

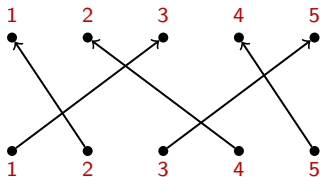
# Two-boundary diagram algebras

Zajj Daugherty  
(Joint work in progress with Arun Ram)

October 17, 2017

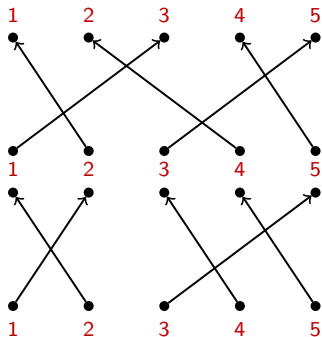
# Motivating example: Schur-Weyl Duality

The **symmetric group**  $S_k$  (permutations) as diagrams:



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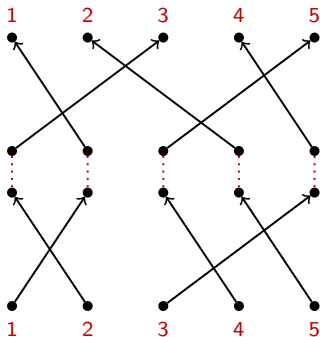
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(with multiplication given by concatenation)

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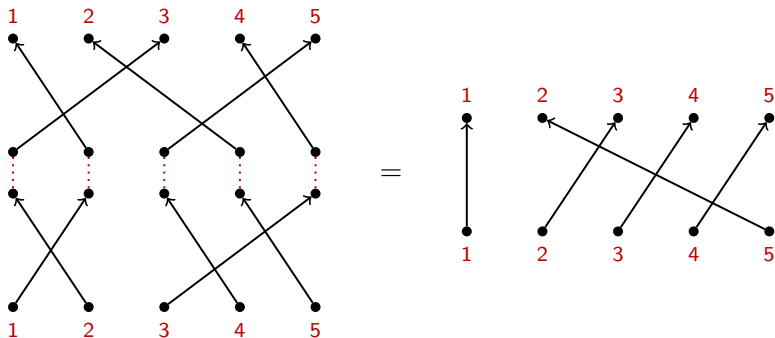
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(with multiplication given by concatenation)

## Motivating example: Schur-Weyl Duality

$GL_n(\mathbb{C})$  acts on  $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$  diagonally.

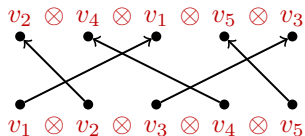
$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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$S_k$  also acts on  $(\mathbb{C}^n)^{\otimes k}$  by place permutation.

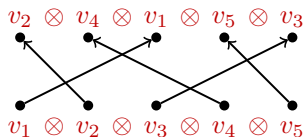


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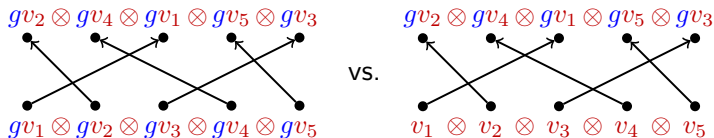
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These actions commute!





## Motivating example: Schur-Weyl Duality

Schur (1901):

$$\underbrace{\text{End}_{\text{GL}_n} \left( (\mathbb{C}^n)^{\otimes k} \right)}_{\text{(all linear maps that commute with } \text{GL}_n)} = \underbrace{\pi(\mathbb{C}S_k)}_{\text{(img of } S_k \text{ action)}} \quad \text{and} \quad \text{End}_{S_k} \left( (\mathbb{C}^n)^{\otimes k} \right) = \underbrace{\rho(\mathbb{C}\text{GL}_n)}_{\text{(img of } \text{GL}_n \text{ action)}}.$$

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Powerful consequence: a duality between representations

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \text{GL}_n\text{-}S_k \text{ bimodule,}$$

where  $G^\lambda$  are distinct irreducible  $\text{GL}_n$ -modules  
 $S^\lambda$  are distinct irreducible  $S_k$ -modules

# More centralizer algebras

Brauer (1937)

Orthogonal and symplectic groups  
(and Lie algebras) acting on  
 $(\mathbb{C}^n)^{\otimes k}$  diagonally centralize  
the **Brauer algebra**:

$$\delta_{b,c} \sum_{i=1}^n v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d$$

with  $\bigcirc = n$

(Diagrams encode maps  $V^{\otimes k} \rightarrow V^{\otimes k}$  that commute with the action of some classical algebra.)

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Temperley-Lieb (1971)

$GL_2$  and  $SL_2$  (and  $\mathfrak{gl}_2$  and  $\mathfrak{sl}_2$ ) acting on  $(\mathbb{C}^2)^{\otimes k}$  diagonally centralize the **Temperley-Lieb algebra**:

$$\delta_{c,d} \sum_{i=1}^2 v_a \otimes v_i \otimes v_i \otimes v_b \otimes v_e$$

with  $\bigcirc = 2$

(Diagrams encode maps  $V^{\otimes k} \rightarrow V^{\otimes k}$  that commute with the action of some classical algebra.)


## Quantum groups and braids

Fix  $q \in \mathbb{C}$ , and let  $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$  be the Drinfeld-Jimbo quantum group associated to Lie algebra  $\mathfrak{g}$ .

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$\mathcal{U} \otimes \mathcal{U}$  has an invertible element  $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$  that yields a map

$$\check{\mathcal{R}}_{VW}: V \otimes W \longrightarrow W \otimes V$$


that (1) satisfies braid relations, and


(2) commutes with the  $\mathcal{U}$ -action on  $V \otimes W$

for any  $\mathcal{U}$ -module  $V$ .

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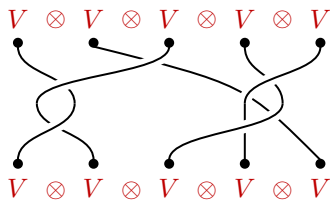
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$W \otimes V$   
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The braid group shares a commuting action with  $\mathcal{U}$  on  $V^{\otimes k}$ :



## Quantum groups and braids

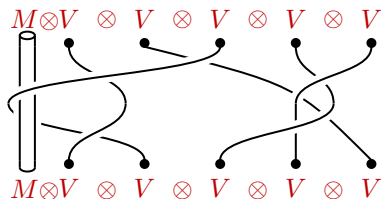
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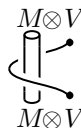
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The **one-pole/affine** braid group shares a commuting action with  $\mathcal{U}$  on  $M \otimes V^{\otimes k}$ :



Around the pole:




$$= \check{R}_{MV} \check{R}_{VM}$$



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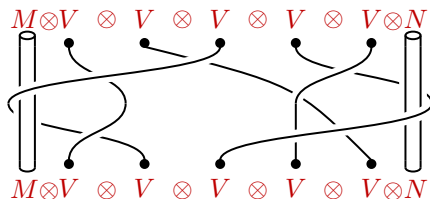
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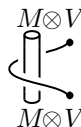
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for any  $\mathcal{U}$ -module  $V$ .

The **two-pole** braid group shares a commuting action with  $\mathcal{U}$  on  $M \otimes V^{\otimes k} \otimes N$ :



Around the pole:



$$= \check{R}_{MV} \check{R}_{VM}$$

Universal

Type B, C, D

Type A

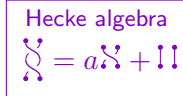
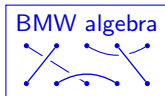
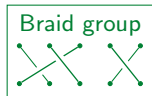
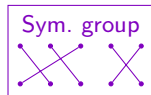
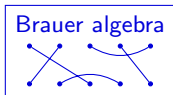
Small Type A

(orthog. & sympl.)

(gen. & sp. linear)

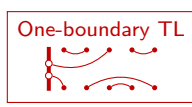
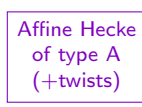
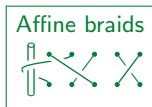
( $GL_2$  &  $SL_2$ )

Lie grp/alg

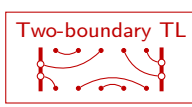
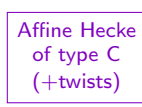
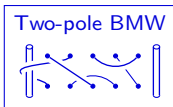
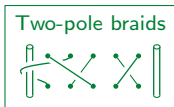


$V = \square$   
 $V \otimes \dots \otimes V$   
 $\Lambda \otimes \dots \otimes \Lambda$

Quantum groups



$M \otimes (V \otimes V) \otimes M$



$N \otimes (V \otimes V) \otimes M$

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(orthog. & simpl.)

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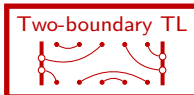
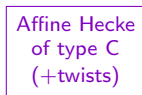
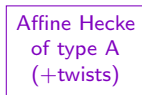
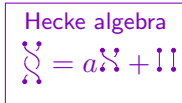
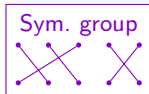
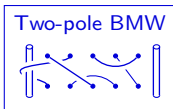
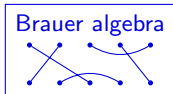
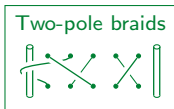
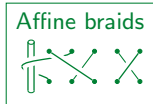
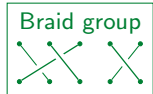
Lie grp/alg

Quantum groups

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 $V \otimes \dots \otimes V$

$M \otimes (V \otimes V)$

$M \otimes (V \otimes_k V)$



## Two-boundary Temperley-Lieb algebras

[MNGB04] Fix  $z, \delta_0, \delta_k \in \mathbb{C}$ . The *two-boundary Temperley-Lieb algebra*  $TL_k$  is a diagram algebra generated over  $\mathbb{C}$  by diagrams

$$e_0 = \left[ \begin{array}{c} 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 1 \end{array} \right], \quad e_k = \left[ \begin{array}{c} k \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ k \end{array} \right], \quad \text{and} \quad e_i = \left[ \begin{array}{c} i \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ i \end{array} \right]$$

for  $i = 1, \dots, k - 1$

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for  $i = 1, \dots, k-1$ , with relations  $e_i e_j = e_j e_i$  for  $|i-j| > 1$ ,

$$e_i e_{i\pm 1} e_i = e_i$$

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$$e_i^2 = \delta_i e_i.$$

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for  $1 \leq i \leq k-1$ ,

Diagrammatic relations for  $e_i e_{i\pm 1} e_i = e_i$ :

- Box 1: A diagram with three strands. The left two strands have two crossings (forming a square), and the right strand is a vertical line. This is equal to a diagram with two strands, each having two crossings.
- Box 2: A diagram with three strands. The left strand has two crossings, and the right two strands have two crossings. This is equal to a diagram with two strands, each having two crossings.
- Box 3: A diagram with three strands. The left strand has two crossings, and the right strand has two crossings. This is equal to a diagram with two strands, each having two crossings.

$$e_i^2 = \delta_i e_i.$$

Diagrammatic relations for  $e_i^2 = \delta_i e_i$ :

- Box 1: A diagram with two crossings on a single strand is equal to  $\delta$  times a diagram with two crossings on a single strand.
- Box 2: A diagram with two crossings on a single strand is equal to  $\delta_0$  times a diagram with two crossings on a single strand.
- Box 3: A diagram with two crossings on a single strand is equal to  $\delta_k$  times a diagram with two crossings on a single strand.

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for  $i = 1, \dots, k-1$ , with relations  $e_i e_j = e_j e_i$  for  $|i-j| > 1$ ,

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for  $1 \leq i \leq k-1$ ,

$$e_i^2 = \delta_i e_i.$$

(Side loops are resolved with a 1 or a  $\delta_i$  depending on whether there are an even or odd number of connections below their lowest point.)



Diagram multiplication:



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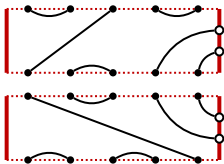


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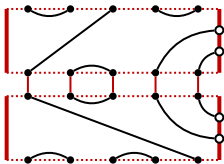
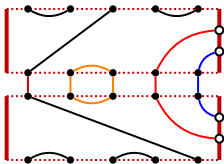
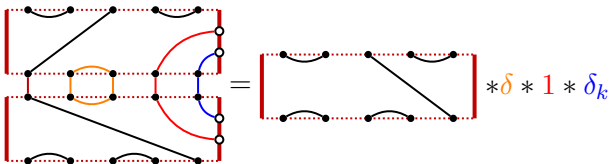


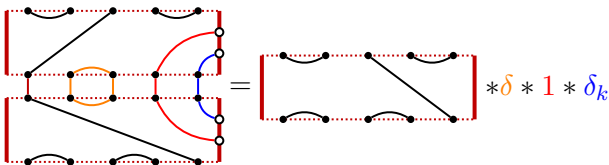
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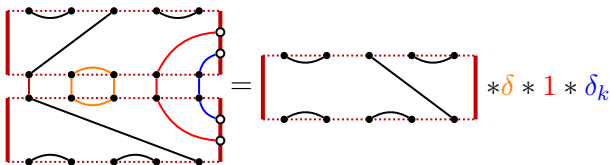
## Diagram multiplication:



In short,  $TL_k$  has basis given by non-crossing diagrams with

- (1)  $k$  connections to the top and to the bottom,
- (2) an even number of connections to the right and to the left, and
- (3) no edges with both ends on the left or both ends on the right.

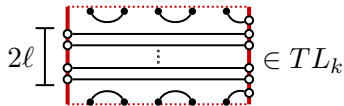
## Diagram multiplication:



In short,  $TL_k$  has basis given by non-crossing diagrams with

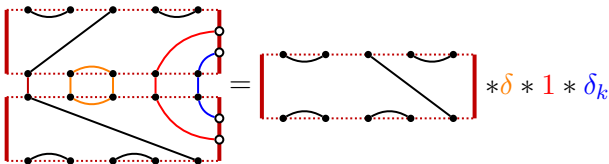
- (1)  $k$  connections to the top and to the bottom,
- (2) an even number of connections to the right and to the left, and
- (3) no edges with both ends on the left or both ends on the right.

However,



So unlike the classical T-L algebras,  $TL_k$  is not finite dimensional!

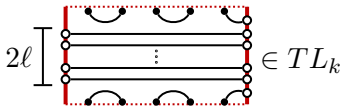
## Diagram multiplication:



In short,  $TL_k$  has basis given by non-crossing diagrams with

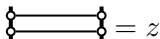
- (1)  $k$  connections to the top and to the bottom,
- (2) an even number of connections to the right and to the left, and
- (3) no edges with both ends on the left or both ends on the right.

However,



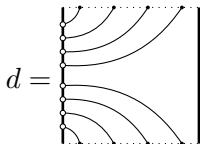
So unlike the classical T-L algebras,  $TL_k$  is not finite dimensional!

Take quotient giving

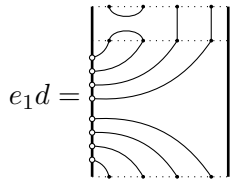
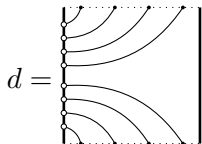




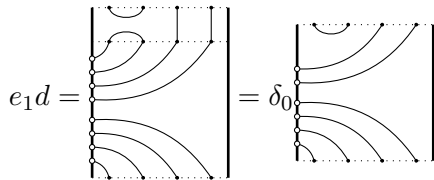
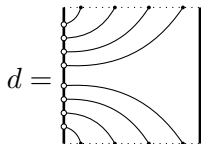
# Representation theory of $TL_k$ : action on diagrams



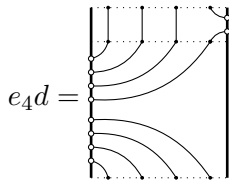
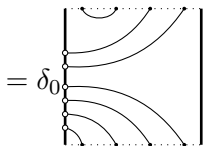
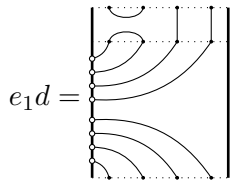
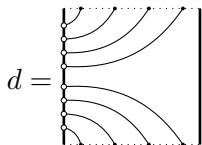
# Representation theory of $TL_k$ : action on diagrams



# Representation theory of $TL_k$ : action on diagrams



# Representation theory of $TL_k$ : action on diagrams



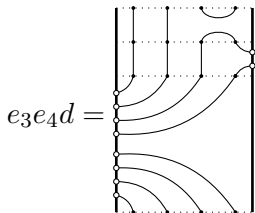
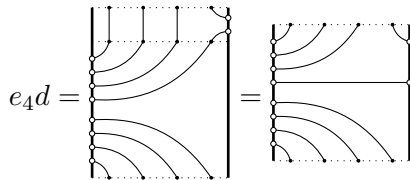
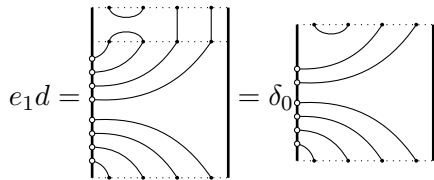
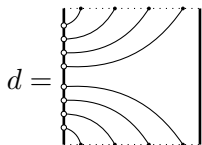
# Representation theory of $TL_k$ : action on diagrams

$$d = \begin{array}{|c|} \hline \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a loop at the top. The strands are connected by arcs.} \\ \hline \end{array}$$

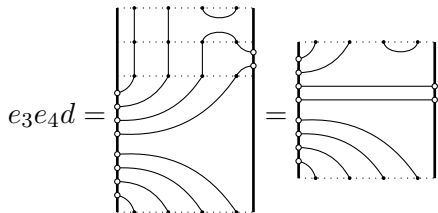
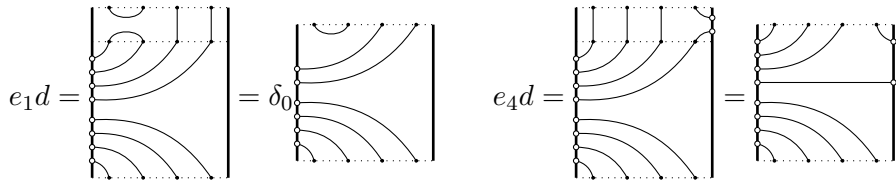
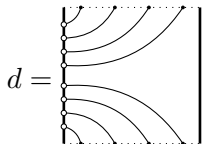
$$e_1 d = \begin{array}{|c|} \hline \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a loop at the top. The strands are connected by arcs.} \\ \hline \end{array} = \delta_0 \begin{array}{|c|} \hline \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a loop at the top. The strands are connected by arcs.} \\ \hline \end{array}$$

$$e_4 d = \begin{array}{|c|} \hline \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a loop at the top. The strands are connected by arcs.} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram with 4 strands on the left and 4 strands on the right, each strand having a loop at the top. The strands are connected by arcs.} \\ \hline \end{array}$$

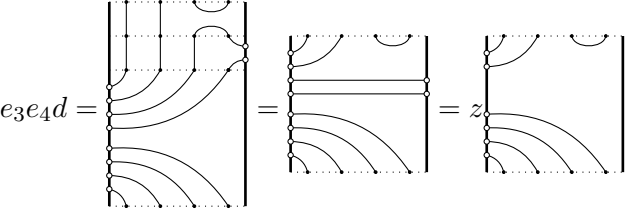
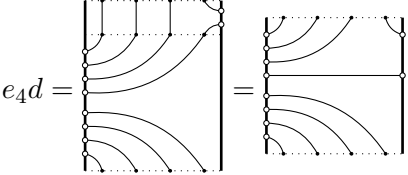
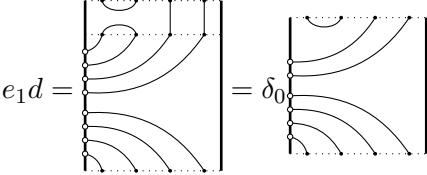
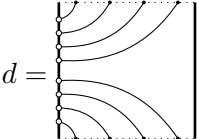
# Representation theory of $TL_k$ : action on diagrams



# Representation theory of $TL_k$ : action on diagrams



# Representation theory of $TL_k$ : action on diagrams





# Representation theory of $TL_k$ : action on diagrams

$$d = \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array}$$

$$e_1 d = \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array} = \delta_0 \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array} \quad e_4 d = \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array}$$

$$e_3 e_4 d = \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array} = z \begin{array}{|c|} \hline \text{Black arcs} \\ \hline \text{Red arcs} \\ \hline \end{array}$$

# Representation theory of $TL_k$ : half diagrams

$$d = \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

$$e_1 d = \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = \delta_0 \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

$$e_4 d = \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

$$e_3 e_4 d = \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = z \left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$$

# Representation theory of $TL_k$ : half diagrams

$$d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_1 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \delta_0 \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_4 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_3 e_4 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = z \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

You can tell when to use

$$\begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = z$$

or not by the parity of connections to the left/right walls.

# Representation theory of $TL_k$ : half diagrams

$$d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_1 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \delta_0 \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_4 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_3 e_4 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = z \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

You can tell when to use

$$\begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = z$$

or not by the parity of connections to the left/right walls.

# Representation theory of $TL_k$ : half diagrams

$$d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_1 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \delta_0 \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_4 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

$$e_3 e_4 d = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = z \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$$

You can tell when to use

$$\begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \begin{array}{|c} \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \end{array} = z$$

or not by the parity of connections to the left/right walls.

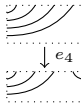
## Generic module:

(act by  $e_i$ , don't make loops)



## Generic module:

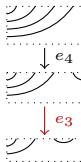
(act by  $e_i$ , don't make loops)



## Generic module:

(act by  $e_i$ , don't make loops)

Red arrows indicate coef of  $z$ .

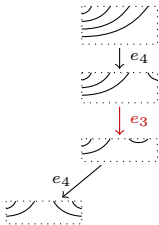




## Generic module:

(act by  $e_i$ , don't make loops)

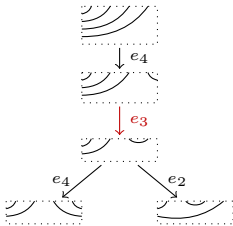
Red arrows indicate coef of  $z$ .



## Generic module:

(act by  $e_i$ , don't make loops)

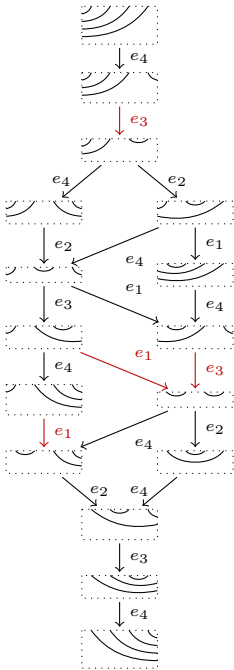
Red arrows indicate coef of  $z$ .



## Generic module:

(act by  $e_i$ , don't make loops)

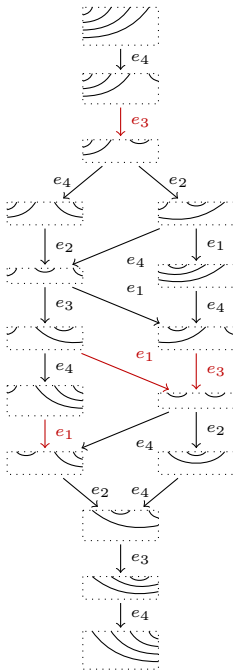
Red arrows indicate coef of  $z$ .



## Generic module:

(act by  $e_i$ , don't make loops)

Red arrows indicate coef of  $z$ .



For what  $z$  does this module split?

Universal

Type B, C, D

Type A

Small Type A

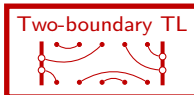
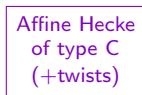
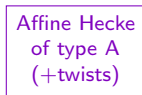
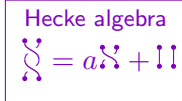
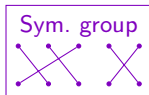
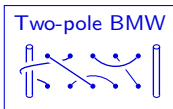
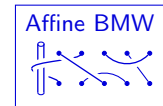
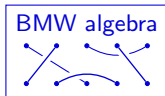
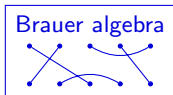
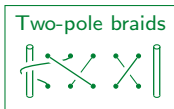
(orthog. & simpl.)

(gen. & sp. linear)

( $GL_2$  &  $SL_2$ )

Lie grp/alg

Quantum groups



$V = \square$   
 $V \otimes \dots \otimes V$

$M \otimes (V \otimes V)$

$M \otimes (V \otimes V \otimes V)$

Universal

Type B, C, D

Type A

Small Type A

(orthog. & simpl.)

(gen. & sp. linear)

(GL<sub>2</sub> & SL<sub>2</sub>)

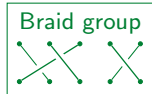
Lie grp/alg

Quantum groups

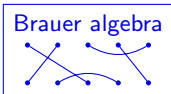
$V = \square$   
 $V \otimes \dots \otimes V$

$M \otimes (V \otimes V)$

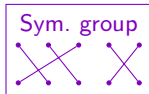
$M \otimes (V \otimes_k V)$



Braid group



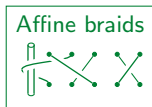
Brauer algebra



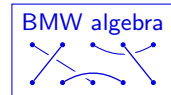
Sym. group



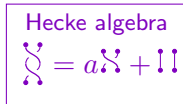
Temperley-Lieb



Affine braids

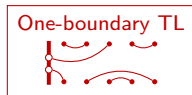


BMW algebra

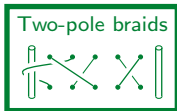


Hecke algebra

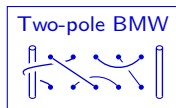
$$S^2 = aS + 1$$



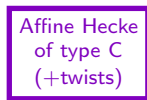
One-boundary TL



Two-pole braids



Two-pole BMW



Affine Hecke  
of type C  
(+twists)



Two-boundary TL

The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \bullet \\ \diagdown \\ \text{---} \\ \diagup \\ \bullet \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \quad \text{and} \quad T_i = \begin{array}{cc} \bullet & \bullet \\ & \diagdown \quad \diagup \\ & \bullet & \bullet \\ & \diagup \quad \diagdown \\ \bullet & \bullet \\ i & i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \bullet \\ \diagdown \\ \text{---} \\ \diagup \\ \bullet \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \quad \text{and} \quad T_i = \begin{array}{cc} \bullet & \bullet \\ & \diagdown \quad \diagup \\ & \text{---} \\ & \diagup \quad \diagdown \\ \bullet & \bullet \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations

$$T_i T_{i+1} T_i = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \diagdown & \diagup & \diagdown \\ \text{---} & \text{---} & \text{---} \\ \diagup & \diagdown & \diagup \\ \bullet & \bullet & \bullet \end{array} = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \diagdown & \diagdown & \diagup \\ \text{---} & \text{---} & \text{---} \\ \diagup & \diagup & \diagdown \\ \bullet & \bullet & \bullet \end{array} = T_{i+1} T_i T_{i+1},$$



The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \text{diagram}, \quad T_0 = \text{diagram} \quad \text{and} \quad T_i = \text{diagram} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations

$$T_i T_{i+1} T_i = \text{diagram} = \text{diagram} = T_{i+1} T_i T_{i+1},$$

$$T_1 T_0 T_1 T_0 = \text{diagram} = \text{diagram} = T_0 T_1 T_0 T_1,$$

The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations

$$T_i T_{i+1} T_i = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} = T_{i+1} T_i T_{i+1},$$

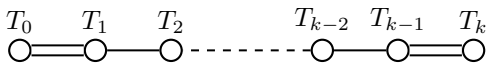
$$T_1 T_0 T_1 T_0 = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = T_0 T_1 T_0 T_1,$$

and, similarly,  $T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$ .

The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations



i.e.

$$T_i T_{i+1} T_i = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} = T_{i+1} T_i T_{i+1},$$

$$T_1 T_0 T_1 T_0 = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = T_0 T_1 T_0 T_1,$$

and, similarly,  $T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$ .

(1) The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array}, \quad T_0 = \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ i \quad i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations  $\begin{array}{c} T_0 \\ \circ \end{array} = \begin{array}{c} T_1 \\ \circ \end{array} - \begin{array}{c} T_2 \\ \circ \end{array} - \dots - \begin{array}{c} T_{k-2} \\ \circ \end{array} - \begin{array}{c} T_{k-1} \\ \circ \end{array} = \begin{array}{c} T_k \\ \circ \end{array}.$

(1) The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \overset{i}{\bullet} \quad \overset{i+1}{\bullet} \\ \diagdown \quad \diagup \\ \underset{i}{\bullet} \quad \underset{i+1}{\bullet} \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations  $\begin{array}{c} T_0 \\ \circ \end{array} = \begin{array}{c} T_1 \\ \circ \end{array} = \begin{array}{c} T_2 \\ \circ \end{array} \text{---} \text{---} \text{---} \begin{array}{c} T_{k-2} \\ \circ \end{array} = \begin{array}{c} T_{k-1} \\ \circ \end{array} = \begin{array}{c} T_k \\ \circ \end{array}.$

(2) Fix constants  $t_0, t_k, t \in \mathbb{C}$ .

The affine type C Hecke algebra  $\mathcal{H}_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

and  $(T_i - t^{1/2})(T_i + t^{-1/2}) = 0$  for  $i = 1, \dots, k-1$ .

(1) The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} i \quad i+1 \\ \diagdown \quad \diagup \\ i \quad i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations  $\begin{array}{c} T_0 \\ \bigcirc \end{array} = \begin{array}{c} T_1 \\ \bigcirc \end{array} = \begin{array}{c} T_2 \\ \bigcirc \end{array} \text{---} \text{---} \text{---} \begin{array}{c} T_{k-2} \\ \bigcirc \end{array} = \begin{array}{c} T_{k-1} \\ \bigcirc \end{array} = \begin{array}{c} T_k \\ \bigcirc \end{array}.$

(2) Fix constants  $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$ .

The affine type C Hecke algebra  $\mathcal{H}_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by the relations  $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$ .

(1) The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} i \quad i+1 \\ \diagdown \quad \diagup \\ i \quad i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations  $T_0 \text{---} T_1 \text{---} T_2 \text{---} \dots \text{---} T_{k-2} \text{---} T_{k-1} \text{---} T_k$ .

(2) Fix constants  $t_0, t_k, t = t_1 = t_2 = \dots = t_{k-1} \in \mathbb{C}$ .

The affine type C Hecke algebra  $\mathcal{H}_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by the relations  $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$ .

(3) Set

$$\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = t_0^{1/2} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad (e_0 = t_0^{1/2} - T_0)$$

$$\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = t_k^{1/2} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \quad (e_k = t_k^{1/2} - T_k)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = t^{1/2} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \quad (e_i = t^{1/2} - T_i)$$

so that  $e_j^2 = z_j e_j$  (for good  $z_j$ ).

(1) The **two-boundary (two-pole) braid group**  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} i \quad i+1 \\ \diagdown \quad \diagup \\ i \quad i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

subject to relations  $T_0 \text{---} T_1 \text{---} T_2 \text{---} \dots \text{---} T_{k-2} \text{---} T_{k-1} \text{---} T_k$ .

(2) Fix constants  $t_0, t_k, t = t_1 = t_2 = \dots = t_{k-1} \in \mathbb{C}$ .

The **affine type C Hecke algebra**  $\mathcal{H}_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by the relations  $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$ .

(3) Set

$$\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = t_0^{1/2} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad (e_0 = t_0^{1/2} - T_0)$$

$$\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = t_k^{1/2} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad (e_k = t_k^{1/2} - T_k)$$

$$\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = t^{1/2} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad (e_i = t^{1/2} - T_i)$$

so that  $e_j^2 = z_j e_j$  (for good  $z_j$ ).

The **two-boundary Temperley-Lieb algebra** is the quotient of  $\mathcal{H}_k$  by the relations  $e_i e_{i\pm 1} e_i = e_i$  for  $i = 1, \dots, k-1$ .



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(2) Fix constants  $t_0, t_k, t = t_1 = t_2 = \dots = t_{k-1} \in \mathbb{C}$ .

The **affine type C Hecke algebra**  $\mathcal{H}_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by the relations  $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$ .

(3) Set

$$\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = t_0^{1/2} \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = t_k^{1/2} \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = t^{1/2} \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$$

so that  $e_j^2 = z_j e_j$ . The **two-boundary Temperley-Lieb algebra** is the quotient of  $\mathcal{H}_k$  by the relations  $e_i e_{i\pm 1} e_i = e_i$  for  $i = 1, \dots, k-1$ .

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$$T_k = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \text{---} \quad \text{---} \\ i \quad i+1 \end{array} \quad \text{for } 1 \leq i \leq k-1.$$

(2) Fix constants  $t_0, t_k, t = t_1 = t_2 = \dots = t_{k-1} \in \mathbb{C}$ .

The **affine type C Hecke algebra**  $\mathcal{H}_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by the relations  $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$ .

(3) Set

$$\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \cdot \quad \cdot \\ \text{---} \end{array} = t_0^{1/2} \begin{array}{c} | \quad | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ | \quad | \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \text{---} \end{array} = t_k^{1/2} \begin{array}{c} | \quad | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = t^{1/2} \begin{array}{c} | \quad | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ | \quad | \\ \text{---} \end{array}$$

so that  $e_j^2 = z_j e_j$ . The **two-boundary Temperley-Lieb algebra** is the quotient of  $\mathcal{H}_k$  by the relations  $e_i e_{i\pm 1} e_i = e_i$  for  $i = 1, \dots, k-1$ .

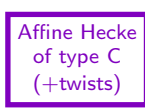
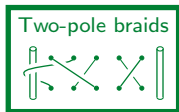
Universal

Type B, C, D

Type A

Small Type A

Qu grp



## Representation theory of $\mathcal{H}_k$

The representations of  $\mathcal{H}_k$  are indexed by pairs  $(\mathfrak{c}, J)$ , where

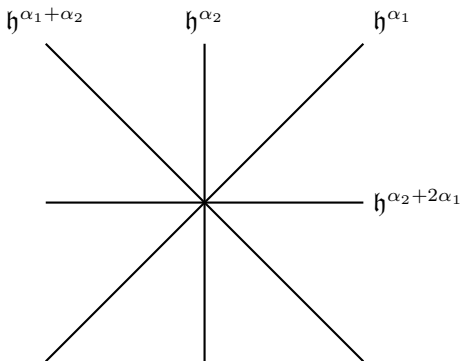
$\mathfrak{c}$  is a point in the fundamental chamber of  
the (finite) type C hyperplane system, and  
 $J$  is a set of choices of positive/negative sides of  
other distinguished hyperplanes intersecting  $\mathfrak{c}$

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Example:  $k = 2$

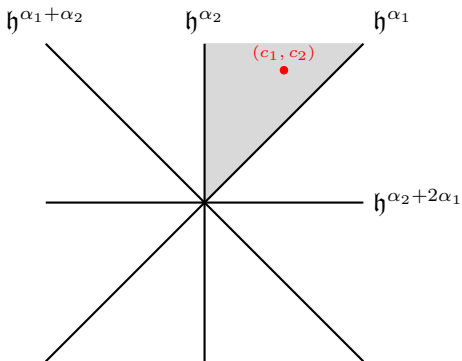


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Example:  $k = 2$

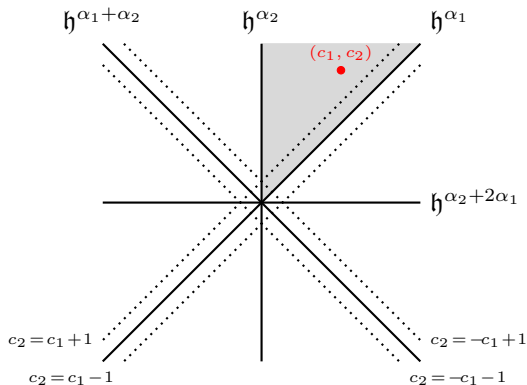


# Representation theory of $\mathcal{H}_k$

The representations of  $\mathcal{H}_k$  are indexed by pairs  $(\mathfrak{c}, J)$ , where

$\mathfrak{c}$  is a point in the fundamental chamber of the (finite) type C hyperplane system, and  $J$  is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting  $\mathfrak{c}$

Example:  $k = 2$

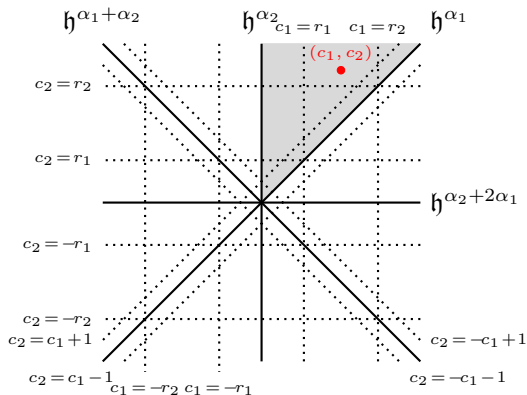


# Representation theory of $\mathcal{H}_k$

The representations of  $\mathcal{H}_k$  are indexed by pairs  $(c, J)$ , where

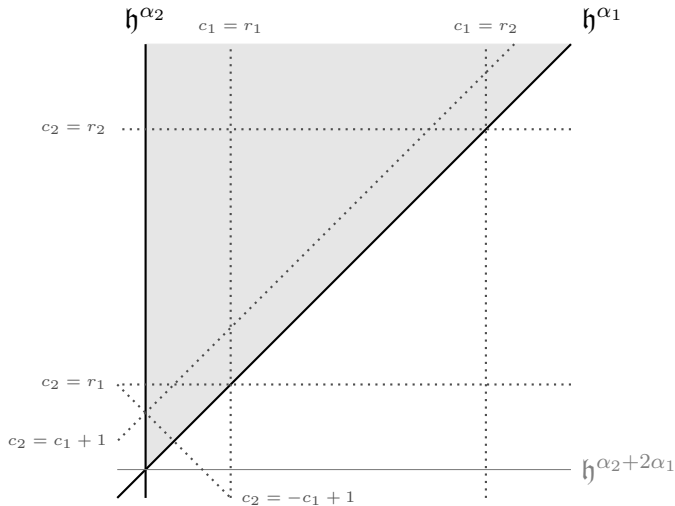
$c$  is a point in the fundamental chamber of the (finite) type C hyperplane system, and  $J$  is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting  $c$

Example:  $k = 2$



The  $r_i$ 's depend on  $\mathcal{H}_k$ 's parameters  $t_0$  and  $t_k$ :  $r_1 = \log_t(t_0/t_k)$ ,  $r_2 = \log_t(t_0 t_k)$

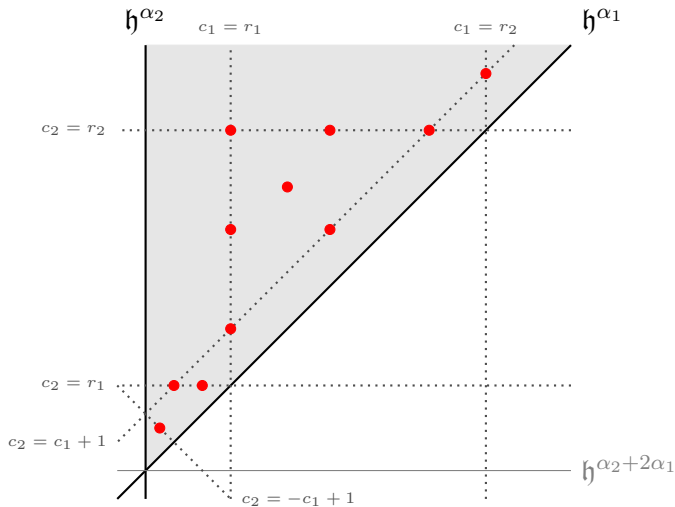
# Representation theory of $\mathcal{H}_k$



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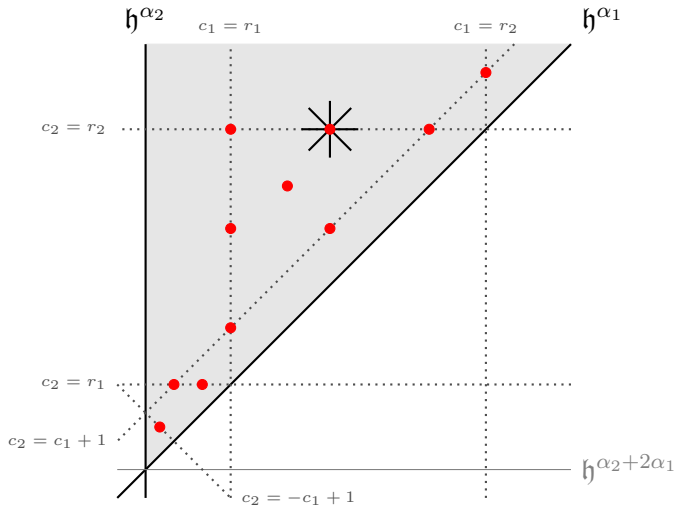


# Representation theory of $\mathcal{H}_k$



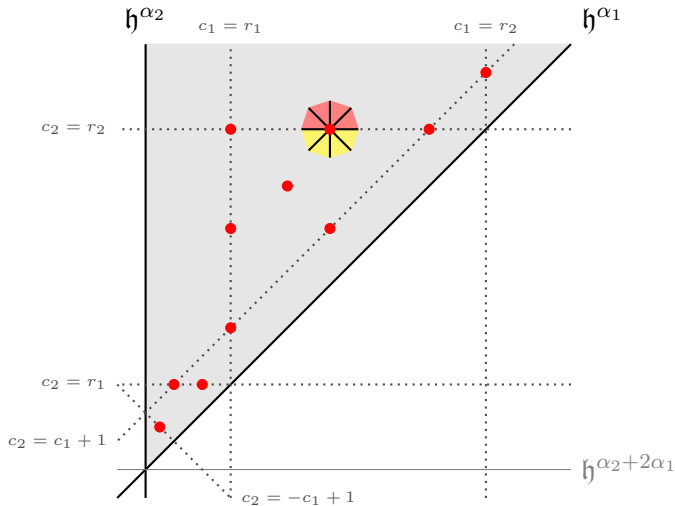
The  $r_i$ s depend on  $\mathcal{H}_k$ 's parameters  $t_0$  and  $t_k$ :  $r_1 = \log_t(t_0/t_k)$ ,  $r_2 = \log_t(t_0 t_k)$

# Representation theory of $\mathcal{H}_k$



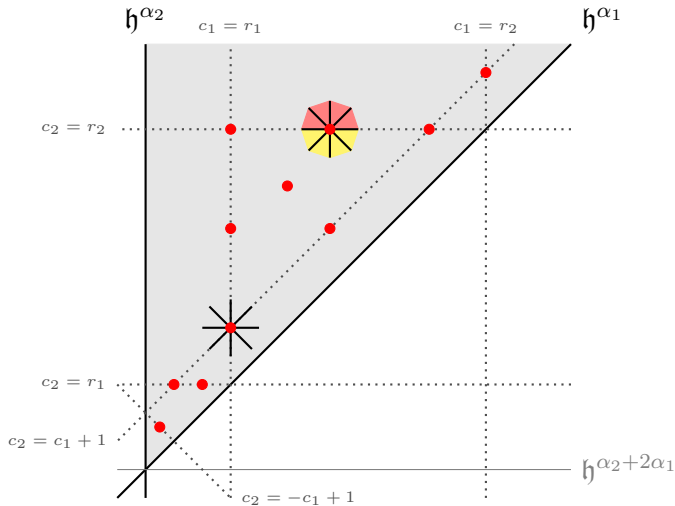
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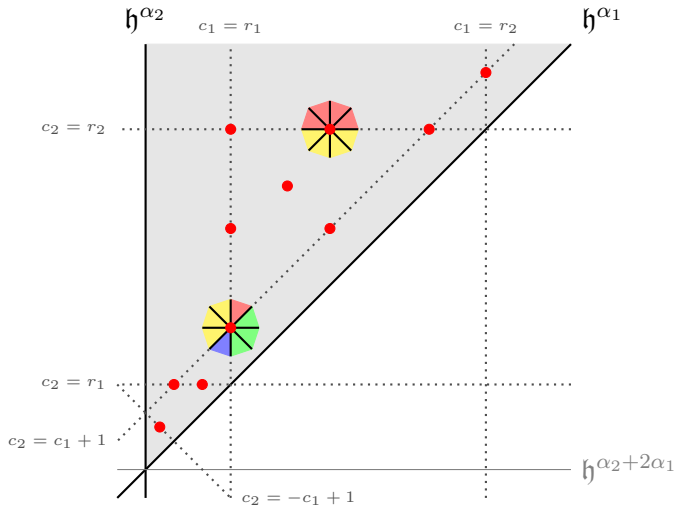
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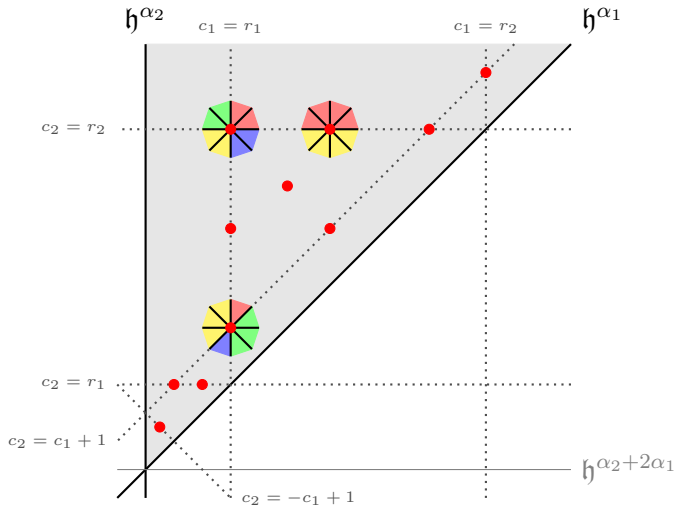
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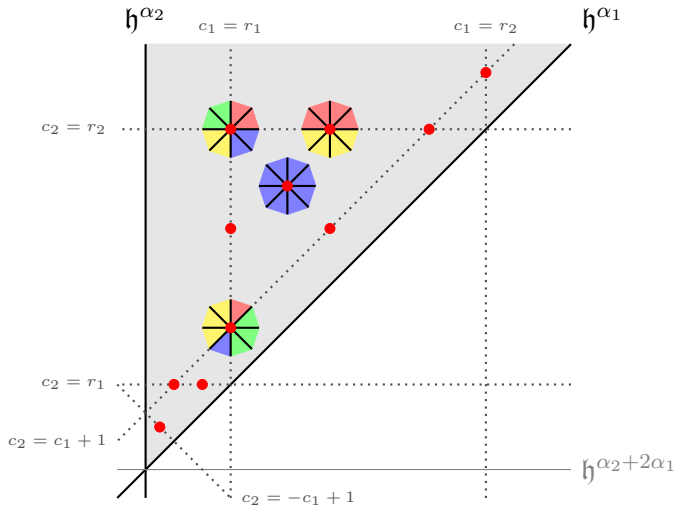
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# Representation theory of $\mathcal{H}_k$

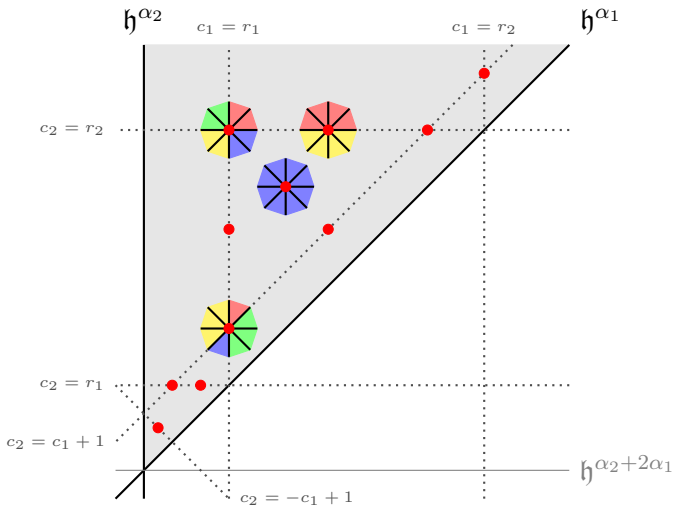


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# Representation theory of $\mathcal{H}_k$



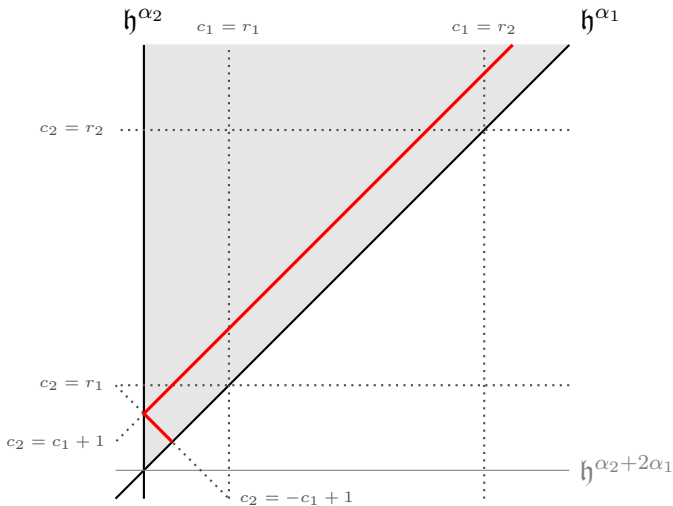
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Thm. (D.-Ram)

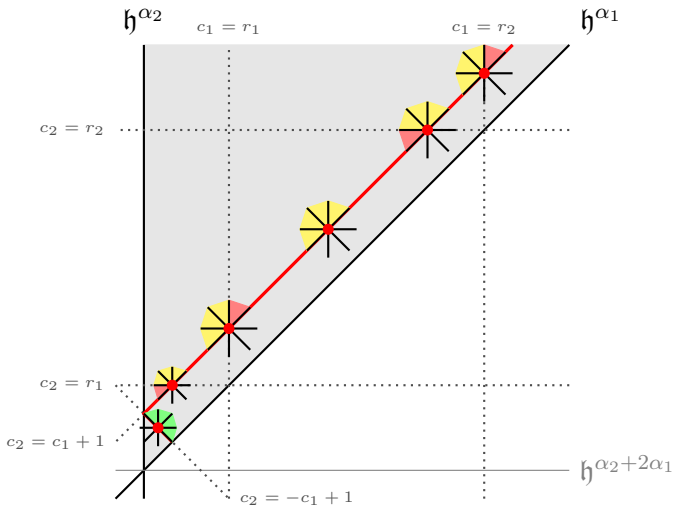
(1) Representations of  $\mathcal{H}_k$  are indexed by pairs  $(\mathbf{c}, J)$ .





Thm. (D.-Ram)

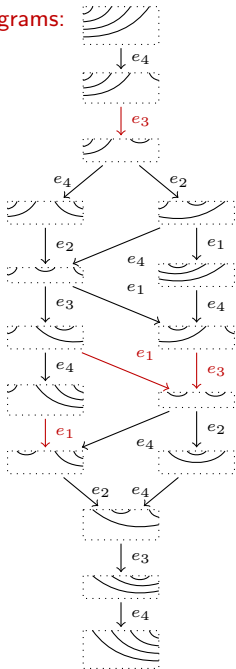
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- (2) The (calibrated) representations of  $\mathcal{H}_k$  that factor through the Temperley-Lieb quotient are (see above).



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Diagrams:



[GN] [DR]

Aff. type  
C Hecke:

