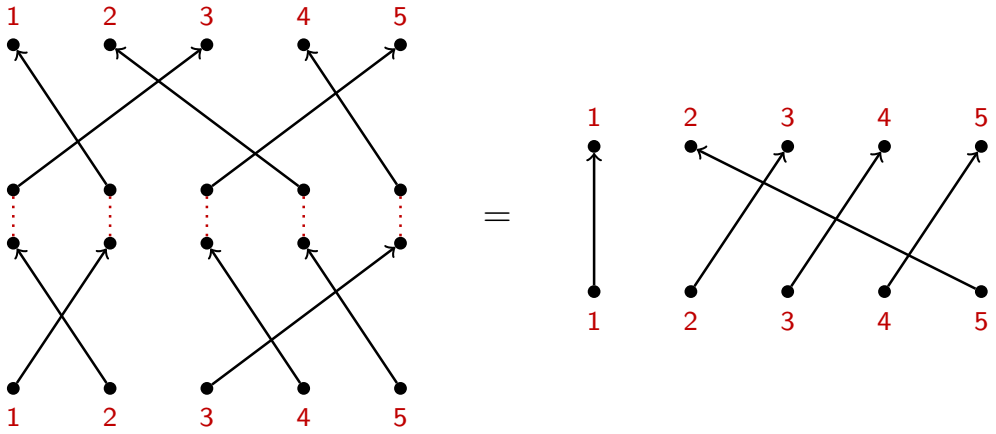
Motivating example: Schur-Weyl Duality

The **symmetric group** S_k (permutations) as diagrams:



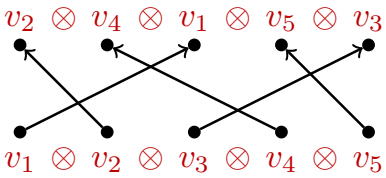
(with multiplication given by concatenation)

Motivating example: Schur-Weyl Duality

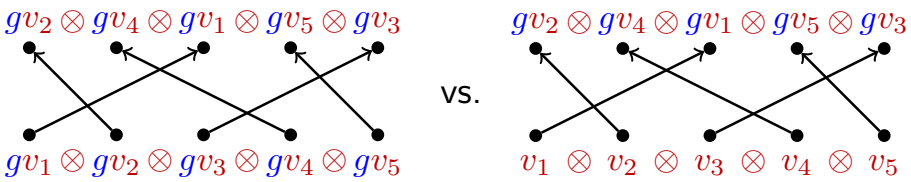
$\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



These actions commute!



Motivating example: Schur-Weyl Duality

Schur (1901):

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\substack{\text{(all linear maps that}\\ \text{commute with } \mathrm{GL}_n)}} = \underbrace{\pi(\mathbb{C}S_k)}_{\substack{\text{(img of } S_k\\ \text{action)}}} \quad \text{and} \quad \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\substack{\text{(img of } \mathrm{GL}_n\\ \text{action)}}}.$$

Powerful consequence: a duality between representations

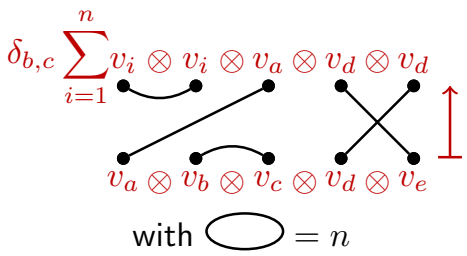
The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

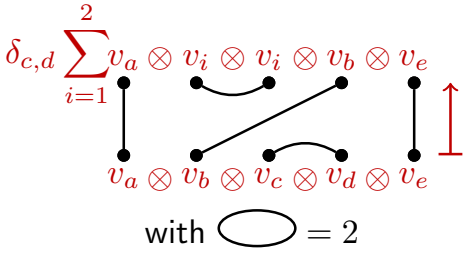
where G^λ are distinct irreducible GL_n-modules
S^λ are distinct irreducible S_k-modules

More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize the **Brauer algebra**:



Temperley-Lieb (1971)
GL_2 and SL_2 (and $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$) acting on $(\mathbb{C}^2)^{\otimes k}$ diagonally centralize the **Temperley-Lieb algebra**:



(Diagrams encode maps $V^{\otimes k} \to V^{\otimes k}$ that commute with the action of some classical algebra.)

Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q\mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

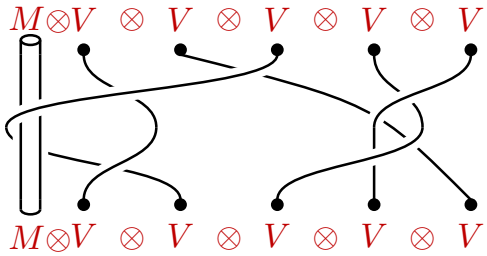
$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\check{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$

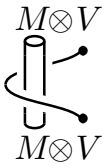


that (1) satisfies braid relations, and
(2) commutes with the $\mathcal{U}$-action on $V \otimes W$
for any $\mathcal{U}$-module V.

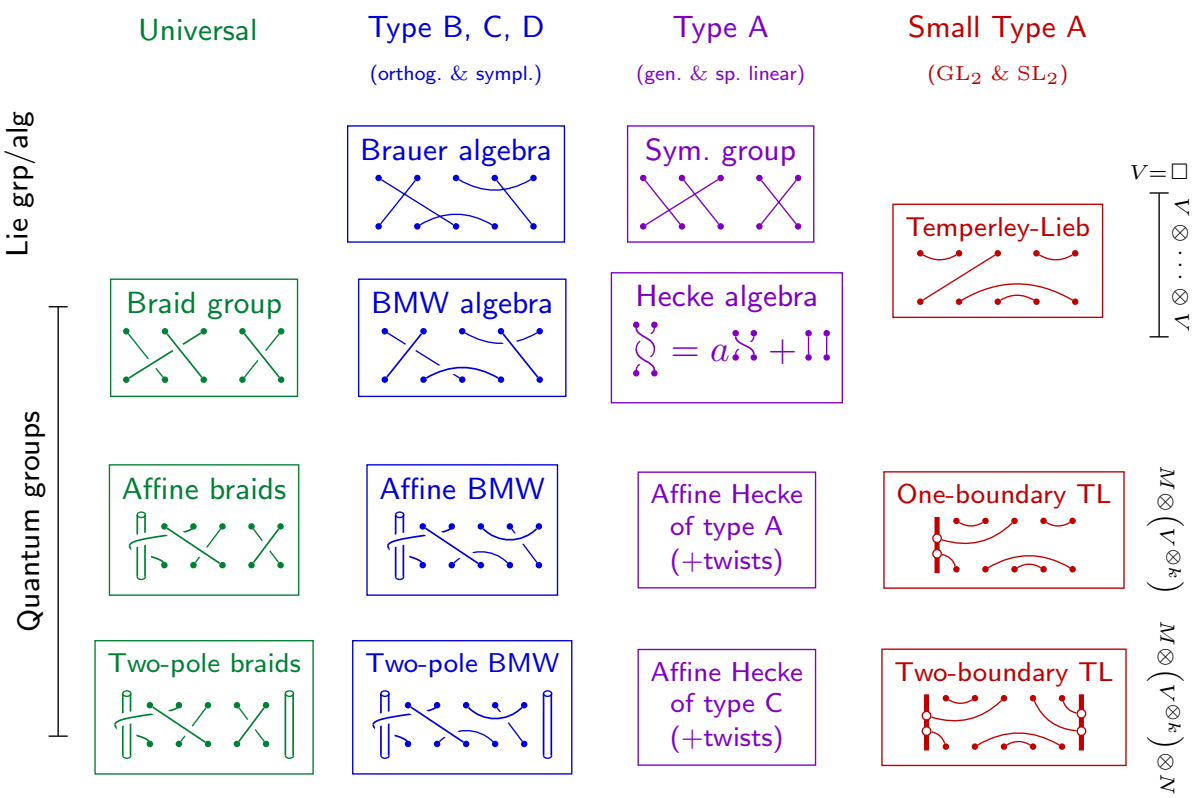
The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k}$:



Around the pole:

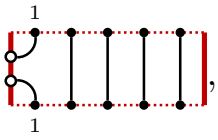


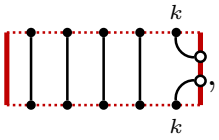
$= \check{R}_{MV}\check{R}_{VM}$

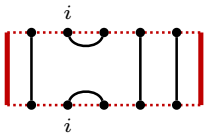


Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The *two-boundary Temperley-Lieb algebra* TL_k is a diagram algebra generated over $\mathbb{C}$ by diagrams

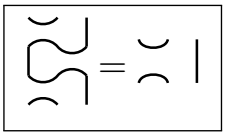
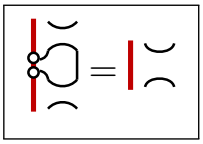
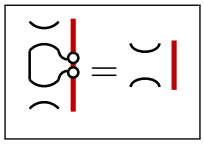
$e_0 =$ 

, $e_k =$ 

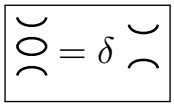
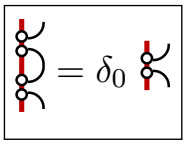
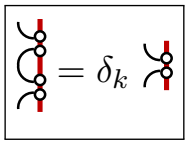
, and $e_i =$ 

for $i = 1, \ldots, k-1$, with relations $e_i e_j = e_j e_i$ for $|i-j| > 1$,

$e_i e_{i\pm1} e_i = e_i$
for $1 \le i \le k-1$,

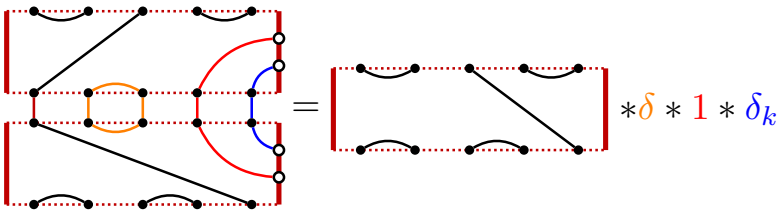
 or  or 

$e_i^2 = \delta_i e_i.$

 or  or 

(Side loops are resolved with a 1 or a δ_i depending on whether there are an even or odd number of connections below their lowest point.)

Diagram multiplication:



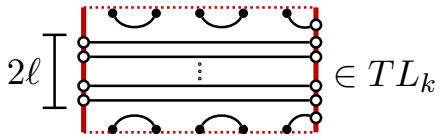
In short, TL_k has basis given by non-crossing diagrams with

(1) k connections to the top and to the bottom,

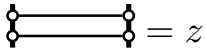
(2) an even number of connections to the right and to the left, and

(3) no edges with both ends on the left or both ends on the right.

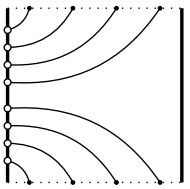
However,

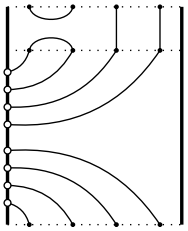
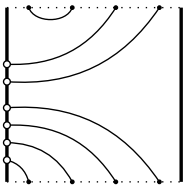


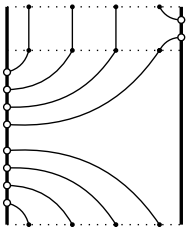
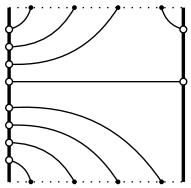
So unlike the classical T-L algebras, TL_k is not finite dimensional! Take quotient giving

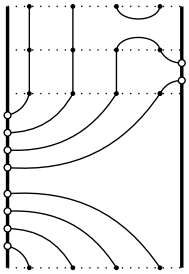
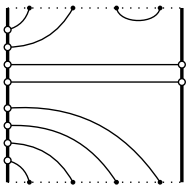
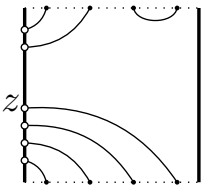


Representation theory of TL_k: action on diagrams

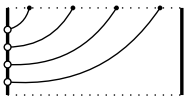
$d =$ 

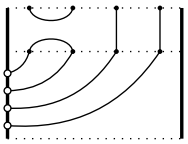
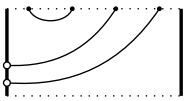
$e_1 d =$  $= \delta_0$ 

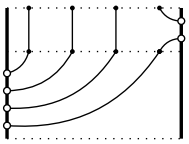
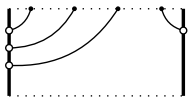
$e_4 d =$  $=$ 

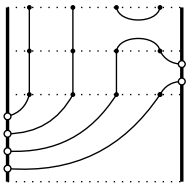
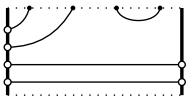
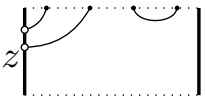
$e_3 e_4 d =$  $=$  $= z$ 

Representation theory of TL_k: half diagrams

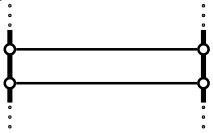
$d =$ 

$e_1 d =$  $= \delta_0$ 

$e_4 d =$  $=$ 

$e_3 e_4 d =$  $=$  $= z$ 

You can tell when to use

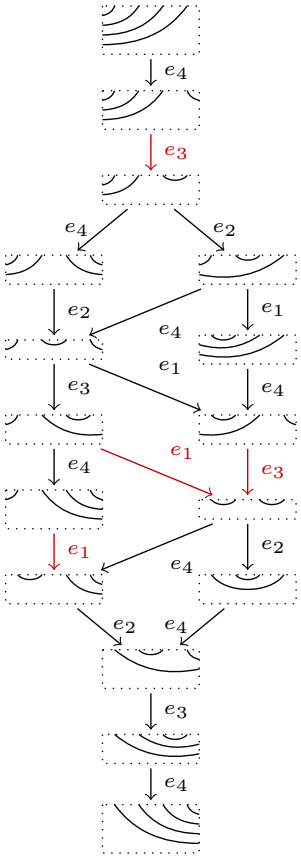
 $= z$

or not by the parity of connections to the left/right walls.

Generic module:

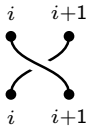
(act by e_i, don't make loops)

Red arrows indicate coef of z.

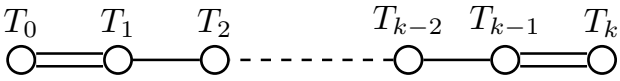


For what z does this module split?

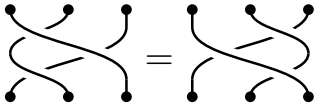
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

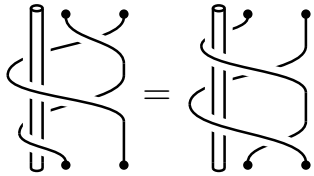
$T_k =$ , $T_0 =$  and $T_i =$  for $1 \leq i \leq k-1$,

subject to relations



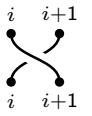
i.e.

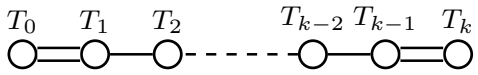
$$T_iT_{i+1}T_i = \text{} = T_{i+1}T_iT_{i+1},$$

$$T_1T_0T_1T_0 = \text{} = T_0T_1T_0T_1,$$

and, similarly, $T_{k-1}T_kT_{k-1}T_k = T_kT_{k-1}T_kT_{k-1}$.

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

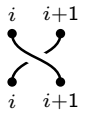
$T_k =$ , $T_0 =$  and $T_i =$  for $1 \leq i \leq k-1$,

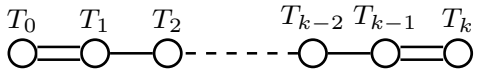
subject to relations .

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$
$$\text{and} \quad (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

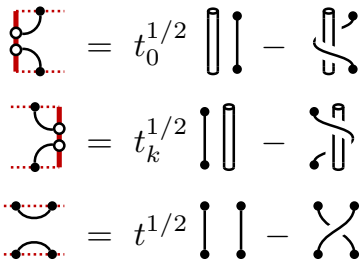
(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$  for $1 \le i \le k-1$,

subject to relations .

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set



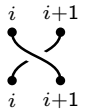
$(e_0 = t_0^{1/2} - T_0)$

$(e_k = t_k^{1/2} - T_k)$

$(e_i = t^{1/2} - T_i)$

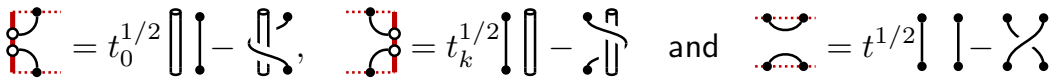
so that $e_j^2 = z_j e_j$ (for good z_j).
The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm1} e_i = e_i$ for $i = 1, \ldots, k-1$.

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$  for $1 \le i \le k-1$.

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set



so that $e_j^2 = z_j e_j$. The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm1} e_i = e_i$ for $i = 1, \ldots, k-1$.

	Universal	Type B, C, D	Type A	Small Type A
Qu grp	Two-pole braids 	Two-pole BMW 	Affine Hecke of type C (+twists)	Two-boundary TL 

$N \otimes (V^{\otimes k}) \otimes M$

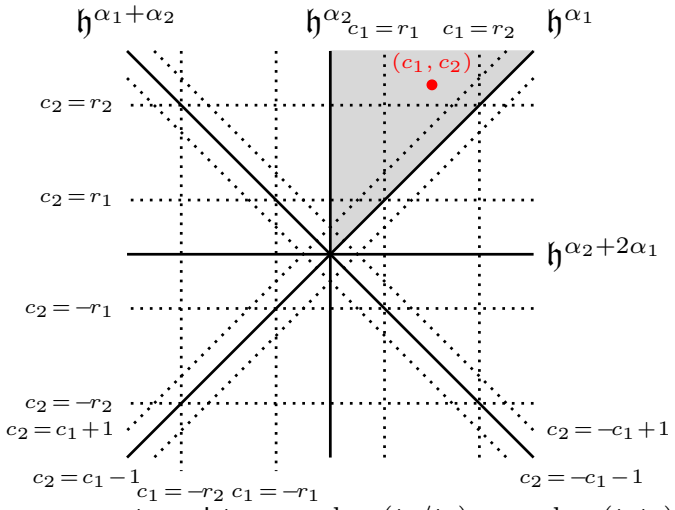
Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$, where

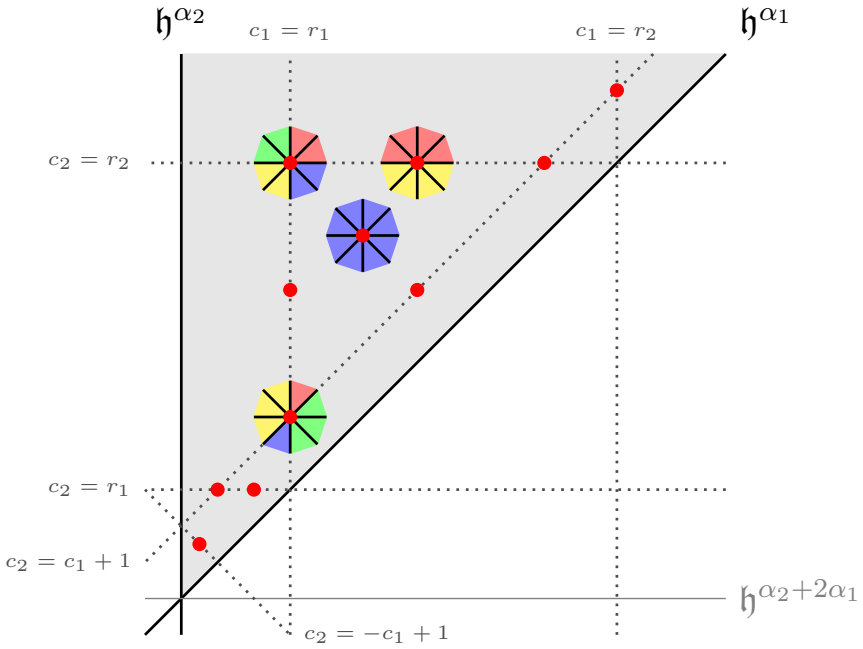
$\mathbf{c}$ is a point in the fundamental chamber of the (finite) type C hyperplane system, and

J is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting $\mathbf{c}$

Example: $k = 2$

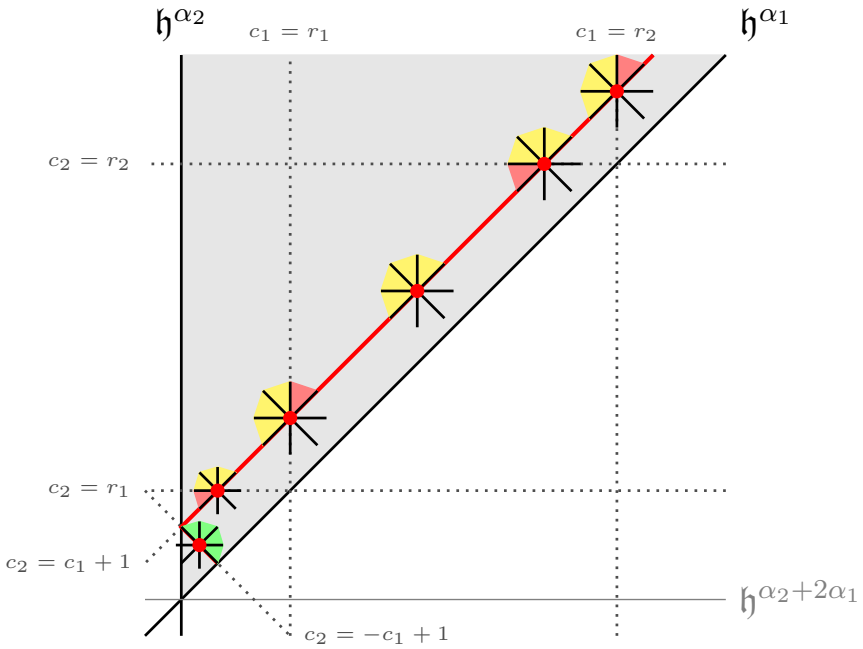


The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$



Thm. (D.-Ram)

(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$.



Thm. (D.-Ram)
(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$.
(2) The (calibrated) representations of $\mathcal{H}_k$ that factor through the Temperley-Lieb quotient are (see above).

