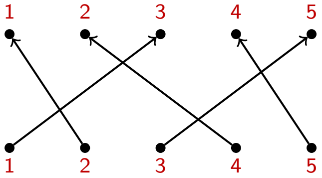
Two-boundary diagram algebras

Zajj Daugherty
(Joint work in progress with Arun Ram)

October 17, 2017

Motivating example: Schur-Weyl Duality

The **symmetric group** S_k (permutations) as diagrams:



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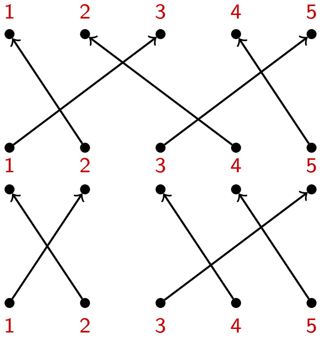

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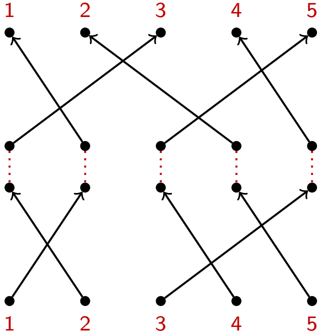

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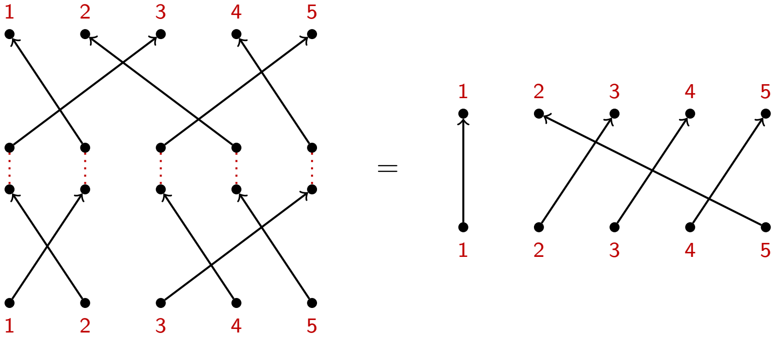

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$\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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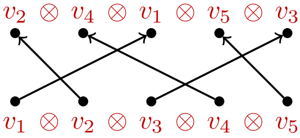

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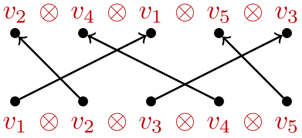
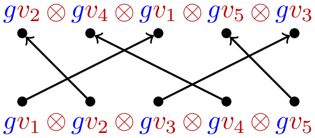
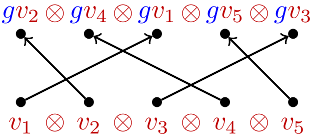


These actions commute!



vs.



Motivating example: Schur-Weyl Duality

Schur (1901):

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\text{(all linear maps that commute with } \mathrm{GL}_n\text{)}} = \underbrace{\pi(\mathbb{C}S_k)}_{\text{(img of } S_k \text{ action)}} \quad \text{and} \quad \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\text{(img of } \mathrm{GL}_n \text{ action)}}.$$

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Powerful consequence: a duality between representations

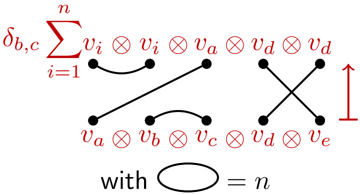
The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

where G^λ are distinct irreducible GL_n-modules
S^λ are distinct irreducible S_k-modules

More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize the **Brauer algebra**:



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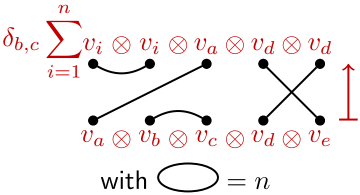
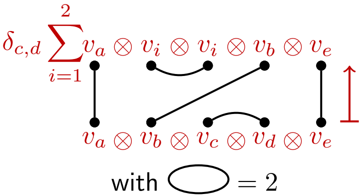


Temperley-Lieb (1971)
GL_2 and SL_2 (and $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$) acting on $(\mathbb{C}^2)^{\otimes k}$ diagonally centralize the **Temperley-Lieb algebra**:



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Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q\mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

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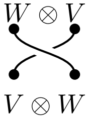
$$\check{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$



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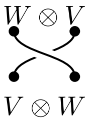

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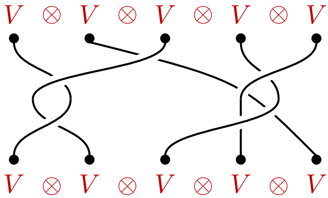

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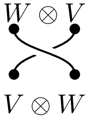

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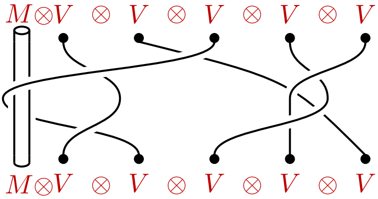
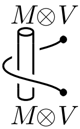
The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k}$:



Around the pole:



$$= \check{R}_{MV}\check{R}_{VM}$$

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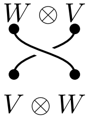

that (1) satisfies braid relations, and
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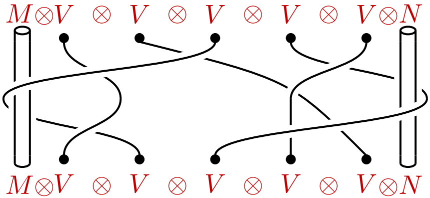
The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k} \otimes N$:



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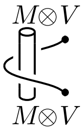
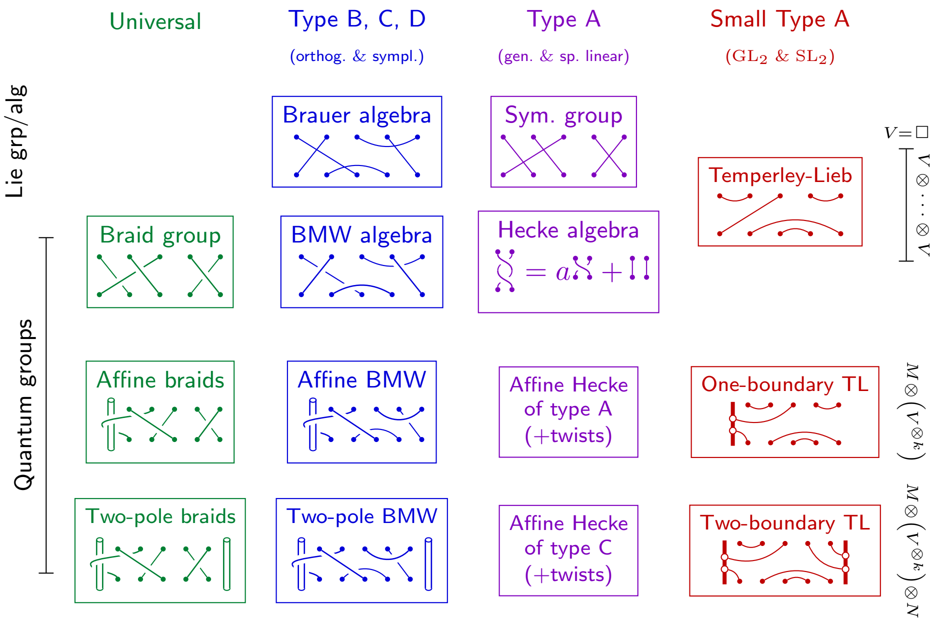


$= \check{R}_{MV} \check{R}_{VM}$



Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL_2 & SL_2)
Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
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Temperley-Lieb
$V \otimes \cdots \otimes V$
Braid group
BMW algebra
Hecke algebra
Affine braids
Affine BMW
Affine Hecke
of type A
(+twists)
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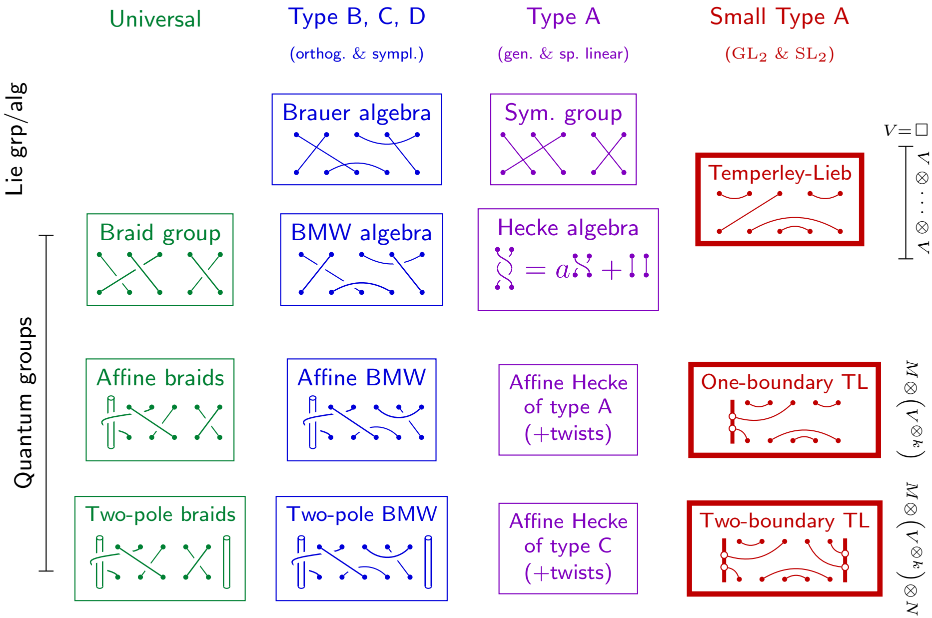

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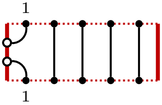
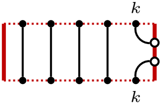
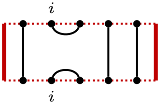
[MNGB04] Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The *two-boundary Temperley-Lieb algebra* TL_k is a diagram algebra generated over $\mathbb{C}$ by diagrams

$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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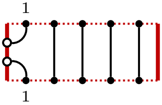
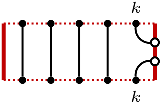
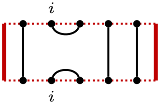
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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$$e_i e_{i\pm 1} e_i = e_i$$

for $1 \leq i \leq k-1$,

$$e_i^2 = \delta_i e_i.$$

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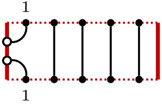
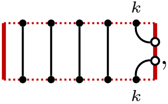
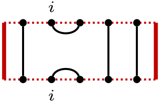
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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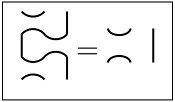
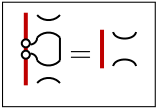
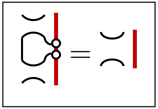
 or  or 

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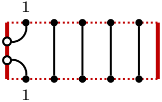
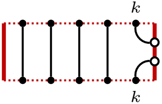
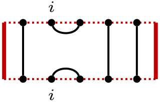
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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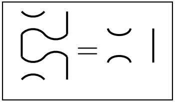

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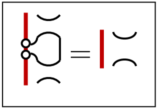

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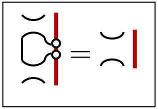

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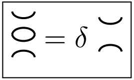

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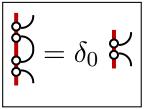

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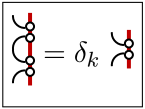

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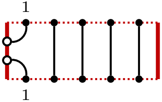
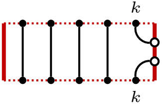
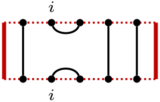
$e_0 =$ 

, $e_k =$ 

, and $e_i =$ 

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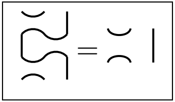
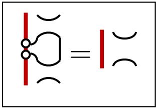
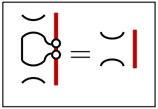
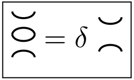

or

or


$e_i^2 = \delta_i e_i$.



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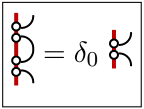

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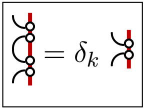


(Side loops are resolved with a 1 or a δ_i depending on whether there are an even or odd number of connections below their lowest point.)

Diagram multiplication:

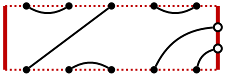


Diagram multiplication:

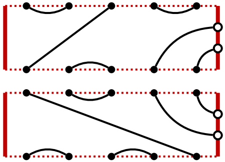


Diagram multiplication:

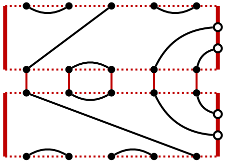


Diagram multiplication:

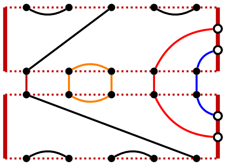


Diagram multiplication:

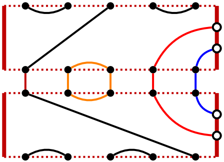
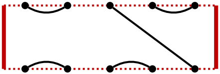
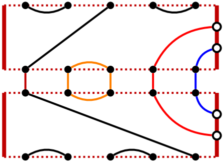
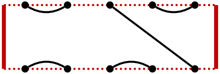
 $=$  $* \delta * 1 * \delta_k$

Diagram multiplication:



$= $  $*\delta * 1 * \delta_k$

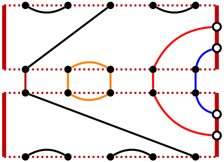
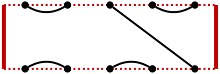
In short, TL_k has basis given by non-crossing diagrams with

(1) k connections to the top and to the bottom,

(2) an even number of connections to the right and to the left, and

(3) no edges with both ends on the left or both ends on the right.

Diagram multiplication:

 $=$  $* \delta * 1 * \delta_k$

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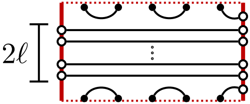
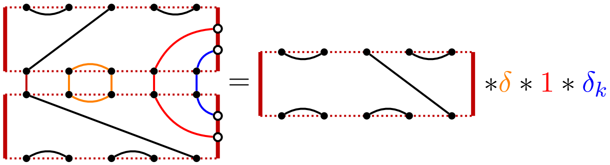
However,



$\in TL_k$

So unlike the classical T-L algebras, TL_k is not finite dimensional!

Diagram multiplication:



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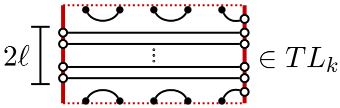
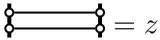
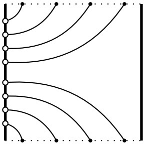
However,



So unlike the classical T-L algebras, TL_k is not finite dimensional!
Take quotient giving

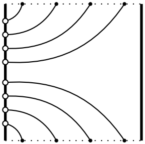


Representation theory of TL_k: action on diagrams

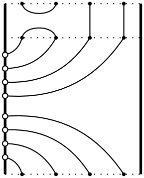
$d =$ 

Representation theory of TL_k: action on diagrams

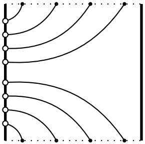
$d =$

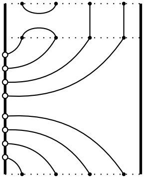
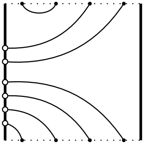


$e_1 d =$

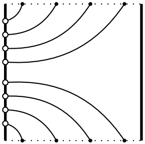


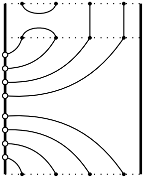
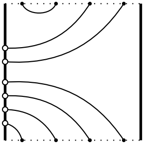
Representation theory of TL_k: action on diagrams

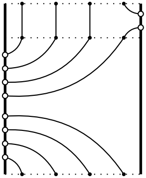
$d =$ 

$e_1 d =$  $= \delta_0$ 

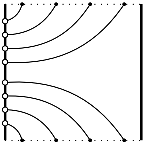
Representation theory of TL_k: action on diagrams

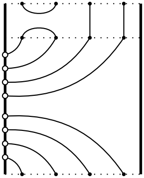
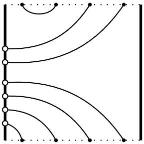
$d =$ 

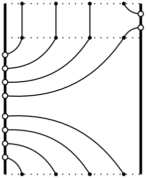
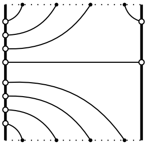
$e_1 d =$  $= \delta_0$ 

$e_4 d =$ 

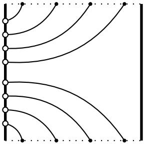
Representation theory of TL_k: action on diagrams

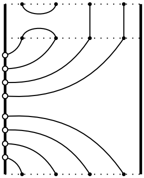
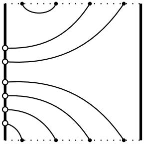
$d =$ 

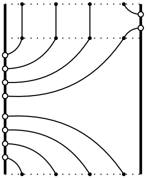
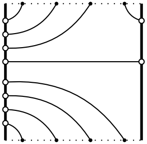
$e_1 d =$  $= \delta_0$ 

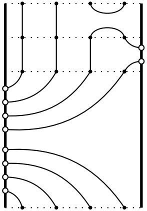
$e_4 d =$  $=$ 

Representation theory of TL_k: action on diagrams

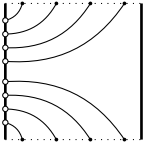
$d =$ 

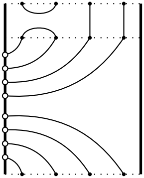
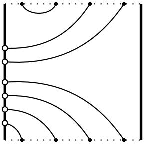
$e_1 d =$  $= \delta_0$ 

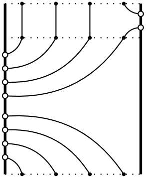
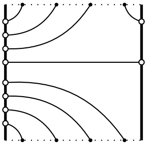
$e_4 d =$  $=$ 

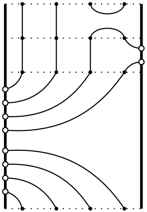
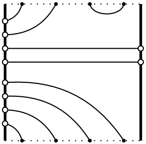
$e_3 e_4 d =$ 

Representation theory of TL_k: action on diagrams

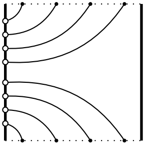
$d =$ 

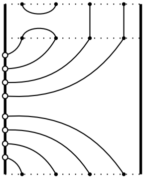
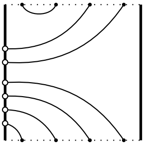
$e_1 d =$  $= \delta_0$ 

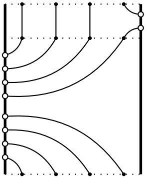
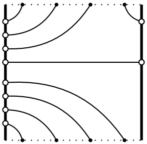
$e_4 d =$  $=$ 

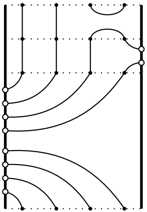
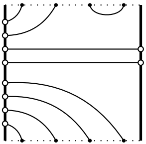
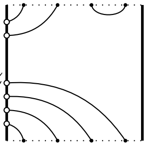
$e_3 e_4 d =$  $=$ 

Representation theory of TL_k: action on diagrams

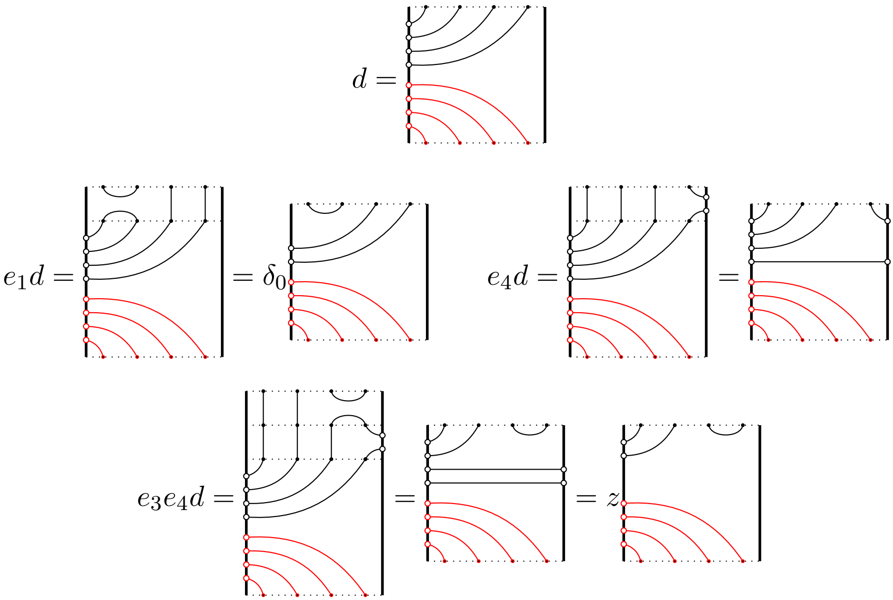
$d =$ 

$e_1 d =$  $= \delta_0$ 

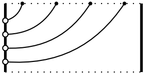
$e_4 d =$  $=$ 

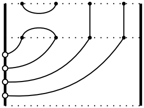
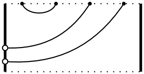
$e_3 e_4 d =$  $=$  $= z$ 

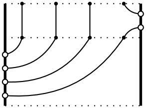
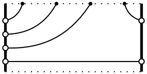
Representation theory of TL_k: action on diagrams

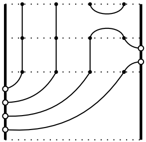
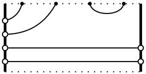
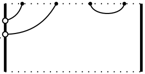


Representation theory of TL_k: half diagrams

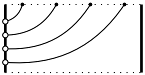
$d =$ 

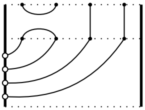
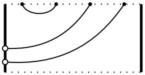
$e_1 d =$  $= \delta_0$ 

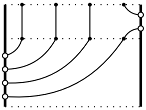
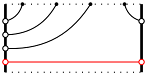
$e_4 d =$  $=$ 

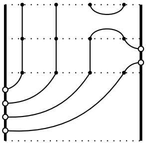
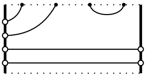
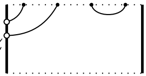
$e_3 e_4 d =$  $=$  $= z$ 

Representation theory of TL_k: half diagrams

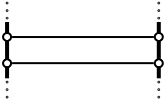
$d =$ 

$e_1 d =$  $= \delta_0$ 

$e_4 d =$  $=$ 

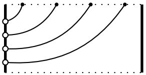
$e_3 e_4 d =$  $=$  $= z$ 

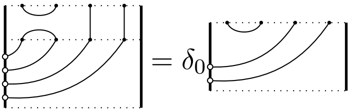
You can tell when to use

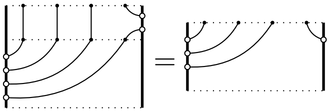
 $= z$

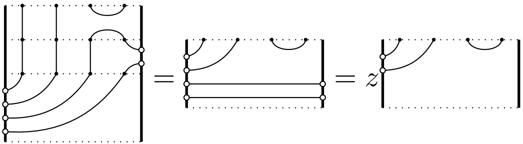
or not by the parity of connections to the left/right walls.

Representation theory of TL_k: half diagrams

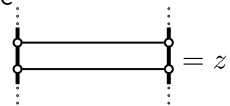
$d =$ 

$e_1 d =$ 

$e_4 d =$ 

$e_3 e_4 d =$ 

You can tell when to use



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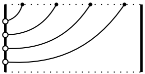
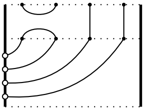
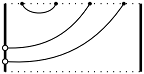
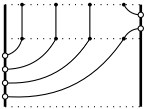
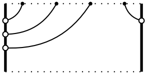
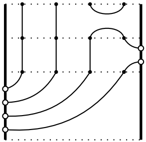
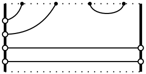
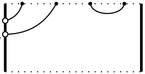
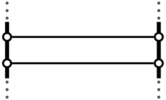
$d =$ 

$e_1 d =$  $= \delta_0$ 

$e_4 d =$  $=$ 

$e_3 e_4 d =$  $=$  $= z$ 

You can tell when to use

 $= z$

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Generic module:

(act by e_i, don't make loops)

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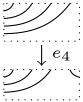

Generic module:

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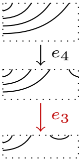
Red arrows indicate coef of z.



Generic module:

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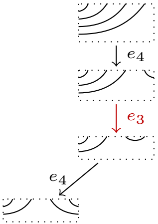

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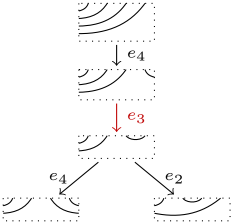

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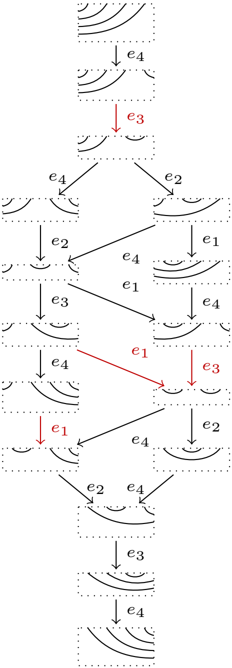

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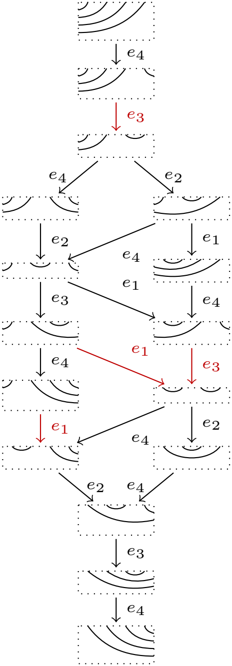
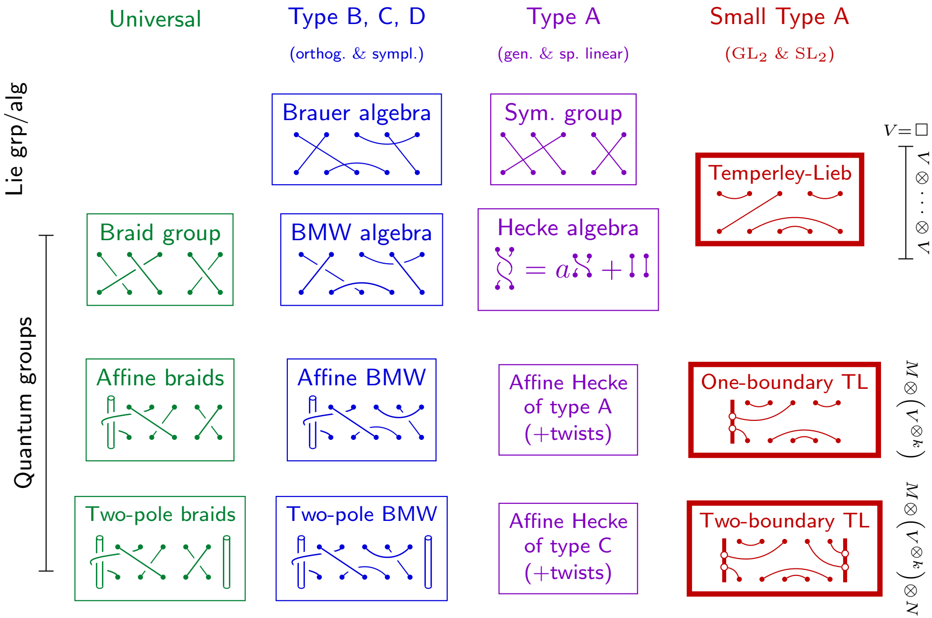
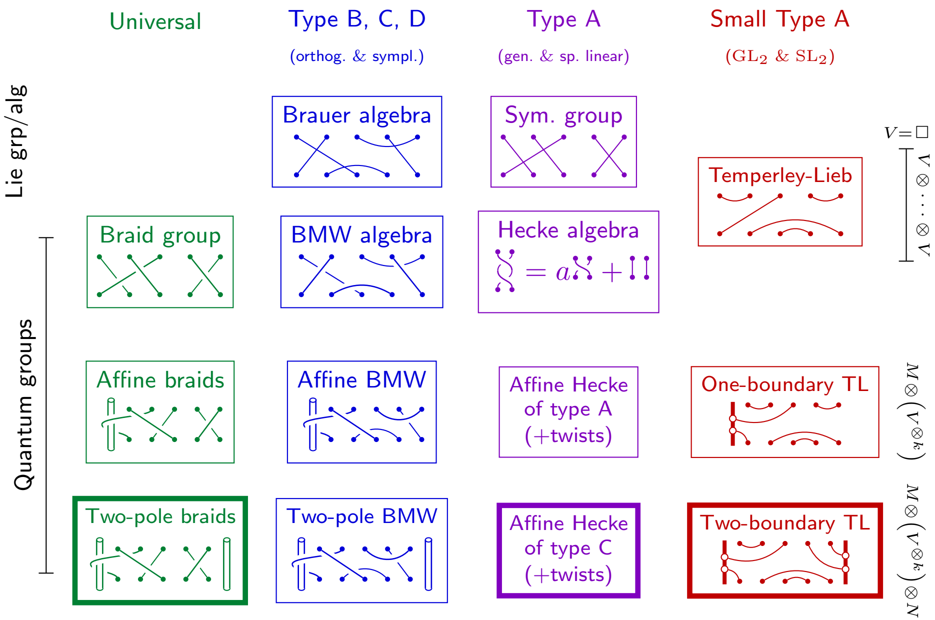


For what z does this module split?



Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL_2 & SL_2)
Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
Temperley-Lieb
$V = \square$
$V \otimes \cdots \otimes V$
Braid group
BMW algebra
Hecke algebra
$= a \; + $
Affine braids
Affine BMW
Affine Hecke of type A (+twists)
One-boundary TL
$\left(V^{\otimes k}\right) \otimes M$
Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two-boundary TL
$N \otimes \left(V^{\otimes k}\right) \otimes M$



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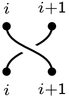
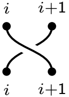
Two-boundary TL
$V = \square$
$V \otimes \cdots \otimes V$
$M \otimes (V^{\otimes k})$
$N \otimes (V^{\otimes k}) \otimes M$

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

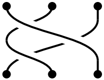
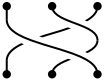
for $1 \leq i \leq k-1$,

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

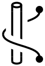
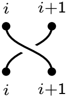
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

subject to relations

$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

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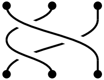
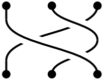
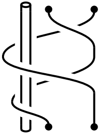
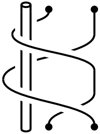
$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$,

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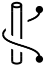
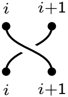
$T_k =$ , $T_0 =$  and $T_i =$  for $1 \leq i \leq k-1$,

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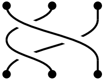
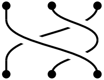
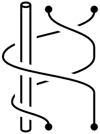
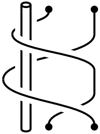
$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$,

and, similarly, $T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$.

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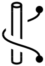
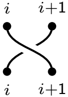
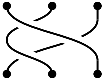
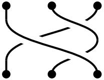
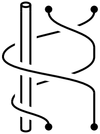
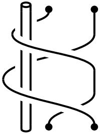
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \le i \le k-1$,

subject to relations

i.e.

$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1}$,

$T_1T_0T_1T_0 =$  $=$  $= T_0T_1T_0T_1$,

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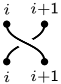
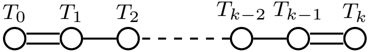
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \le i \le k-1$,

subject to relations 

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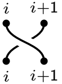
(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $\quad T_0 =$  and $\quad T_i =$ 

for $1 \le i \le k-1$,

subject to relations 

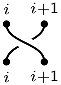
.

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

$$\text{and} \quad (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$  , $T_0 =$  and $T_i =$ 

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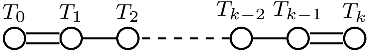
subject to relations 

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(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
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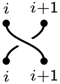
$T_k =$ , $T_0 =$  and $T_i =$ 

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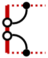
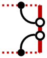
(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

 $= t_0^{1/2}$  $-$  $(e_0 = t_0^{1/2} - T_0)$

 $= t_k^{1/2}$  $-$  $(e_k = t_k^{1/2} - T_k)$

 $= t^{1/2}$  $-$  $(e_i = t^{1/2} - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).

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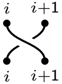
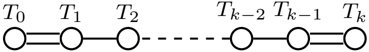
$T_k =$ , $\quad T_0 =$  and $\quad T_i =$ 

for $1 \leq i \leq k-1$,

subject to relations 

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(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
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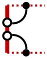
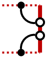
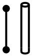
 $= t_0^{1/2}$  $-$  $\qquad (e_0 = t_0^{1/2} - T_0)$

 $= t_k^{1/2}$  $-$  $\qquad (e_k = t_k^{1/2} - T_k)$

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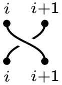
$T_k =$ , $T_0 =$  and $T_i =$ 

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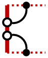
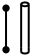
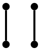
 $= t_0^{1/2}$  $-$ ,  $= t_k^{1/2}$  $-$  and  $= t^{1/2}$  $-$ 

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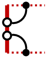 $= t_0^{1/2}$  $-$ ,  $= t_k^{1/2}$  $-$  and  $= t^{1/2}$  $-$ 

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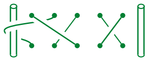
	Universal	Type B, C, D	Type A	Small Type A
Qu grp	Two-pole braids 	Two-pole BMW 	Affine Hecke of type C (+twists)	Two-boundary TL 

$M \otimes \left(V^{\otimes k}\right) \otimes N$

Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$, where

$\mathbf{c}$ is a point in the fundamental chamber of the (finite) type C hyperplane system, and

J is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting $\mathbf{c}$

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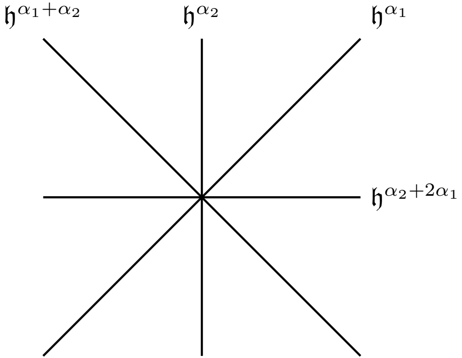

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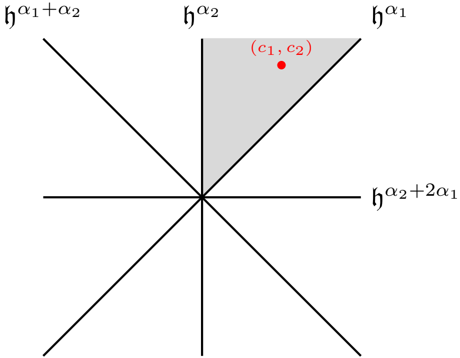

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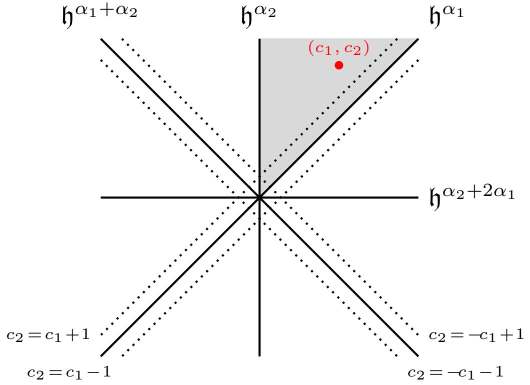

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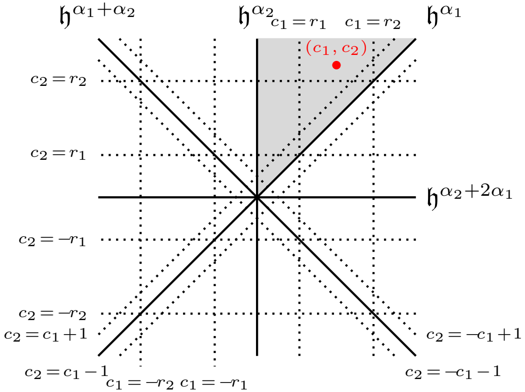

The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

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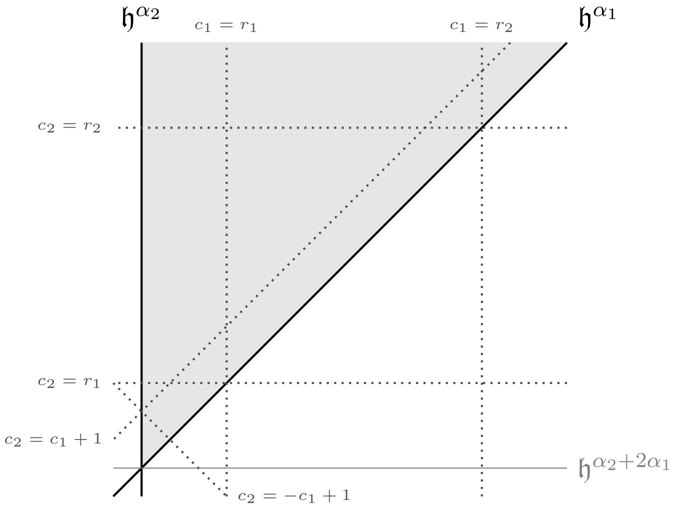

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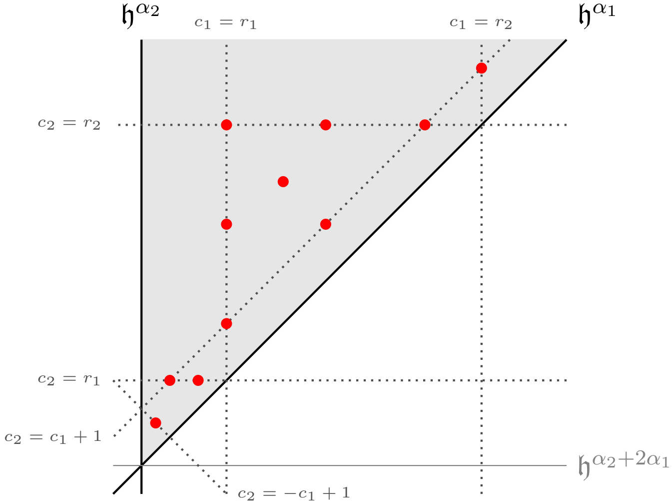

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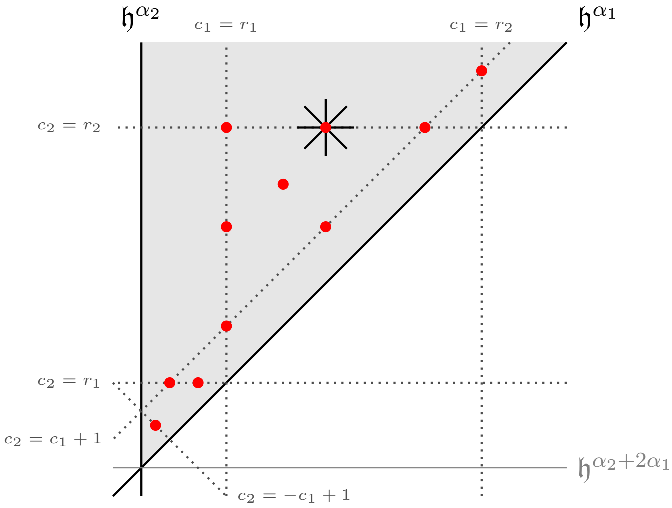

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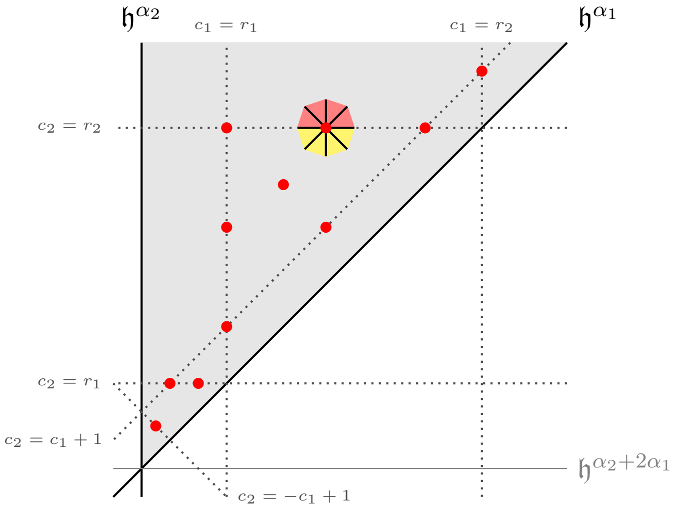

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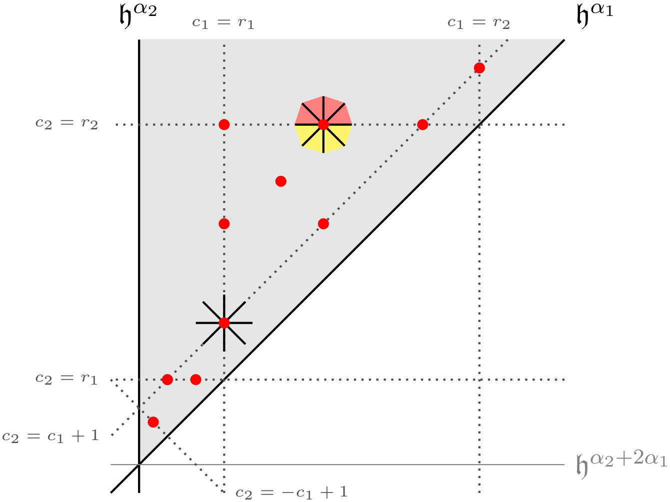

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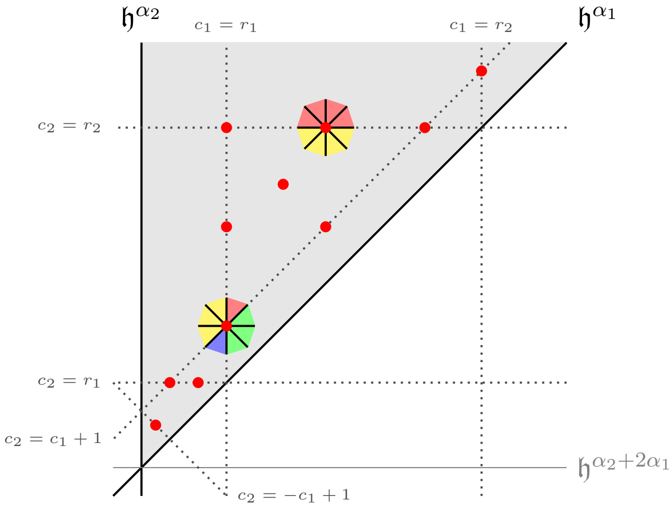

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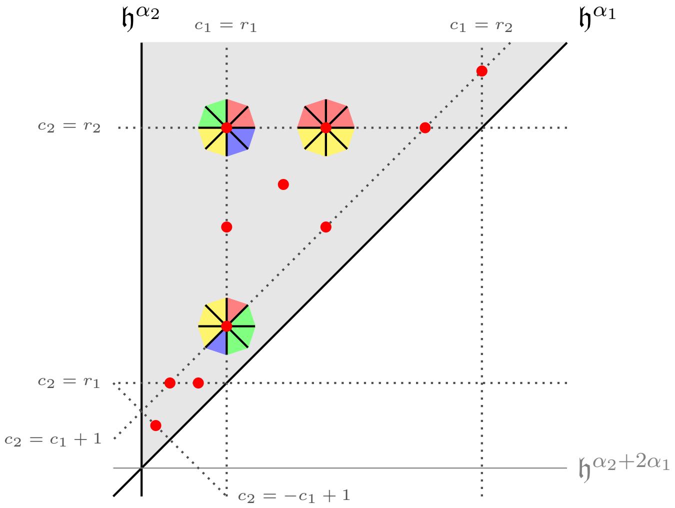

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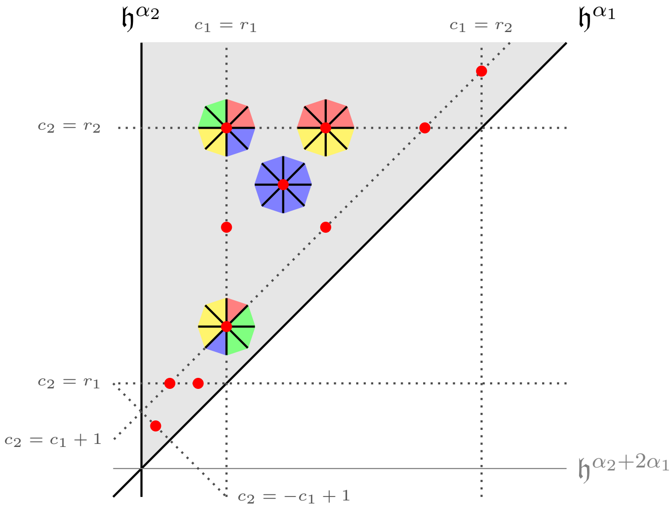

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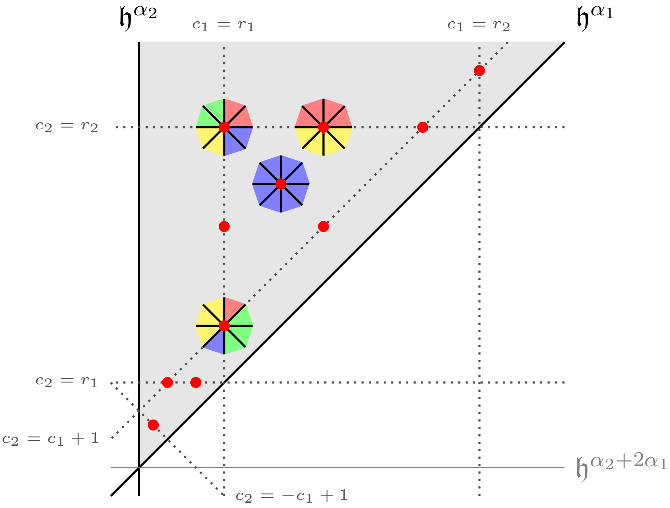

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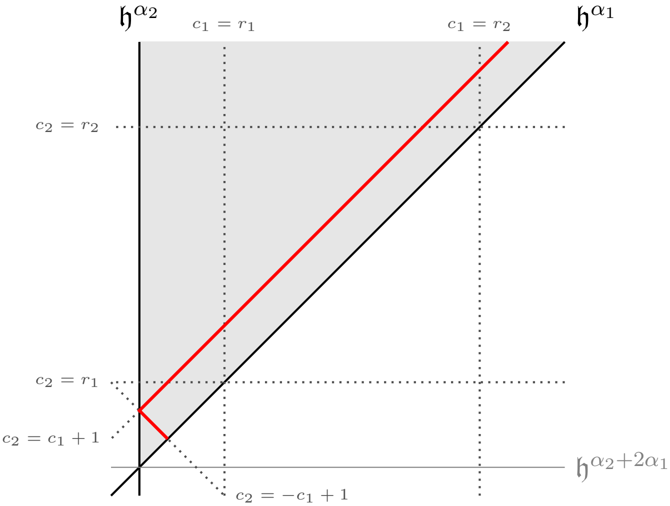
(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$.



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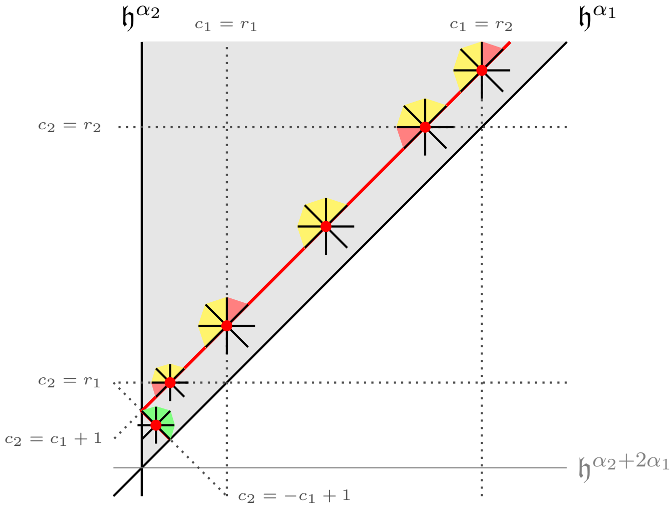

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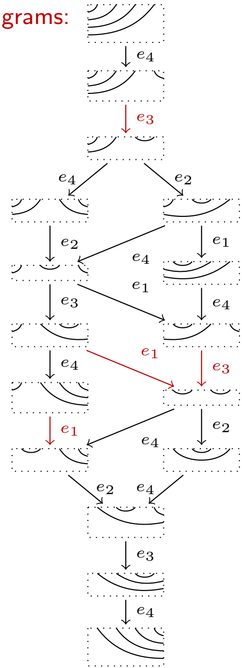
Diagrams:



[GN] [DR]

Aff. type C Hecke: