The affine signed Brauer algebra

Zajj Daugherty
Joint with M. Balagovic, I. Entova-Aizenbud, I. Halacheva, J. Hennig, M. S. Im, G. Letzter, E. Norton, V. Serganova, and C. Stroppel

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Lie superalgebras

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a super Lie bracket

$$[,]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

satisfying

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$$

and

$$[x, [y, z]]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]],$$

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Let $V = V_0 \oplus V_1 = \mathbb{C}^{m|n}$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$. For (homogeneous) $v \in V_i$, write $\bar{v} = i$ for its degree.

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$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\middle|\; A \in \mathrm{End}(V_0), D \in \mathrm{End}(V_1) \right\},$$

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Bracket: $[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx$.

Lie superalgebras

Let $\beta : V \otimes V \to \mathbb{C}$ be an odd, nondegenerate, homogeneous, bilinear form satisfying

$$\beta(v,w) = (-1)^{\bar{v}\bar{w}}\beta(w,v) \quad \text{(supersymmetric)}.$$

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Goal: Study the representation theory of $\mathfrak{p}(n)$. In particular, study the category $\mathcal{F}_n$ of finite-dimensional integrable representations. Highest weight category!

Translation functors

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Namely, you study the action of $\mathcal{U}\mathfrak{g}$ on

$$M \otimes V \otimes V \otimes \cdots \otimes V = M \otimes V^{\otimes d},$$

where V is $\mathfrak{g}$'s favorite module, and M is another simple module, by constructing operators in $\mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$ that commute with the $\mathfrak{g}$-action. Many commuting operators are generated by taking coproducts of central elements.

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Example: If $\mathfrak{g} = \mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, then the commuting operators generate the degenerate affine Brauer algebra; when $\mathfrak{g} = \mathfrak{sl}(V)$, you get the graded Hecke algebra of type A.

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Example: $V \otimes V$

The algebra $\mathrm{End}_{\mathfrak{p}(V)}(V \otimes V)$ is 3-dimensional with basis 1,

$$s : v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v, \quad \text{and} \quad e = \beta^* \circ \beta : v \otimes w \mapsto \beta(v, w)c,$$

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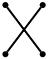
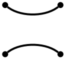
$s =$  and $e =$  .

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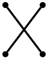
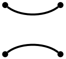
$s =$  and $e =$  .

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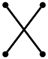
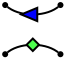
$s =$  and $e =$  . (marked Brauer)

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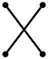
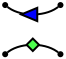
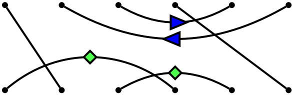
$s =$  and $e =$  . (marked Brauer)

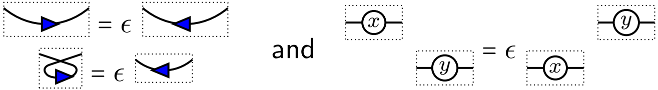
Relation: $e \circ s = e = -s \circ e$. Also, $e^2 = 0$. (non-semisimple case)

(Moon 2003, Kujawa-Tharp 2014) The marked Brauer algebra $B_d(\delta, \epsilon)$, $\epsilon = \pm 1$, is the space spanned by marked Brauer diagrams

$d =$ 

caps get one ◇ each,
cups get one ▶ or ◀ each,
no two markings at same height.

with equivalence up to isotopy except for the local relations



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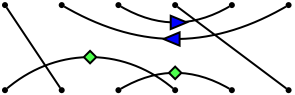
$d =$ 

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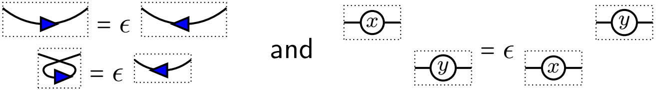

for any adjacent markings $\textcircled{x}$ and $\textcircled{y}$ (meaning no markings of height between these two). Multiplication is given by vertical concatenation, with relations  $= \delta$,

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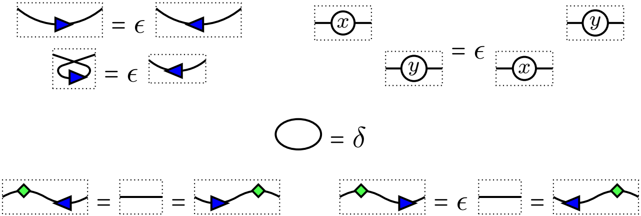

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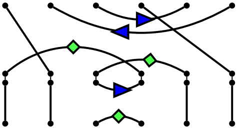

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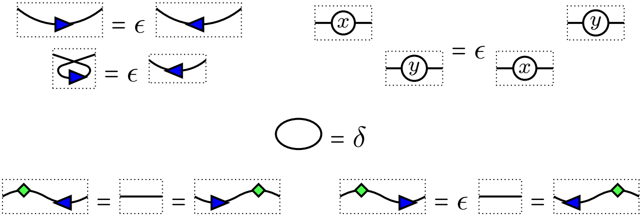

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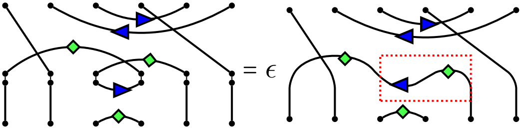

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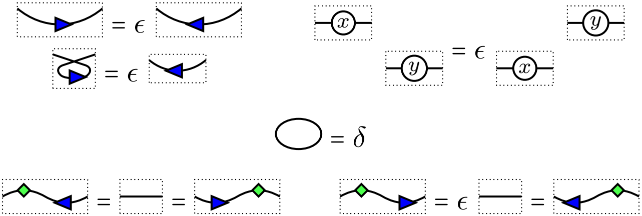

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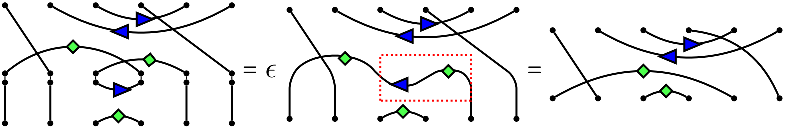

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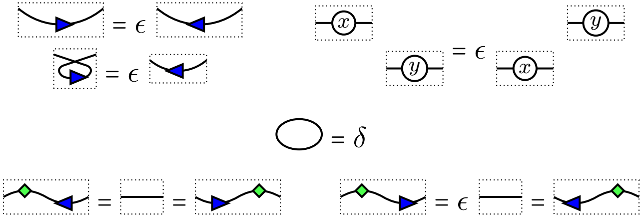

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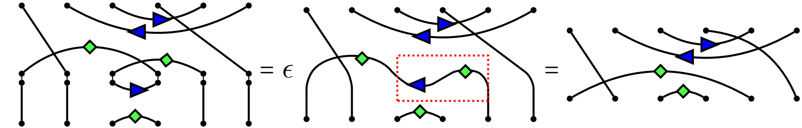


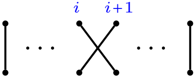
Note:

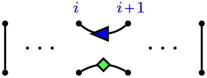
(1) $B_d(\delta, 1)$ = classical Brauer.

(2) If $\epsilon = -1$, then multiplication is well-defined exactly when $\delta = 0$. This case is called the signed Brauer algebra.

Centralizer algebras

The marked Brauer algebra $B_d(\delta, \epsilon)$ is generated by

$s_i =$ 

and $e_i =$ 

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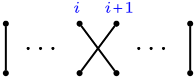
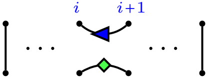
$s_i =$ 

and $e_i =$ 

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Let $\mathfrak{g} = \mathfrak{gl}(V)^\beta$ (β-invariants) with $V = V_0 \oplus V_1$ (for β non-deg, homog, super symmetric, bilinear).

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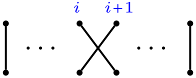
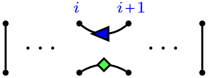
$s_i =$ 

and $e_i =$ 

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for $i = 1, \dots, k-1$.

Let $\mathfrak{g} = \mathfrak{gl}(V)^\beta$ (β-invariants) with $V = V_0 \oplus V_1$ (for β non-deg, homog, super symmetric, bilinear). If

$$\beta^* : \mathbb{C} \to V \otimes V \quad \text{and} \quad \begin{aligned} s : V \otimes V &\to V \otimes V \\ u \otimes v &\mapsto (-1)^{\bar{u}\bar{v}} v \otimes u \end{aligned}$$

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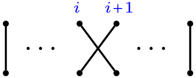
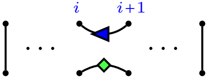
$s_i =$ 

and $e_i =$ 

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then the map

$$e_i \mapsto 1^{\otimes i-1} \otimes \beta^* \beta \otimes 1^{d-i-1}, \quad s_i \mapsto 1^{\otimes i-1} \otimes s \otimes 1^{d-i-1},$$

for $i = 1, \ldots, d-1$, gives

$$B_d(\delta, \epsilon) \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes d})$$

when $\delta = \dim V_0 - \dim V_1$ and $\epsilon = (-1)^{\bar{\beta}}$ [KT14].

Classical: Jucys-Murphy elements and the Casimir

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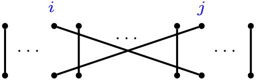
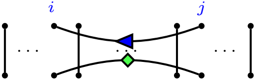
$s_{i,j} =$ 

and $e_{i,j} =$ 

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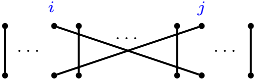
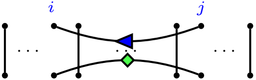
$s_{i,j} =$ 

and $e_{i,j} =$ 

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The Brauer algebra $B_d(\delta) = B_d(\delta, 1)$ has Jucys-Murphy elements

$$x_j = c + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad c \in \mathbb{C},\ j = 1, \ldots, d,$$

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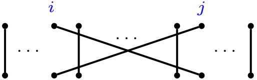
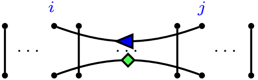
$s_{i,j} =$ 

and $e_{i,j} =$ 

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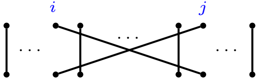
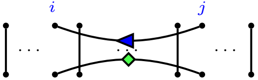
$s_{i,j} =$ 

and $e_{i,j} =$ 

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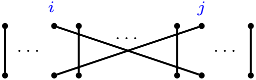
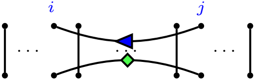
$s_{i,j} =$ 

and $e_{i,j} =$ 

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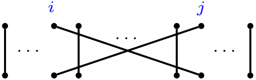
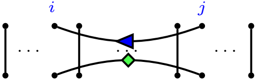
$s_{i,j} =$ 

and $e_{i,j} =$ 

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Classical: Action on $M \otimes V^{\otimes d}$

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Then letting the finite part act on $V^{\otimes d}$ as before, and as the identity on M, we have a representation

$$\mathbb{W}_d(\delta) \to \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d}).$$

Periplectic J-M elements and action on tensor space

But back to periplectic land...

The center is trivial, so there's no Casimir element in $\mathcal{U}\mathfrak{p}(n)$, and there's no split Casimir in $\mathcal{U}\mathfrak{p}(n) \otimes \mathcal{U}\mathfrak{p}(n)$.

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Key observation: With respect to the inner product

$$\langle x, y \rangle = \mathrm{str}(xy) = \mathrm{tr}\left(xy|_{V_0}\right) - \mathrm{tr}\left(xy|_{V_1}\right),$$

on $\mathfrak{gl}(n|n)$, we have

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Then for any $\mathfrak{p}(n)$-module M, Ω acts on $M \otimes V$, and that action commutes with the action of $\mathfrak{p}(n)$. [BDEHHILNSS]

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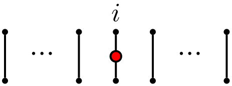
$y_i =$ 

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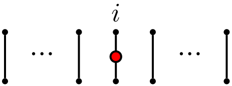
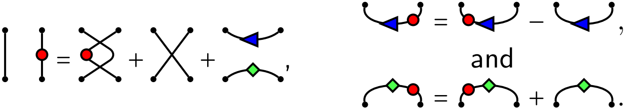
y_i = 

Define the affine signed Brauer algebra $\mathsf{s}\mathbb{W}_d$ as the algebra generated by $B_d(0,-1)$ and $y_1, \ldots, y_d$, together with relations



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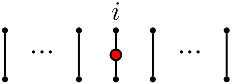
$y_i =$ 

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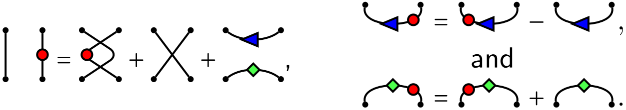

Some results: ([BDEHHILNSS-2])

- Presentation of s$\mathbb{W}_d$ and related algebras/categories.
- Action on tensor space and translation functors.
- Filtrations and specializations.
- Basis and spanning sets.
- Center.

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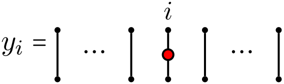

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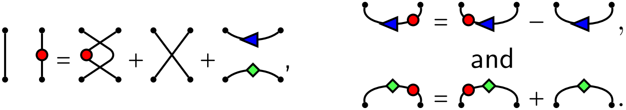

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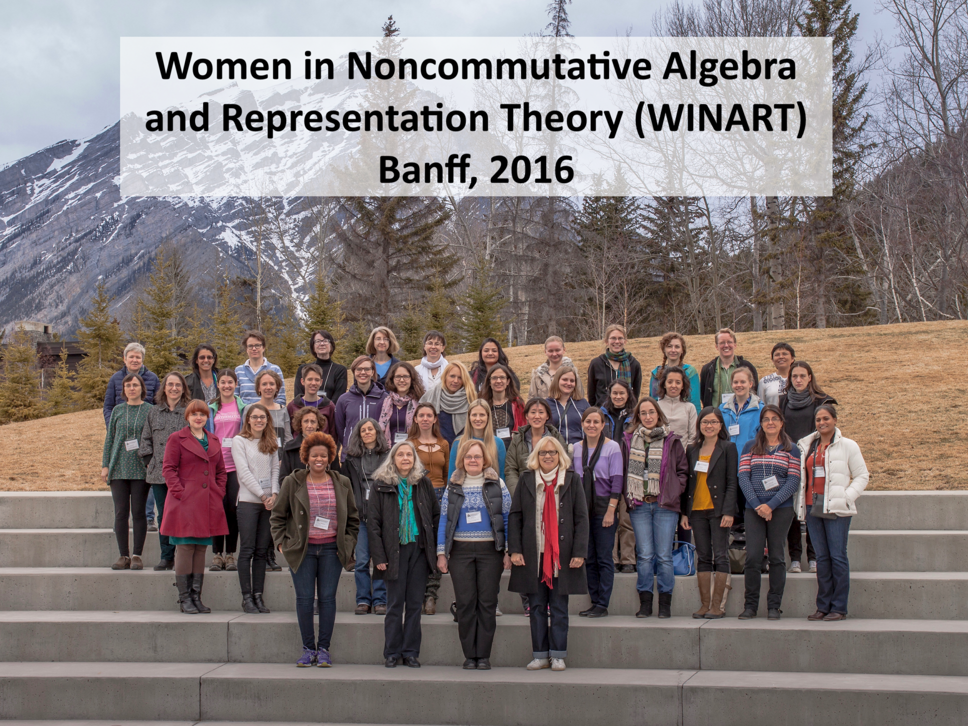
Many other result about the category of finite-dimensional integrable $\mathfrak{p}(V)$-modules itself in [BDEHHILNSS-1] as well!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