Representations of the periplectic Lie superalgebra $\mathfrak{p}(n)$

Zajj Daugherty
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$$[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

satisfying

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$$

and

$$[x, [y, z]]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]],$$

where x, y, z are each homogeneous, and $\bar{x}$ means degree.

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Ex. If β is odd, we need $m = n$. Then $\mathfrak{g} = \mathfrak{p}(V) = \mathfrak{p}(n)$ the periplectic Lie superalgebra.

Periplectic Lie superalgebra

Let $V = V_0 \oplus V_1 = \mathbb{C}^{n|n}$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$.
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Goal: Study the representation theory of $\mathfrak{p}(n)$. In particular, study the category $\mathcal{F}_n$ of finite-dimensional integrable representations.

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Define the thick Kac module (corresponding to λ) as

$$\mathbf{K}(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}}^{\mathfrak{p}(n)} V(\lambda),$$

and the thin Kac module (corresponding to λ) as

$$\mathcal{K}(\lambda) = \mathrm{Coind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{p}(n)} V(\lambda).$$

$(\dim(\mathbf{K}(\lambda)) = 2^n \dim(\mathcal{K}(\lambda)))$

The category $\mathcal{F}_n$ is a highest weight category (in the sense of Cline, Parshall, and Scott), with two natural highest weight structures:

1. With standard modules $\mathbf{K}(\lambda)$, costandard modules $\mathcal{K}(\lambda)$, and partial order on dominant weights given by

$$\mu \geq \lambda \quad \text{whenever} \quad \mu_i \leq \lambda_i \text{ for each } i;$$

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Theorem

(1) *(BGG reciprocity) The multiplicity of $\mathbf{K}(\lambda)$ in $P(\lambda)$ (as summands) is equal to the multiplicity of $L(\lambda)$ in $\mathcal{K}(\lambda)$ (as subquotients), and vice versa (with a shift).*

(2) *If $\mathbf{K}(\mu)$ appears in $P(\lambda)$, then $\mu \geq \lambda$.*

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[Disclaimer: there is a parity-switching involution Π on $\mathfrak{p}(V)$ and all of its modules. Many of these and the following results are technically "M or $\Pi^\ell M$" or "$\Pi^\ell \phi$" in place of an operator ϕ. See paper for details.]

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Key observation: With respect to the inner product

$$\langle x, y\rangle = \mathrm{str}(xy) = \mathrm{tr}\left(xy|_{V_0}\right) - \mathrm{tr}\left(xy|_{V_1}\right),$$

on $\mathfrak{gl}(n|n)$, we have

$$\mathfrak{p}(n)^* = \mathfrak{p}(n)^{\perp} \qquad \text{so} \qquad \mathfrak{gl}(n|n) = \mathfrak{p}(n) \oplus \mathfrak{p}(n)^*.$$

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We define the split (fake) Casimir element of

$$\mathfrak{p}(n) \otimes \mathfrak{p}(n)^* \subset \mathfrak{p}(n) \otimes \mathfrak{gl}(n|n)$$

by

$$\Omega = 2\sum b \otimes b^*,$$

where the sum is over a basis of $\mathfrak{p}(n)$ and its dual in $\mathfrak{p}(n)^*$.

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Moreover, it commutes with the action of $\mathfrak{p}(n)$.

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Let $\{v_i \mid i = 1, \ldots, n\}$ and $\{v_{i'} \mid i = 1, \ldots, n\}$ be orthogonal bases of V_0 and V_1, with $\beta(v_i, v_{j'}) = \delta_{ij}$. Then

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The algebra $\mathrm{End}_{\mathfrak{g}}(V \otimes V)$ is 3-dimensional with basis 1,

$$s : v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v, \quad \text{and} \quad e = \beta^* \circ \beta : v \otimes w \mapsto \beta(v, w) c.$$

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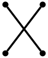
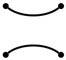
$s =$  and $e =$  .

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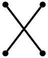
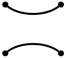
$s =$  and $e =$  .

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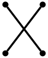
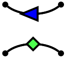
$s =$  and $e =$  . (marked Brauer)

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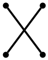
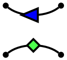
$s =$  and $e =$  . (marked Brauer)

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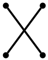
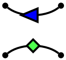
$s =$  and $e =$  . (marked Brauer)

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Then, on $V \otimes V$, we have $\Omega = s + e$. (same as classical case)

Translation functors

Consider the following endofunctor of $\mathcal{F}_n$,

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If

$$y_j = \sum_{i<j} \Omega_{i,j} = (\Delta^j \otimes 1)(\Omega) \otimes 1^{d-j},$$

then $y_1, y_2, \ldots, y_d$ form a commutative family of endomorphisms of

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Theorem

Θ'_k is exact, and is 0 for $k \notin \mathbb{Z}$. So $\Theta' = \bigoplus_{k \in \mathbb{Z}} \Theta'_k$.

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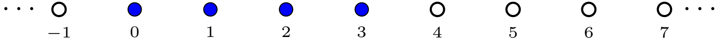

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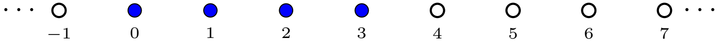

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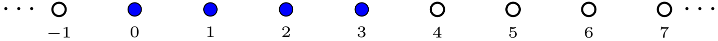

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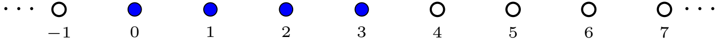

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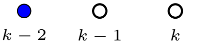

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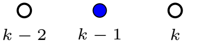

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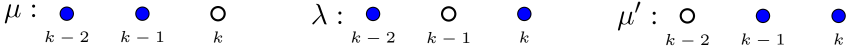

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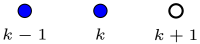
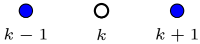
λ : 

μ : 

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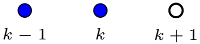

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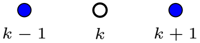

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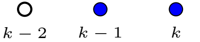

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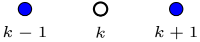

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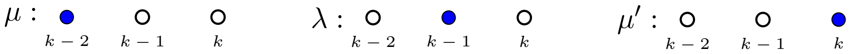

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Up to natural isomorphisms (e.g. on the Grothendeik group), we have

$$\Theta_k^2 = 0, \qquad \Theta_k\Theta_j = \Theta_j\Theta_k, \quad \textit{for } |j-k| > 1,$$
$$\textit{and} \qquad \Theta_k\Theta_{k\pm1}\Theta_k = \Theta_k.$$

Namely, the functors Θ_k generate the infinite Temperley-Lieb algebra $TL_\infty(0)$ (the parameter is $q+q^{-1}$, where $q=i$).

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$$\textit{left adjoint to } \Theta_{k-1} \quad \textit{and} \quad \textit{right adjoint to } \Theta_{k+1}.$$

Theorem

Up to natural isomorphisms (e.g. on the Grothendeik group), we have

$$\Theta_k^2 = 0, \qquad \Theta_k \Theta_j = \Theta_j \Theta_k, \quad \textit{for } |j-k| > 1,$$
$$\textit{and} \qquad \Theta_k \Theta_{k \pm 1} \Theta_k = \Theta_k.$$

Namely, the functors Θ_k generate the infinite Temperley-Lieb algebra $TL_\infty(0)$ (the parameter is $q + q^{-1}$, where $q = i$).

(The maps on the Grothendieck group induced by the Θ'_k's also satisfy these relations. At the level of $\mathcal{F}_n$, though, it's the Θ_k that we want.)

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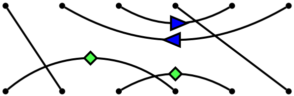
Next paper:

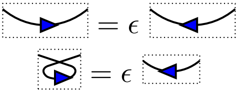
- Cyclotomic marked Brauer algebras.

(Moon 2003, Kujawa-Tharp 2014) The marked Brauer algebra $B_d(\delta, \epsilon)$, $\epsilon = \pm 1$, is the space spanned by marked Brauer diagrams

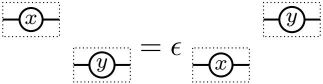
$d =$ 

caps get one ◇ each,
cups get one ▶ or ◀ each,
no two markings at same height.

with equivalence up to isotopy except for the local relations



and



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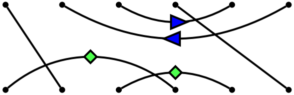
$d =$ 

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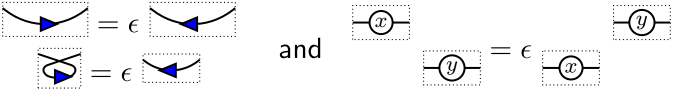

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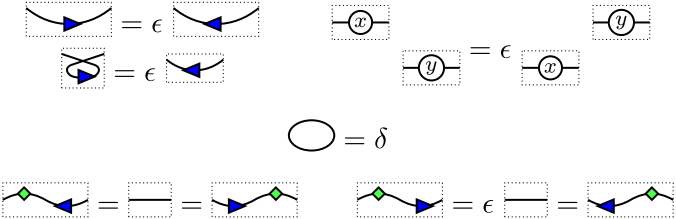
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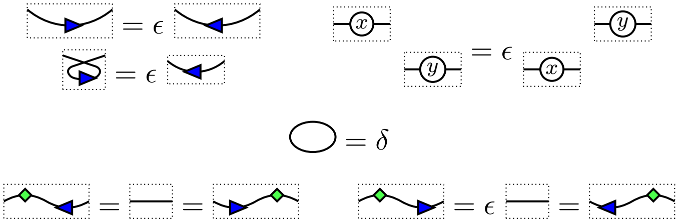

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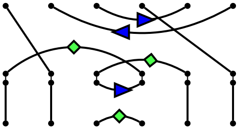

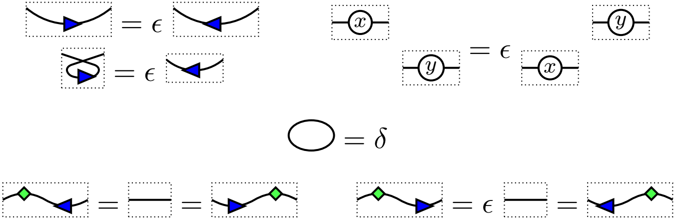

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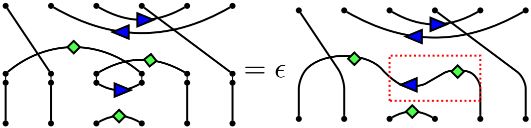

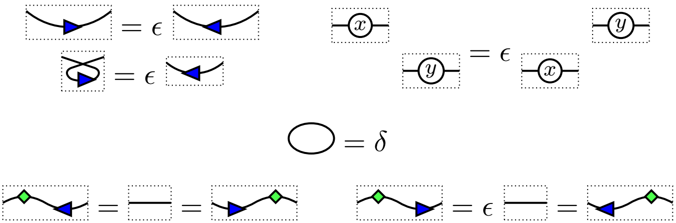

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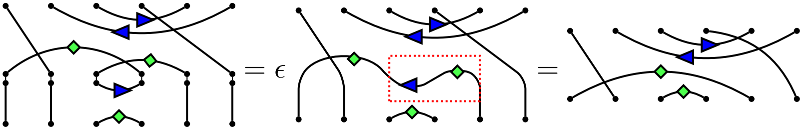

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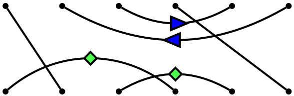
$d =$ 

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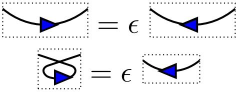

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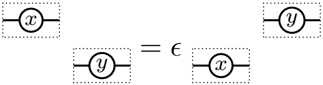

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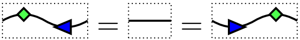

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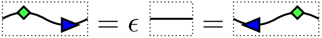
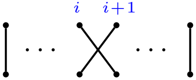


.

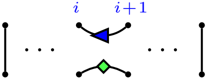
Note:
(1) $B_d(\delta, 1) = B_d(\delta)$.
(2) If $\epsilon = -1$, then multiplication is well-defined exactly when $\delta = 0$.

Centralizer algebras

The marked Brauer algebra $B_d(\delta, \epsilon)$ is generated by

$s_i =$ 

and

$e_i =$ 

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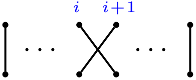
$s_i =$ 

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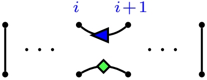
$e_i =$ 

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Back in our more general setting $\mathfrak{g} = \mathfrak{gl}(V)^\beta$ with $V = V_0 \oplus V_1$ (for β non-deg, homog, super symmetric, bilinear).

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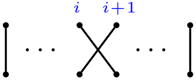
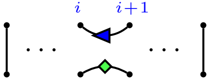
$s_i =$ 

and $e_i =$ 

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$$\beta^* : \mathbb{C} \to V \otimes V \quad \text{and} \quad \begin{aligned} s : V \otimes V &\to V \otimes V \\ u \otimes v &\mapsto (-1)^{\bar{u}\bar{v}} v \otimes u \end{aligned}$$

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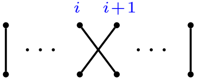
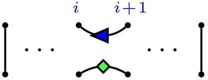
$s_i =$ 

and $e_i =$ 

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then the map

$$e_i \mapsto 1^{\otimes i-1} \otimes \beta^* \beta \otimes 1^{k-i-1}, \quad s_i \mapsto 1^{\otimes i-1} \otimes s \otimes 1^{k-i-1},$$

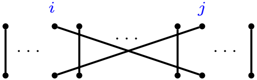
for $i = 1, \ldots, k-1$, gives

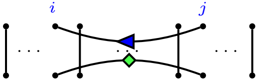
$$B_d(\delta, \epsilon) \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(V^{\otimes d})$$

when $\delta = \mathrm{dim} V_0 - \mathrm{dim} V_1$ and $\epsilon = (-1)^{\bar{\beta}}$ [KT14].

Jucys-Murphy elements and the Casimir

For $i < j$, let

$s_{i,j} =$ 

and $e_{i,j} =$ 

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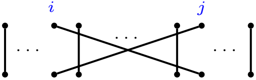
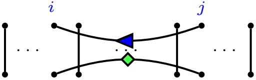
$s_{i,j} =$ 

and $e_{i,j} =$ 

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The Brauer algebra $B_d(\delta) = B_d(\delta, 1)$ has Jucys-Murphy elements

$$x_j = c + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad c \in \mathbb{C},\ j = 1, \ldots, k,$$

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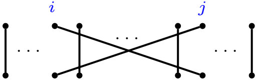
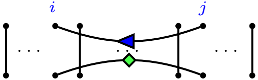
$s_{i,j} =$ 

and $e_{i,j} =$ 

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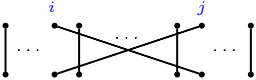
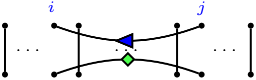
$s_{i,j} =$ 

and $e_{i,j} =$ 

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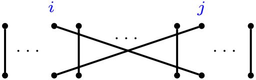
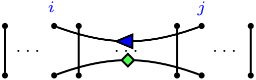
$s_{i,j} =$ 

and $e_{i,j} =$ 

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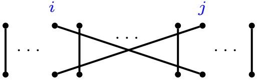
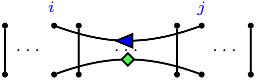
$s_{i,j} =$ 

and $e_{i,j} =$ 

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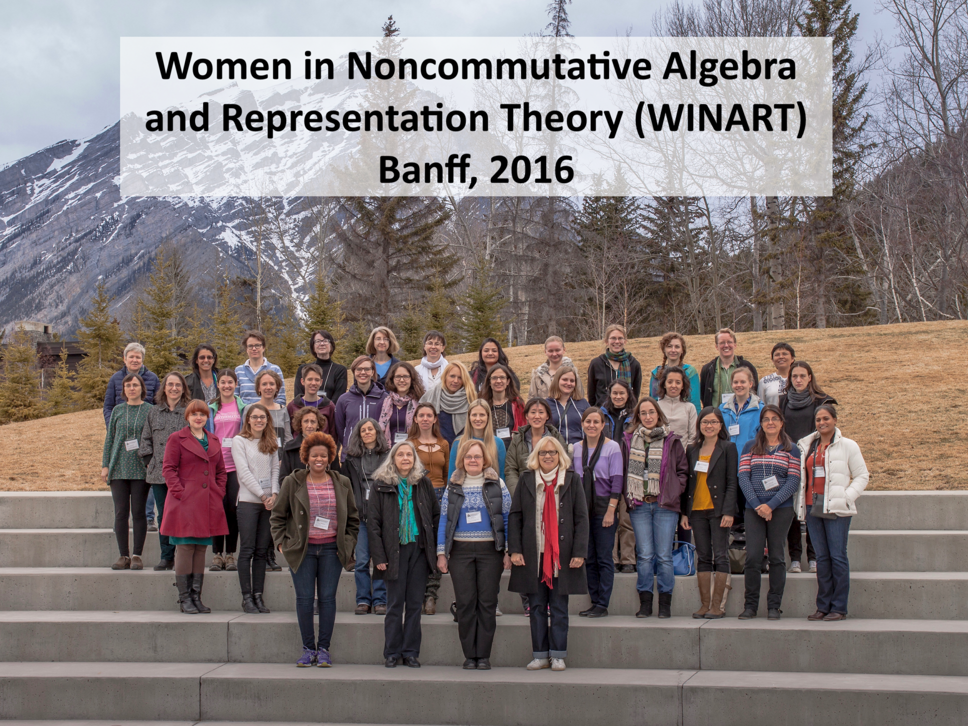
Women in Noncommutative Algebra
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Banff, 2016