sp () $= 3!(1 \cdot 2 \cdot 1)$

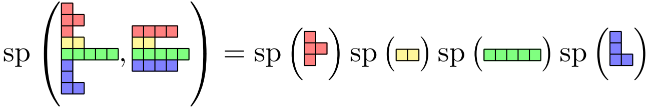
Then, for α refining β, the coefficient of M_β in ψ^*_α is $1/\mathrm{sp}(\alpha, \beta)$, where

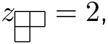
sp () = sp () sp () sp () sp ()

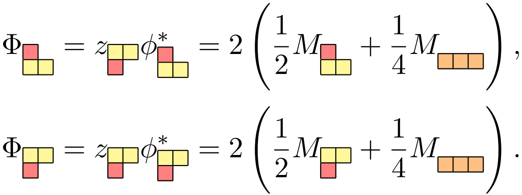
$$= 3!(1 \cdot 2 \cdot 1) \cdot 1!(2) \cdot 1!(5) \cdot 3!(1 \cdot 1 \cdot 2)$$

Computing coefficients





As another example, 



Computing coefficients

$$\mathrm{sp}\left((1,2,1)\right) = \ell(\gamma)!\prod_k \gamma_j = 3!(1\cdot 2\cdot 1)$$

$$\mathrm{sp}\left((1,2,1,2,5,1,1,2),\,(4,2,5,4)\right) = \mathrm{sp}\left((1,2,1)\right)\,\mathrm{sp}\left((2)\right)\,\mathrm{sp}\left((5)\right)\,\mathrm{sp}\left((1,1,2)\right)$$

As another example, $z_{(2,1)} = 2$,

$$\Phi_{(1,2)} = z_{(2,1)}\phi^*_{(1,2)} = 2\left(\frac{1}{2}M_{(1,2)} + \frac{1}{4}M_{(3)}\right),$$

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$$\Phi_{(1,2)} + \Phi_{(2,1)} = M_{(1,2)} + M_{(2,1)} + M_{(3)}$$

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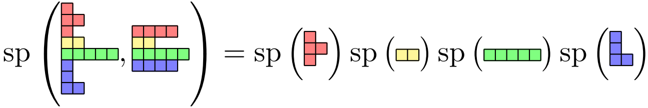

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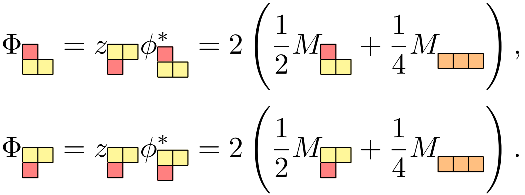

So

$$\Phi_{(1,2)} + \Phi_{(2,1)} = M_{(1,2)} + M_{(2,1)} + M_{(3)}$$

$$= m_{(2,1)} + m_{(3)} = m_{(2)} m_{(1)} = p_2 p_1 = p_{(2,1)}.$$

Computing coefficients

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Theorem (BDHMN)

Type 2 QSym *power sums refine* Sym *power sums:*

$$p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Phi_\alpha.$$

Other results

We also give

- Transition matrices between type 1 and type 2, and to fundamentals.
- Combinatorial proofs of product formulas for type 1 and 2:

$$\Psi_\alpha \Psi_\beta = \frac{z_\alpha z_\beta}{z_{\alpha\cdot\beta}} \sum_{\gamma \in \alpha ⧢ \beta} \Psi_{\mathrm{wd}(\gamma)}, \quad \Phi_\alpha \Phi_\beta = \frac{z_\alpha z_\beta}{z_{\alpha\cdot\beta}} \sum_{\gamma \in \alpha ⧢ \beta} \Phi_{\mathrm{wd}(\gamma)}.$$

- Some comments about plethysm, and other algebraic connections.

See arXiv:1710.11613 for more details.

Thanks!