Quasisymmetric power sums

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Joint work with
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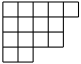
Some combinatorics

Partitions:

 $= (5,4,4,2) = \lambda$

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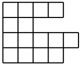 $= (4,2,5,4) = \alpha$

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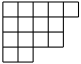 $= (5, 4, 4, 2) = \lambda$

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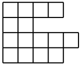 $= (4, 2, 5, 4) = \alpha$

For a composition α,

let $|\alpha|$ be the size (# boxes) of α;

let $\ell(\alpha)$ be the length (# parts) of α; and

let $\tilde{\alpha}$ be the rearrangement of the parts of α into decreasing order.

For example, $|\alpha| = 15$, $\ell(\alpha) = 4$, and $\tilde{\alpha} = \lambda$.

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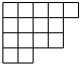 $= (5, 4, 4, 2) = \lambda$

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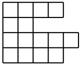 $= (4, 2, 5, 4) = \alpha$

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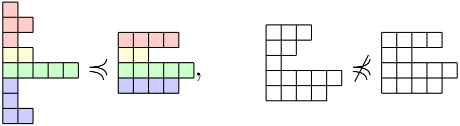
For compositions α and β, we say α refines β, written $\alpha \preccurlyeq \beta$, if β can be built by combining adjacent parts of α. For example,



Symmetric functions

Consider the complex polynomial ring in variables $x_1, x_2, \ldots, x_n$, and let S_n act by permutation of the variables. Then define

$$\mathrm{Sym}_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}.$$

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This is a graded ring, with homogeneous components

$$\mathrm{Sym}_n^k = \{\text{homogeneous } p \in \mathrm{Sym}_n \text{ of deg. } k\}.$$

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Think: symmetric functions in $\mathbb{C}[\![x_1, x_2, \ldots]\!]$.

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Lots of favorite bases: Any basis of Sym can be indexed by integer partitions $\lambda \vdash n$.

Favorite bases of Sym

Monomial symmetric functions:

$$m_\lambda = \sum_{\substack{\tilde{\alpha}=\lambda \\ i_1<i_2<\cdots<i_\ell}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

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Scalar product: $\langle , \rangle : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathbb{C}$ defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu},$$

so that the homogeneous and monomial functions are dual.

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Schur functions:

$$s_\lambda = \sum_{\substack{\text{ss tabl. } T \\ \text{of shape } \lambda}} x^{\mathrm{wt}(T)} = \sum_\mu K_{\lambda\mu} m_\mu,$$

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Note

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \quad \text{and} \quad \langle e_\lambda, \omega(m_\mu) \rangle = \delta_{\lambda,\mu}$$

where ω is the involution on Sym sending $e_\lambda \to h_\lambda$.

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We have

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$$

where z_λ is the size of the stabilizer of a permutation of cycle type λ:

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Generating functions

$$H(t) = \sum_{k\geq 0} h_k t^k = \prod_{i\geq 1} (1 - x_i t)^{-1}$$

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$$P(t) = \sum_{k \geq 0} p_k t^k = \frac{d}{dt} \ln(H(t)) = \frac{d}{dt} \ln(1/E(-t))$$

Variations on Sym

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Think: The elementary symmetric functions $e_1, e_2, \ldots$ generate Sym, and, aside from commuting, are algebraically independent. Now, we're lifting to an algebra where the elementary functions no longer commute. So the abelianization

$$\mathcal{A}b : \mathrm{NSym} \to \mathrm{Sym}$$

is surjective (with kernel generated by commutators).

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- Noncom. power sums: two choices, ψ and ϕ!

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then $\mathbf{H}(t) = 1/\mathbf{E}(-t)$. (Recall: $H(t) = 1/E(-t)$ in Sym).

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- Noncom. power sums: two choices, ψ and ϕ!

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Not the same! (No unique notion of log derivative for power series with noncommutative coefficients.) But

$$\mathcal{A}b(\psi_\alpha) = p_{\tilde{\alpha}} = \mathcal{A}b(\phi_\alpha)$$

Variations on Sym

The ring of quasisymmetric functions QSym is a subring of $\mathbb{C}[\![x_1, x_2, \dots]\!]$ consisting of series where the coefficients on the monomials

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Extending linearly gives a natural surjective map $\mathrm{QSym} \to \mathrm{Sym}$.

Dual Hopf algebras

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Question: What is dual to ψ? to ϕ?

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Define

$$\Psi_\alpha = z_{\tilde{\alpha}} \psi^*_\alpha, \quad \text{so that} \quad \langle \psi_\alpha, \Psi_\beta \rangle = z_\alpha \delta_{\alpha\beta}.$$

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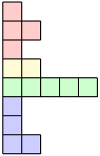

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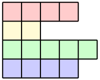

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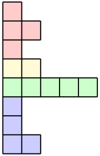

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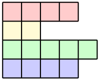
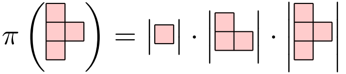


First, for each block, we compute the product of the partial sums:



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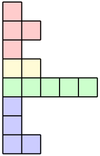

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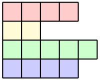

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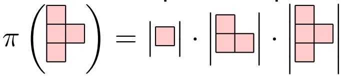

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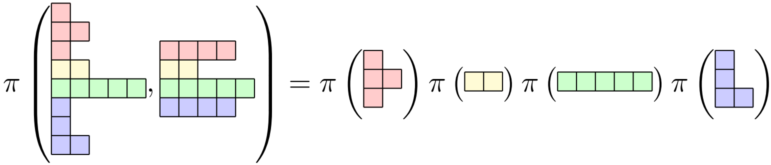
Then, for α refining β, the coefficient of M_β in ψ^*_α is $1/\pi(\alpha, \beta)$, where



$$= (1 \cdot 3 \cdot 4)(2)(5)(1 \cdot 2 \cdot 4)$$

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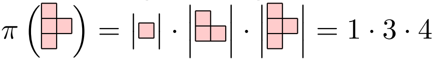

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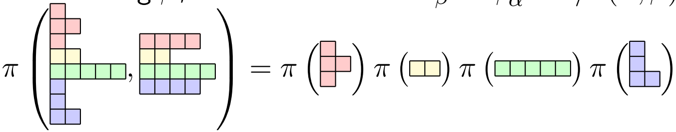
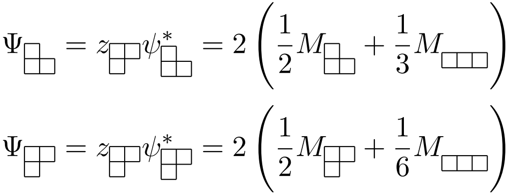


As another example, z $= 2$,



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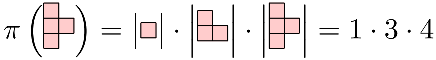

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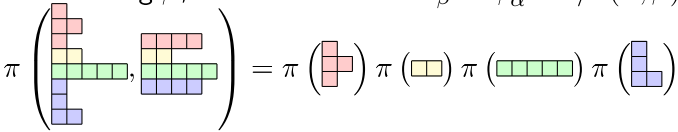

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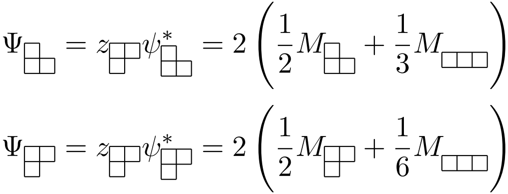

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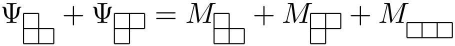

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Theorem (BDHMN)

Type 1 QSym *powers sum to* Sym *powers:*

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Theorem: $p_\lambda = \sum_{\tilde{\alpha}=\lambda} \Psi_\alpha, \quad$ where $\quad \Psi_\alpha = z_{\tilde{\alpha}} \sum_{\alpha \preccurlyeq \beta} \frac{1}{\pi(\alpha,\beta)} M_\beta.$

Proof outline: For compositions α and β, define $\mathcal{O}_{\alpha,\beta}$ be the set of ordered set partitions $(B_1, \cdots, B_{\ell(\beta)})$ of $\{1, \cdots, \ell(\alpha)\}$ satisfying

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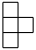
$\alpha =$  and $\beta =$ ,

then $\mathcal{O}_{\alpha,\beta}$ contains $(\{1,3\},\{2\})$ and $(\{2\},\{1,3\})$.

Theorem: $p_\lambda = \sum_{\tilde{\alpha}=\lambda} \Psi_\alpha$, where $\Psi_\alpha = z_{\tilde{\alpha}} \sum_{\alpha \preccurlyeq \beta} \frac{1}{\pi(\alpha,\beta)} M_\beta$.

Proof outline: For compositions α and β, define $\mathcal{O}_{\alpha,\beta}$ be the set of ordered set partitions $(B_1, \cdots, B_{\ell(\beta)})$ of $\{1, \cdots, \ell(\alpha)\}$ satisfying

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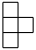
For example, if

$\alpha =$  and $\beta =$ ,

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It has been shown that

$$p_\lambda = \sum_{\text{part'n } \mu} |\mathcal{O}_{\lambda,\mu}| m_\mu, \quad \text{so that} \quad p_\lambda = \sum_{\text{comp } \beta} |\mathcal{O}_{\lambda,\beta}| M_\beta.$$

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where $S_n^\lambda = \{\sigma \in S_n \text{ of cycle type } \lambda\}$.

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is the permutation sending

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  Non-example: $571428369 \quad \rightarrow \quad (5)(7)\|(14)\|(2)(836)(9)$

$$\begin{aligned}\mathrm{Cons}_{(1,2,1)\preccurlyeq(1,2,1)} &= \{1234, 1243, 1342, 2134, 2143, 2341, 3124,\\ &\qquad 3142, 3241, 4123, 4132, 4231\},\end{aligned}$$

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Lemma

Fix $\alpha \preccurlyeq \beta$ of size n Then

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Example: $\alpha = (2, 3, 2, 2)$, $\beta = (5, 4)$, $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

Split σ according to β: $\underbrace{73962}_{\sigma^{(1)}} \| \underbrace{8451}_{\sigma^{(1)}}$

For each i, "rotate" $\sigma^{(i)}$ into consistency with to $\alpha \preccurlyeq \beta$, and record rotations. . .

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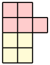
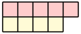
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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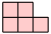
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

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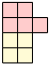
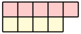
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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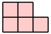
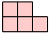
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block:

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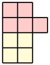
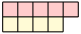
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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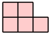
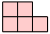
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73\textcolor{red}{9}62$

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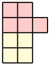
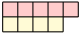
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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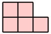
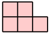
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73962 \xrightarrow{\text{rotate left by 3}} 62739$, $s_2^{(1)} = 3$

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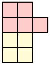
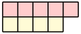
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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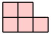
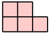
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73{\color{red}9}62 \xrightarrow{\text{rotate left by 3}} 6273{\color{red}9}$, $s_2^{(1)} = 3$

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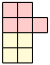
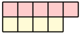
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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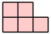
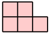
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73{\color{red}9}62 \xrightarrow{\text{rotate left by 3}} 6273{\color{red}9}$, $s_2^{(1)} = 3$

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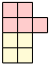
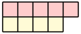
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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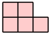
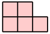
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73962 \xrightarrow{\text{rotate left by 3}} 62739$, $s_2^{(1)} = 3$

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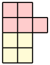
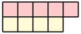
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ $(\in \mathrm{Cons}_{\alpha\preccurlyeq\beta})$.

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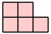
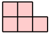
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73962 \xrightarrow{\text{rotate left by 3}} 62739$, $s_2^{(1)} = 3$

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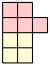
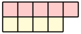
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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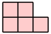
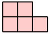
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73962 \xrightarrow{\text{rotate left by 3}} 62739$, $s_2^{(1)} = 3$

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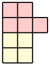
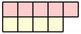
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha\preccurlyeq\beta}$).

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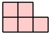
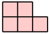
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73{\color{red}9}62 \xrightarrow{\text{rotate left by 3}} 6273{\color{red}9}$, $s_2^{(1)} = 3$

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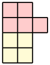
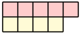
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ $(\in \mathrm{Cons}_{\alpha \preccurlyeq \beta})$.

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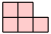
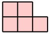
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73{\color{red}9}62 \xrightarrow{\text{rotate left by 3}} 6273{\color{red}9}$, $s_2^{(1)} = 3$

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 block: ${\color{red}8}451 \xrightarrow{\text{rotate left by 1}} 451{\color{red}8}$, $s_2^{(2)} = 1$

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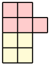
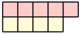
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha\preccurlyeq\beta}$).

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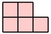
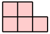
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73{\color{red}9}62 \xrightarrow{\text{rotate left by 3}} 6273{\color{red}9}$, $s_2^{(1)} = 3$

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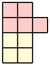
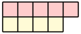
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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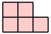
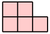
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73962 \xrightarrow{\text{rotate left by 3}} 62739$, $s_2^{(1)} = 3$

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 block: $45|18$

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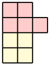
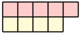
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha\preccurlyeq\beta}$).

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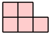
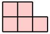
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73{\color{red}9}62 \xrightarrow{\text{rotate left by 3}} 6273{\color{red}9}$, $s_2^{(1)} = 3$

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 block: $4{\color{red}5}|18 \xrightarrow{\text{rotate left by 0}} 4{\color{red}5}|18$

Then there is a bijection

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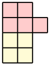
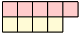
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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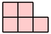
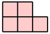
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73{\color{red}9}62 \xrightarrow{\text{rotate left by 3}} 6273{\color{red}9}$, $s_2^{(1)} = 3$

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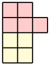
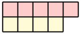
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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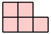
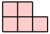
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73962 \xrightarrow{\text{rotate left by 3}} 62739$, $s_2^{(1)} = 3$

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 block: $45|18 \xrightarrow{\text{rotate left by 0}} 45|18$, $s_1^{(2)} = 0$

So $739628451 \mapsto (267394518, ((2,3),(0,1)))$.

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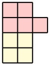
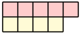
Example: $\alpha =$ , $\beta =$ , $\sigma = 739628451$ ($\in \mathrm{Cons}_{\alpha \preccurlyeq \beta}$).

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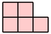
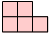
$i = 1$: $\sigma^{(1)} = 73962$, β_1 parts of α: 

 block: $73962 \xrightarrow{\text{rotate left by 3}} 62739$, $s_2^{(1)} = 3$

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So $739628451 \mapsto (267394518, ((2,3),(0,1)))$. Invertible!

Lemma

Fix $\alpha \preccurlyeq \beta$ *of size* n *Then*

$$n! = |\mathrm{Cons}_{\alpha \preccurlyeq \beta}| \cdot \pi(\alpha, \beta).$$

Proof: Let

$$A_{\alpha \preccurlyeq \beta} = \bigotimes_{i=1}^{\ell(\beta)} \left(\bigotimes_{j=1}^{\ell(\alpha^{(i)})} \mathbb{Z}/a_j^{(i)}\mathbb{Z} \right), \quad \text{where } a_j^{(i)} = \sum_{r=1}^{j} \alpha_r^{(i)},$$

so that $|A_{\alpha \preccurlyeq \beta}| = \pi(\alpha, \beta)$. Then there is a bijection

$$S_n \to \mathrm{Cons}_{\alpha \preccurlyeq \beta} \times A_{\alpha \preccurlyeq \beta}.$$

□

Lemma

Fix $\alpha \preccurlyeq \beta$ *of size* n *Then*

$$n! = |\mathrm{Cons}_{\alpha \preccurlyeq \beta}| \cdot \pi(\alpha, \beta).$$

Proof: Let

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Lemma

Fix $\alpha \preccurlyeq \beta$ *of size* n *Then*

$$|\mathcal{O}_{\alpha \preccurlyeq \beta}| \cdot |S_n^{\lambda}| = \sum_{\substack{\alpha \preccurlyeq \beta \\ \tilde{\alpha} = \lambda}} |\mathrm{Cons}_{\alpha \preccurlyeq \beta}|.$$

(Similar proof.)

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(Similar proof.)

Therefore

$$|\mathcal{O}_{\lambda\beta}| \cdot |S_n^{\lambda}| = \sum_{\substack{\alpha \preccurlyeq \beta \\ \tilde{\alpha} = \lambda}} \frac{n!}{\pi(\alpha, \beta)},$$

so that

$$p_{\lambda} = \sum_{\text{comp } \beta} |\mathcal{O}_{\lambda,\beta}| M_{\beta} = \sum_{\tilde{\alpha} = \lambda} \Psi_{\alpha}, \quad \text{where} \quad \Psi_{\alpha} = z_{\tilde{\alpha}} \sum_{\alpha \preccurlyeq \beta} \frac{1}{\pi(\alpha, \beta)} M_{\beta},$$

as desired.

Type 2

In Sym the power sum basis is (essentially) self-dual:

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}.$$

In NSym, the type 2 power sum basis is defined by the generating function relation

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Define

$$\Phi_\alpha = z_{\tilde{\alpha}} \phi^*_\alpha, \qquad \text{so that} \qquad \langle \phi_\alpha, \Phi_\beta \rangle = z_\alpha \delta_{\alpha\beta}.$$

Computing coefficients

$$\Phi_\alpha = z_{\tilde{\alpha}} \sum_{\beta \succcurlyeq \alpha} \frac{1}{\mathrm{sp}(\alpha,\beta)} M_\beta.$$

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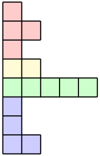
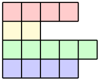
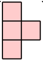


refines



First, for each block, we compute $\mathrm{sp}(\gamma) = \ell(\gamma)! \prod_k \gamma_j$:

$\mathrm{sp}\Big($  $\Big) = 3!(1 \cdot 2 \cdot 1)$

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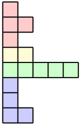
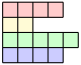
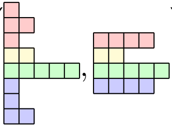
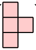
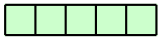
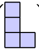
 refines 

First, for each block, we compute $\mathrm{sp}(\gamma) = \ell(\gamma)! \prod_k \gamma_j$:

sp () $= 3!(1 \cdot 2 \cdot 1)$

Then, for α refining β, the coefficient of M_β in ψ^*_α is $1/\mathrm{sp}(\alpha,\beta)$, where

sp () = sp () sp () sp () sp ()

$$= 3!(1 \cdot 2 \cdot 1) \cdot 1!(2) \cdot 1!(5) \cdot 3!(1 \cdot 1 \cdot 2)$$

Computing coefficients

$$\mathrm{sp}\left(\ydiagram{1,2,1}\right) = \ell(\gamma)!\prod_k \gamma_j = 3!(1\cdot 2\cdot 1)$$

$$\mathrm{sp}\left(\ydiagram{1,2,1,2,5,1,1,2}, \ydiagram{4,2,5,4}\right) = \mathrm{sp}\left(\ydiagram{1,2,1}\right)\mathrm{sp}\left(\ydiagram{2}\right)\mathrm{sp}\left(\ydiagram{5}\right)\mathrm{sp}\left(\ydiagram{1,1,2}\right)$$

As another example, $z_{\ydiagram{2,1}} = 2$,

$$\Phi_{\ydiagram{1,2}} = z_{\ydiagram{2,1}}\phi^*_{\ydiagram{1,2}} = 2\left(\frac{1}{2}M_{\ydiagram{1,2}} + \frac{1}{4}M_{\ydiagram{3}}\right)$$

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So

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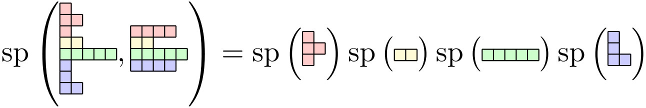

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Theorem (BDHMN)

Type 2 QSym *powers sum to* Sym *powers:*

$$p_\lambda = \sum_{\tilde{\alpha}=\lambda} \Phi_\alpha.$$