Centralizers of the Lie superalgebra $\mathfrak{p}(n)$, where loops go to die

Zajj Daugherty

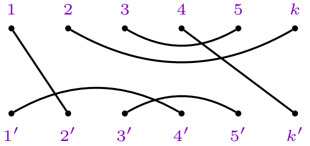
The City College of New York

Joint with Martina Balagovic, Maria Gorelik, Iva Halacheva, Johanna Hennig, Mee Seong Im, Gail Letzter, Emily Norton, Vera Serganova, and Catharina Stroppel

Women in Noncommutative Algebra and Representation Theory at BIRS, Spring 2016

The classical Brauer algebra

The Brauer algebra $B_k(\delta)$ is the space spanned by Brauer diagrams

$d =$ 

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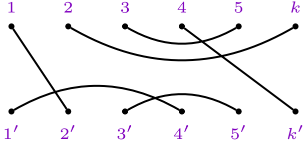
$d =$ 

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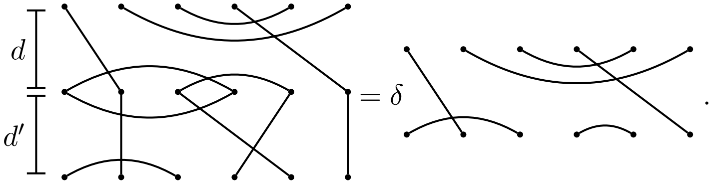
$dd' =$ 

Action on tensor space

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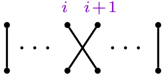
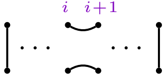
$s_i =$ 

and $e_i =$ 

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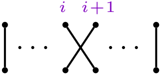
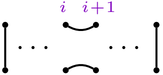
$s_i =$ 

and $e_i =$ 

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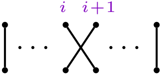
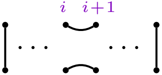
$s_i =$ 

and $e_i =$ 

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Let V be a f.d. vector space, with $\beta : V \otimes V \to \mathbb{C}$ a non-degenerate symmetric (resp. skew symmetric) bilinear form on V, and β^* its dual. Then the map $B_k(\delta) \to \mathrm{End}(V^{\otimes k})$ that sends

$$s_i \mapsto 1^{\otimes i-1} \otimes s \otimes 1^{k-i-1}, \qquad e_i \mapsto 1^{\otimes i-1} \otimes \beta^*\beta \otimes 1^{k-i-1},$$

where $s(u \otimes v) = v \otimes u$, is a map

$$B_k(\delta) \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes k})$$

when $\mathfrak{g} = \mathfrak{so}(V)$ (resp. $\mathfrak{sp}(V)$), $\delta = \mathrm{dim} V$ (resp. $-\mathrm{dim} V$).

Lie superalgebras and action on tensor space (still)

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$$\beta(u, v) = (-1)^{\bar{u}\bar{v}} \beta(v, u) \quad \text{(supersymmetric).}$$

Then

$$\mathfrak{g} = \{x \in \mathrm{End}(V) \mid \beta(xu, v) + (-1)^{\bar{x}\bar{u}} \beta(v, xu)\}$$

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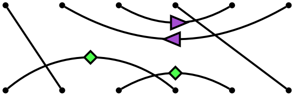
$d =$ 

caps get one ◆ each,
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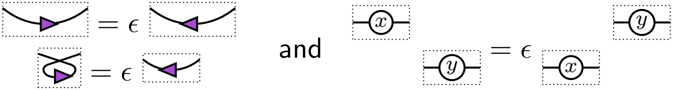
no two markings at same height.

with equivalence up to isotopy except for the local relations



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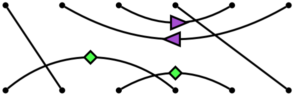
$d =$ 

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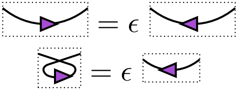

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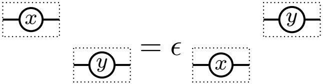

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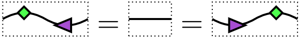

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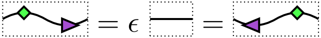

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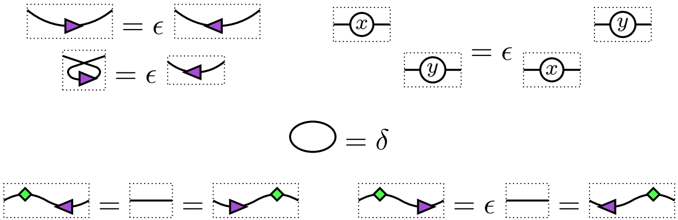

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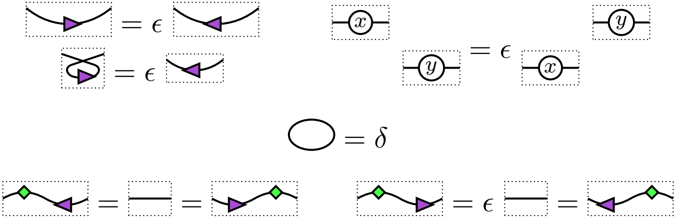

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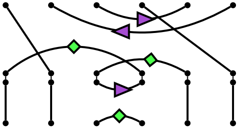

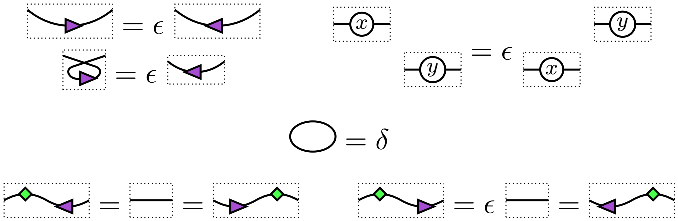

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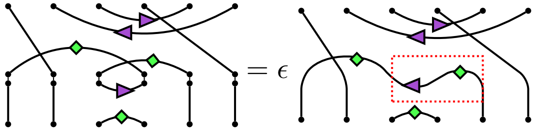

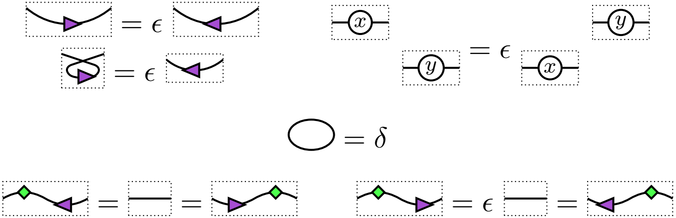

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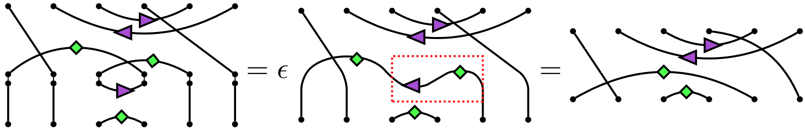

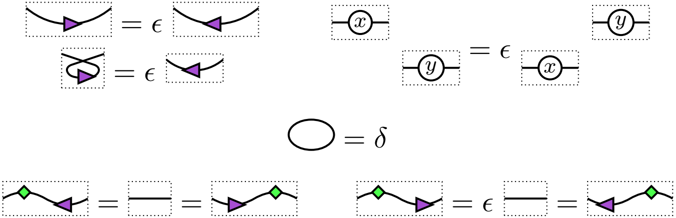

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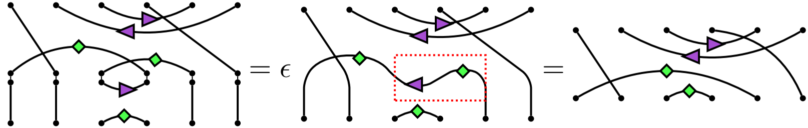

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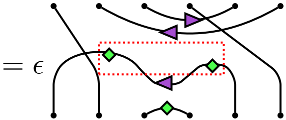

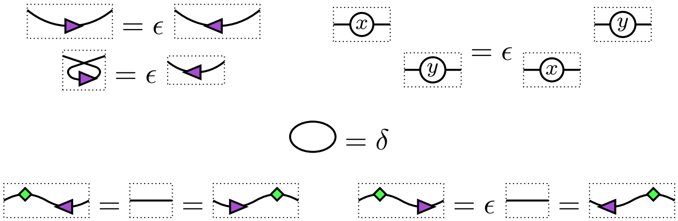

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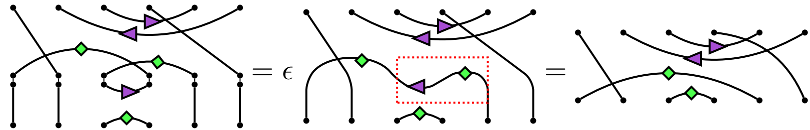

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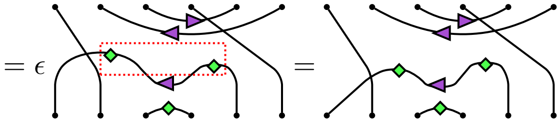

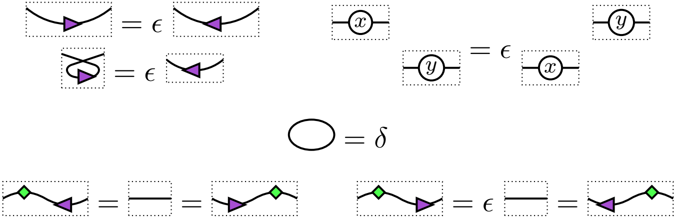

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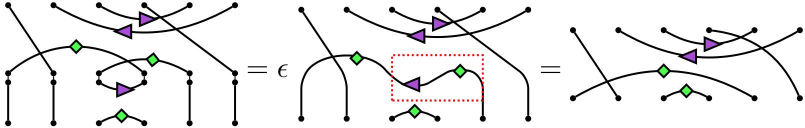

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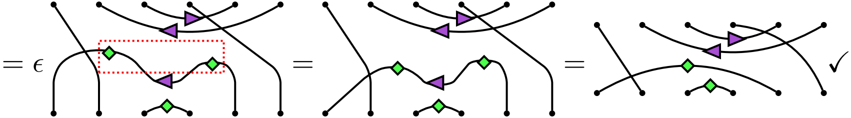

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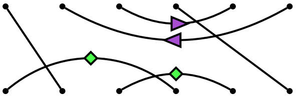
$d =$ 

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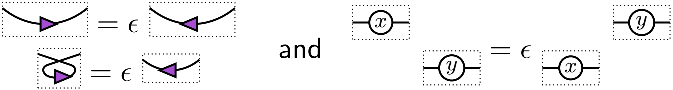

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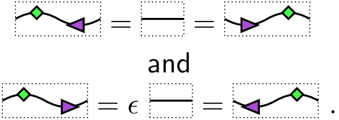

Note:
(1) $B_k(\delta, 1) = B_k(\delta)$.
(2) If $\epsilon = -1$, then multiplication is well-defined exactly when $\delta = 0$.

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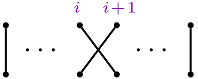
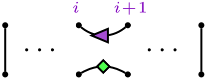
$s_i =$ 

and $e_i =$ 

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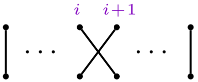
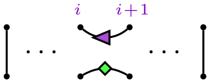
$s_i =$ 

and $e_i =$ 

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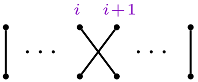
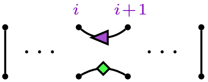
$s_i =$ 

and $e_i =$ 

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$$\beta^* : \mathbb{C} \to V \otimes V \quad \text{and} \quad \begin{aligned} s : V \otimes V &\to V \otimes V \\ u \otimes v &\mapsto (-1)^{\bar{u}\bar{v}} v \otimes u, \end{aligned}$$

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when $\delta = \mathrm{dim} V_0 - \mathrm{dim} V_1$ and $\epsilon = (-1)^{\bar{\beta}}$ [KT14].

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Specifically, with $n = \dim V_0 = \dim V_1$,

$$\mathfrak{p}(V) \cong \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n|n) \mid B = B^t, C = -C^t \right\}.$$

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The representation theory of $\mathfrak{p}(V)$ is still mysterious. In particular, $B_k(0, -1)$ was first defined by Moon in 2003 to help study $\mathfrak{p}(V)$; Kujawa and Tharp aimed to push further, getting that $V^{\otimes k}$ decomposes into the sum of indecomposables indexed by partitions of $k, k-2, k-4, \cdots > 0$.

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The representation theory of $\mathfrak{p}(V)$ is still mysterious. In particular, $B_k(0,-1)$ was first defined by Moon in 2003 to help study $\mathfrak{p}(V)$; Kujawa and Tharp aimed to push further, getting that $V^{\otimes k}$ decomposes into the sum of indecomposables indexed by partitions of $k, k-2, k-4, \cdots > 0$. Moon calculated the highest weight vectors for $\mathfrak{p}(V)$ in $V \otimes V$ and $V \otimes V \otimes V$ in detail.

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Specifically

$$V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V,$$

where $\mathrm{Sym}^2 V$ and $\bigwedge^2 V$ are both indecomposible, but not simple:

$$0 \to L(\square\square) \to \mathrm{Sym}^2 V \xrightarrow{\beta} \mathbb{C} \to 0$$

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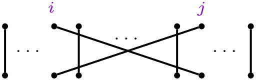
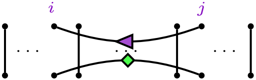
Jucys-Murphy elements and the Casimir

For $i < j$, let

$s_{i,j} =$  and $e_{i,j} =$  .

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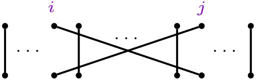
$s_{i,j} =$ 

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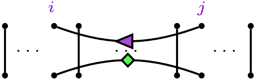
$e_{i,j} =$ 

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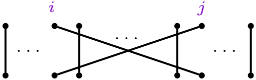
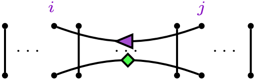
$s_{i,j} =$ 

and $e_{i,j} =$ 

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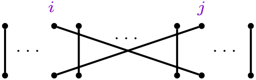
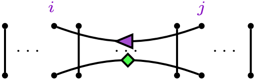
$s_{i,j} =$ 

and $e_{i,j} =$ 

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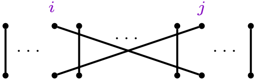
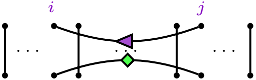
$s_{i,j} =$ 

and $e_{i,j} =$ 

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Define the degenerate affine version $\mathcal{B}_k(\delta)$ by

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For the marked Brauer algebra,

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(b) Non-semisimple actions! In $V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V$,

$$e_1 : \mathrm{Sym}^2 V \xrightarrow{\beta} \mathbb{C} \xrightarrow{\beta^*} \bigwedge^2 (V)$$

has non-trivial image. So, for example, the action of $B_3(0,-1)$ on $V^{\otimes 3}$ does not restrict to a closed action on $(\mathrm{Sym}^2 V) \otimes V$.

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To do: What are the minimal polynomials for γ? What happens at the next step $K(\lambda) \otimes V \otimes V$ when M_i doesn't split? What are the dimensions?

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Try 3: Kac modules of two types: $K(\lambda)$ (small) and $\tilde{K}(\lambda)$ (big) .
Let $\phi = \sum_{\alpha \in \Delta(\mathfrak{g}_{-1})} \alpha$ and let $V(\lambda)$ be the simple $\mathfrak{g}_0$-module of highest weight λ. Define

$$K(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V(\lambda - \phi) \qquad \tilde{K}(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}}^{\mathfrak{g}} V(\lambda).$$

Then $K(\lambda) \otimes V \cong M_1 \oplus \cdots \oplus M_n$ where

$$0 \to K(\lambda + \varepsilon_i) \to M_i \to K(\lambda - \varepsilon_i) \to 0,$$

whenever $\lambda \pm \varepsilon_i$ are dominant, or replace $K(*)$ with 0 whenever they're not (similar statement for $\tilde{K}$). Proof uses eigenvalues of γ on $K(\lambda) \otimes V$ and $\tilde{K}(\lambda) \otimes V$, which are combinatorial in terms of boxes added/removed (good), but do not differentiate between adding or removing (not as great).
To do: What are the minimal polynomials for γ? What happens at the next step $K(\lambda) \otimes V \otimes V$ when M_i doesn't split? What are the dimensions?