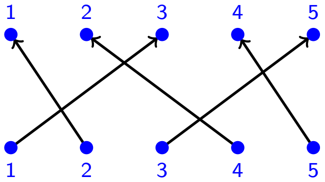
Permutations, partitions, lattices, and some linear algebra: a taste of combinatorial representation theory.

Zajj Daugherty
City College of New York

April 8, 2016

Permutations and the symmetric group

Permutation diagrams:



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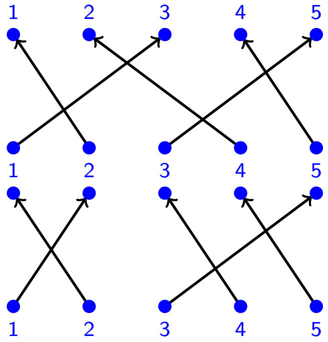

Permutations “multiply” by stacking and resolving.

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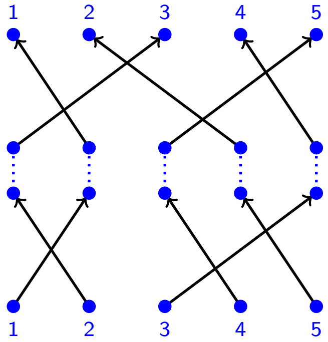

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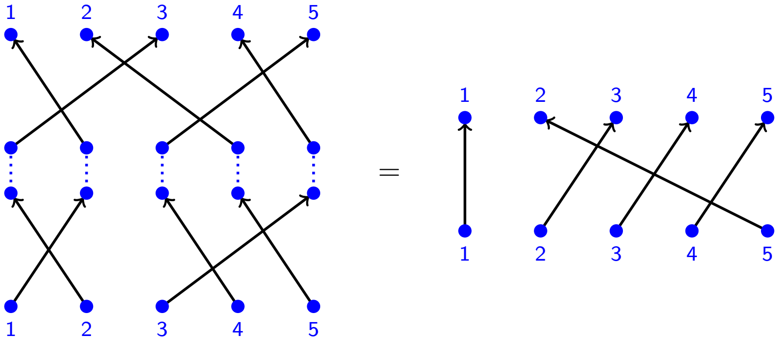

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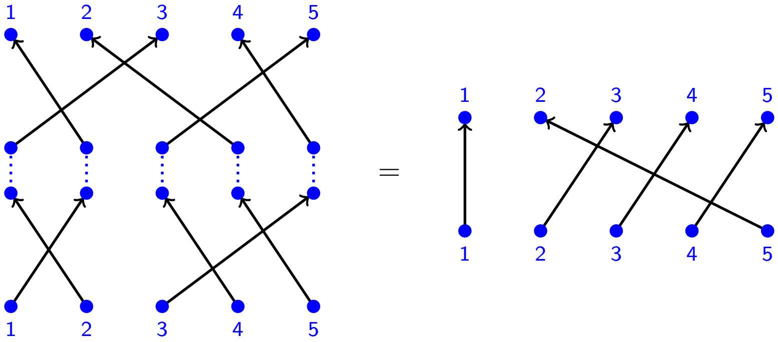


Permutations "multiply" by stacking and resolving.

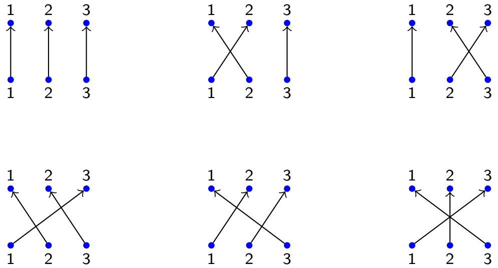
The symmetric group S_n is the group of permutations of $1, \dots, n$ with multiplication given by stacking and resolving diagrams.

Some examples:

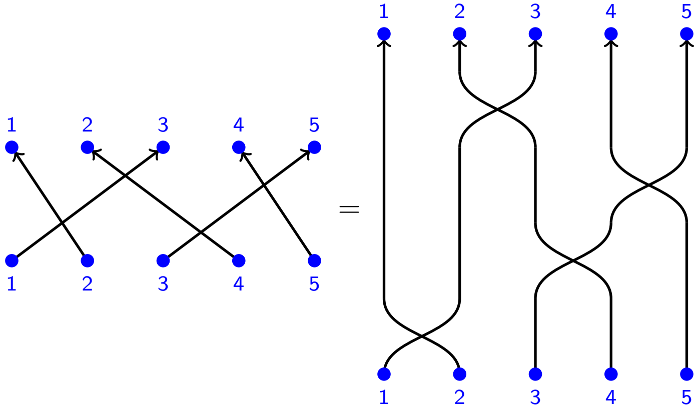
S_1 :

S_2 :

S_3 :



I can build any permutation by multiplying some sequence of adjacent transpositions. For example,



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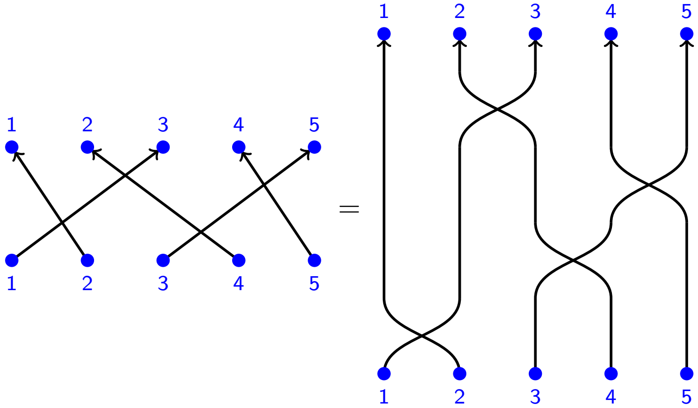

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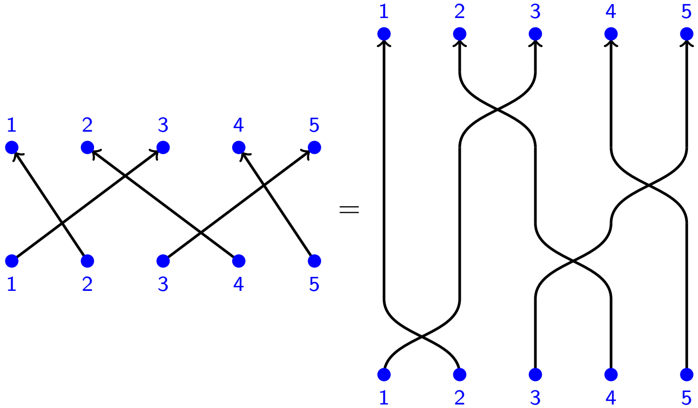

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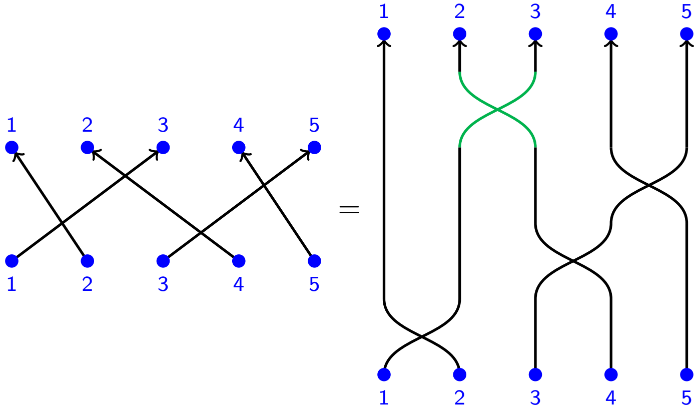

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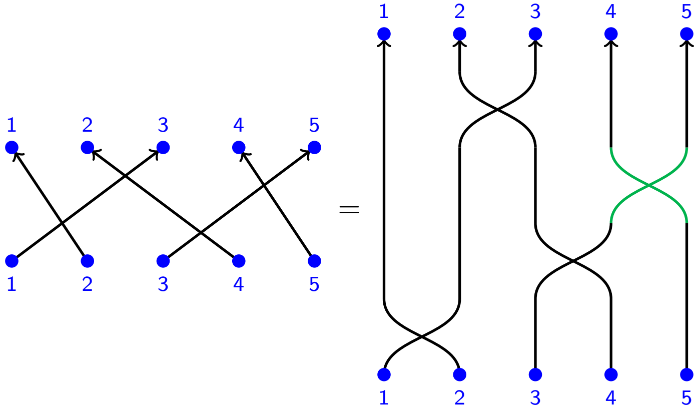

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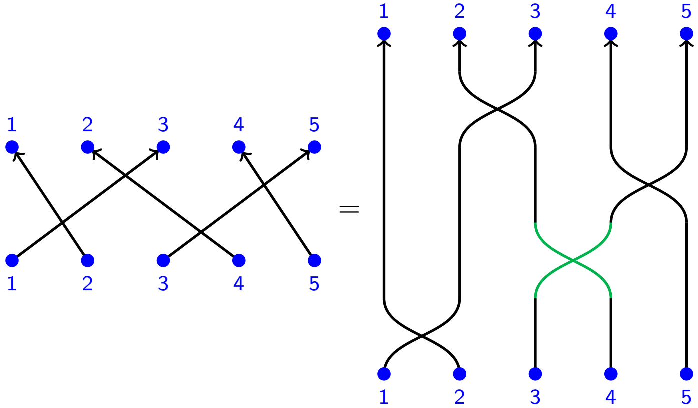

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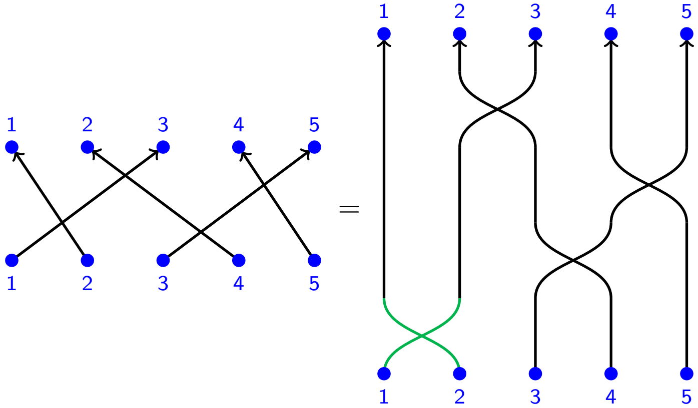

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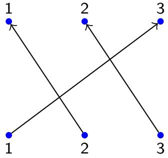
Example: Permutation representation of the symmetric group.



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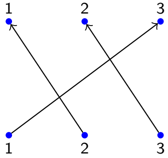

Pick a basis for $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

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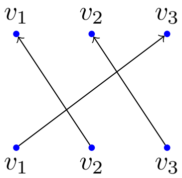

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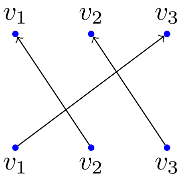

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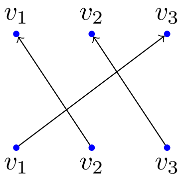

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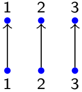
Aside: we actually have a representation of the group ring

$$\mathbb{R}S_n = \left\{ \sum_{\sigma \in S_n} r_\sigma \sigma \mid r_\sigma \in \mathbb{R} \right\},$$ with multiplication like polynomials.



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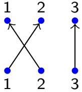

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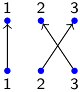

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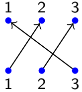

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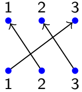

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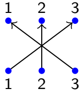

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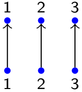

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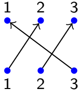

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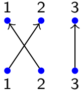

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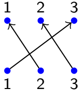

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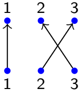

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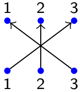

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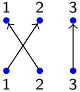

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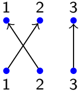

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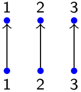
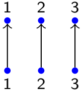


$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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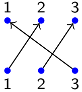

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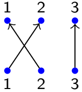

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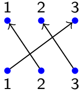

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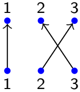

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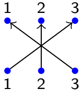

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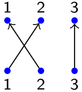

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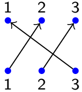

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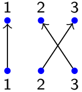

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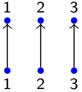

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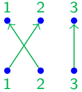

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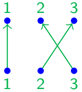
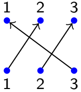


$$\mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad s_2$$



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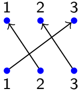

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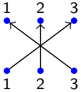

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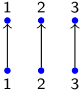
Note that I only had to tell you the matrices for s_1 and s_2! This is because the representation has the same multiplication rules as the permutations, and every permutation can be built out of these transpositions.



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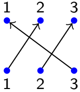

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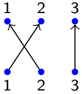

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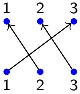

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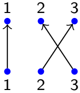

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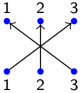

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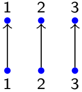

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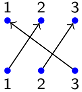

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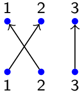

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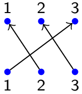

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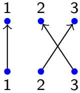

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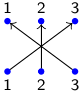

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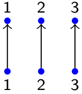

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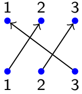

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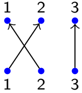

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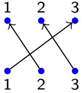

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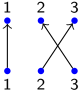

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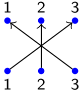

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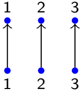

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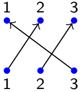

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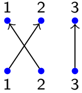

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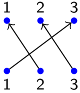

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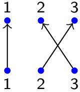

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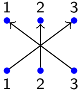

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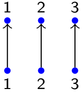

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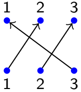

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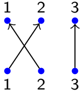

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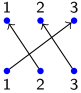

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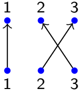

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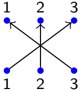

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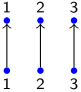

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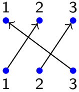

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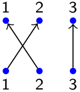

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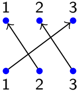

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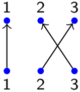

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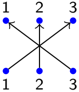

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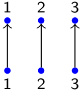

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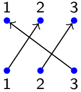

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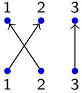

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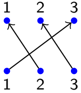

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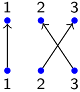

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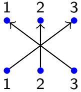

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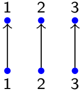

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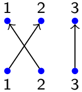

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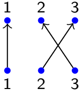

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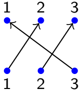

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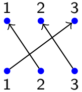

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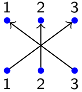

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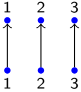
Start with the permutation representation P with basis $\{v_1, v_2, v_3\}$.



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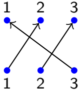

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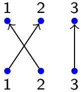

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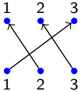

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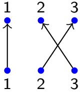

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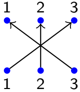

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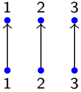

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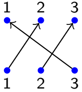

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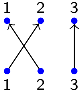

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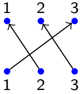

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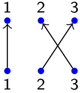

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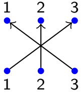

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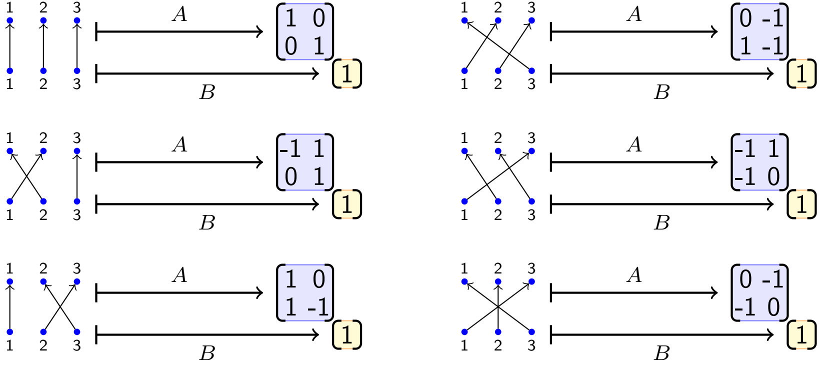

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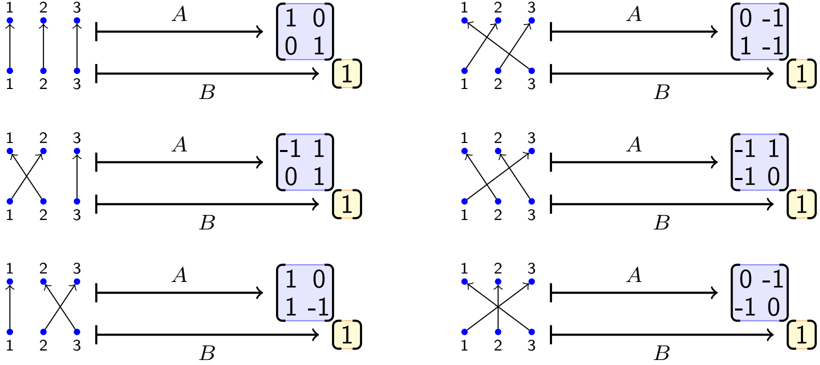

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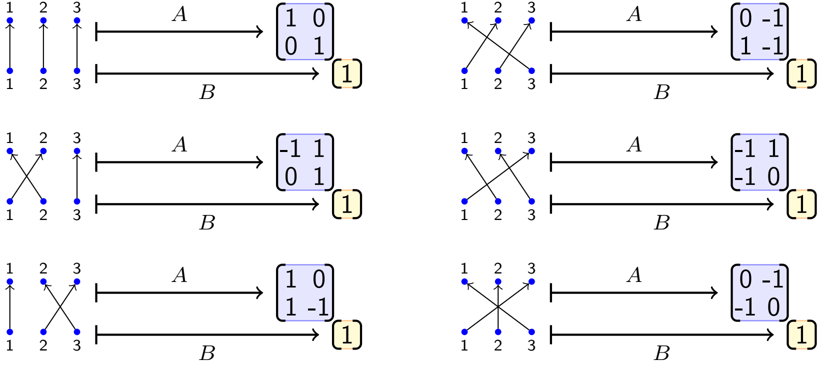

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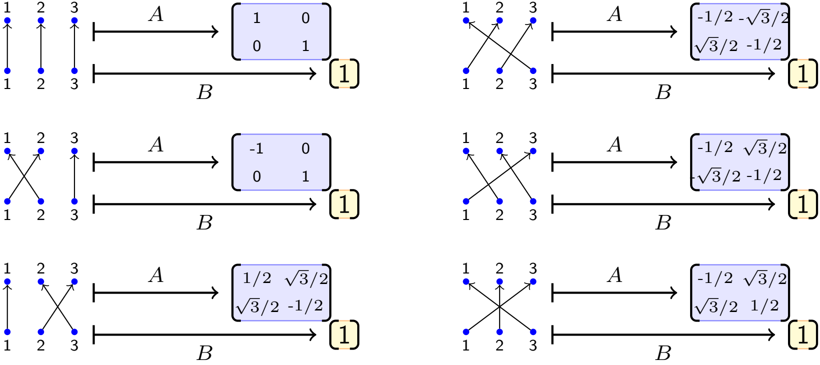

Start with the permutation representation P with basis $\{v_1, v_2, v_3\}$. Change to basis

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How about some combinatorics?

Let n be a non-negative integer.
A partition λ of n is a non-ordered list of positive integers which sum to n.

Example: the partitions of 3 are (3), $(2, 1)$, and $(1, 1, 1)$.

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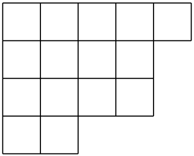
Example: the partitions of 3 are (3), $(2,1)$, and $(1,1,1)$.

We draw partitions as n boxes piled up and to the left, where the parts are the number of boxes in a row:

$$\lambda = (5,4,4,2) =$$



Young's lattice:

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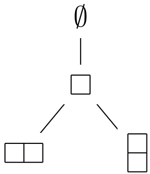

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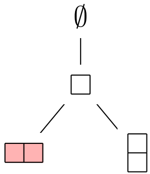

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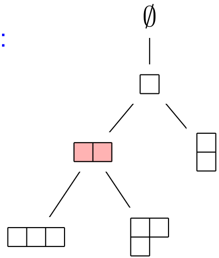

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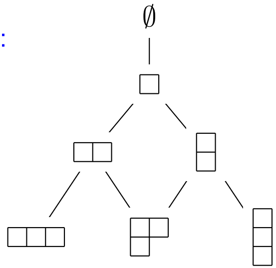

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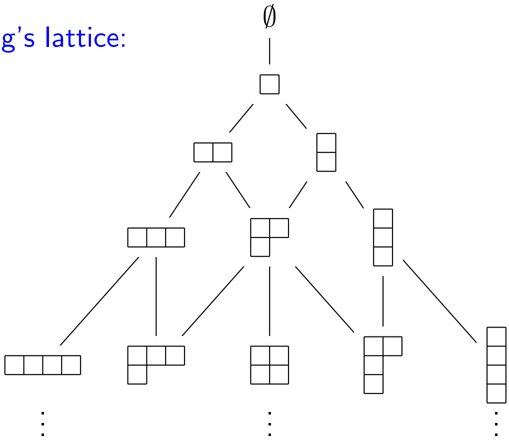

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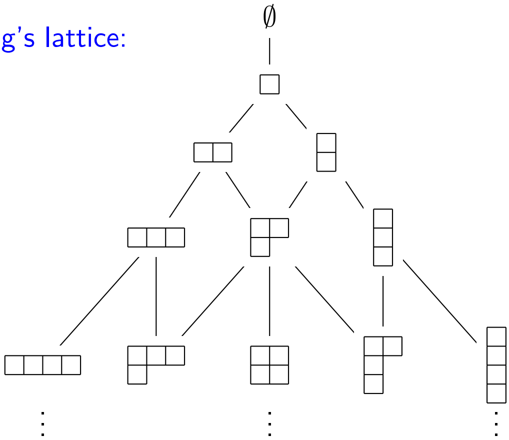

λ-Tableau: a path from $\emptyset$ down to a partition λ.

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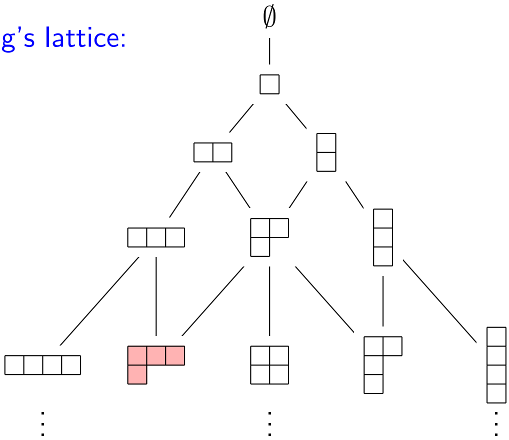

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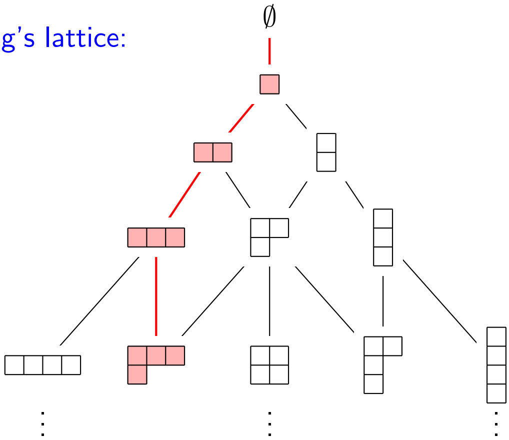

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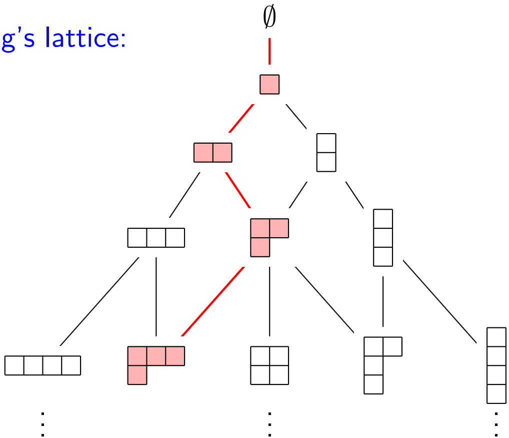

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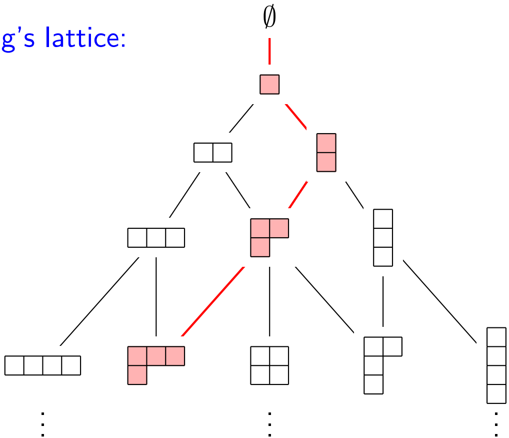

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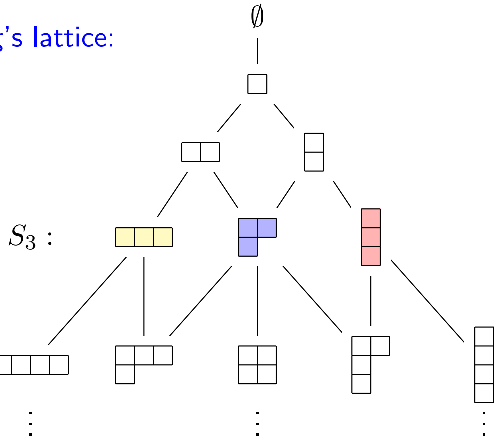

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Theorem 1: (Up to isomorphism) the simple S_n-representations are indexed by partitions of n.

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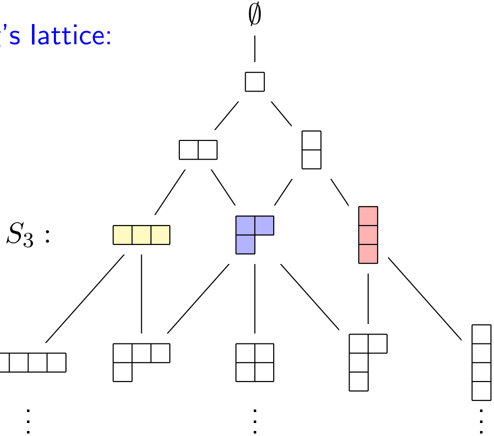

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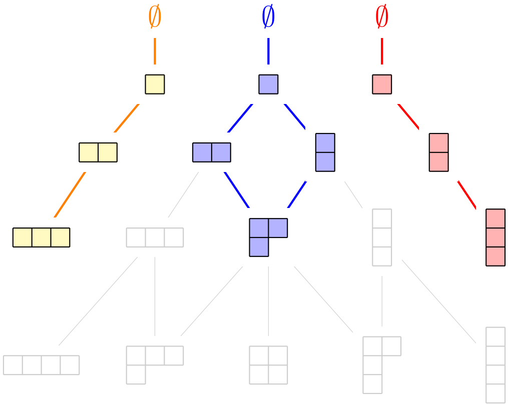
Theorem 2: If λ is a partition of n, then the corresponding representation has basis indexed by λ-tableaux, and matrices determined by other combinatorial data about those paths.



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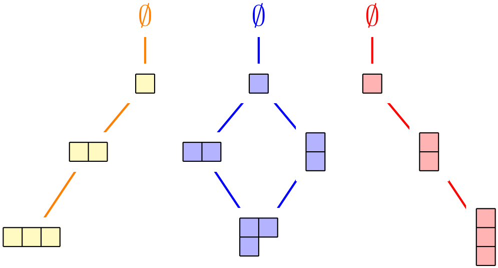

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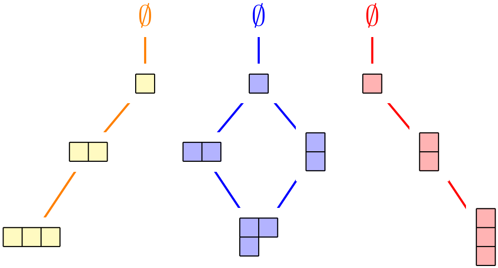

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The content of a box in a partition is its diagonal number:

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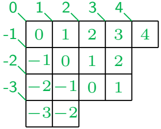

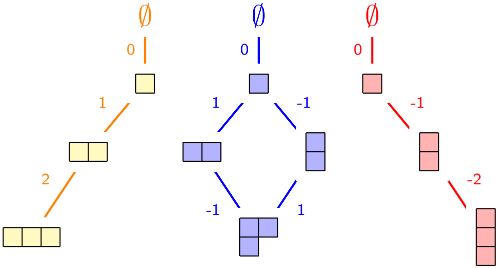

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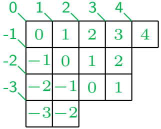
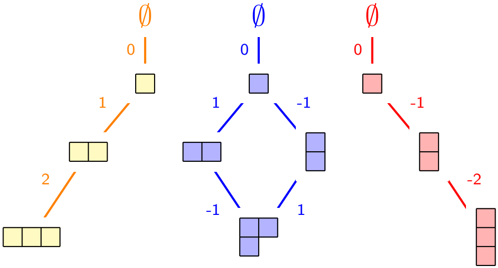
$\lambda = (5, 4, 4, 2) =$ 



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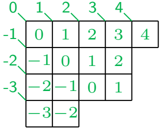

Again:

* each partition is secretly a representation
* each path is secretly a basis vector

Now: entries in matrices for s_1, s_2, ..., are given by expressions in the contents of boxes added.

The rule for s_i:

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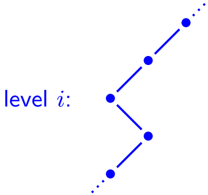
Suppose v goes with the path



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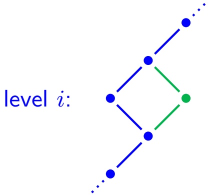

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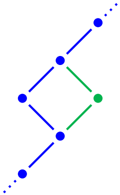

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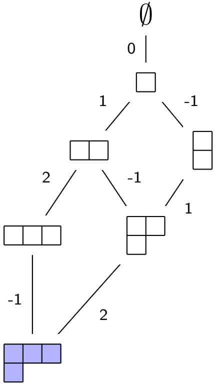

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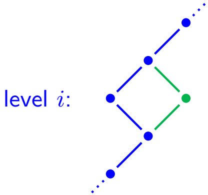

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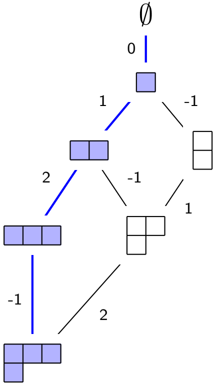

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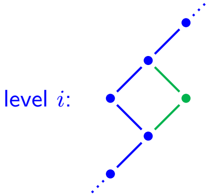

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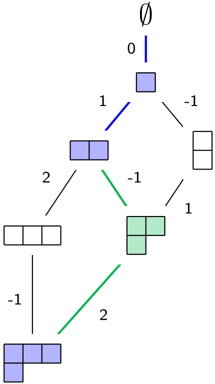

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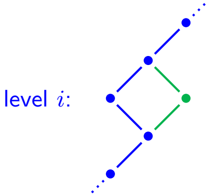

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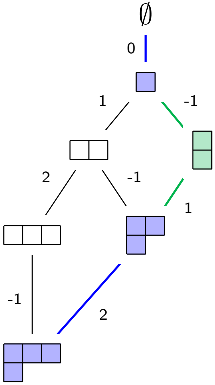

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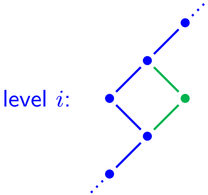

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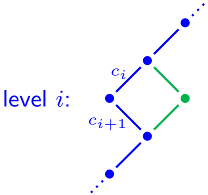

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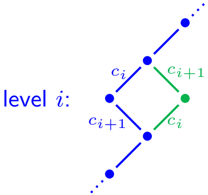

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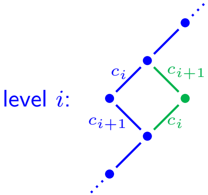

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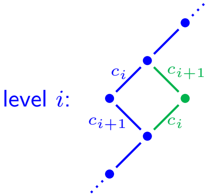

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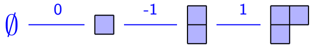
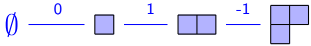
v: 

u: 

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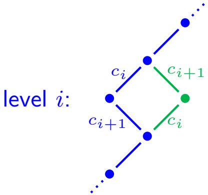

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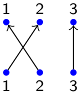
$s_1 =$ 

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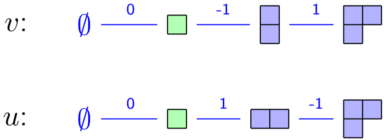

	v	u
v		
u		

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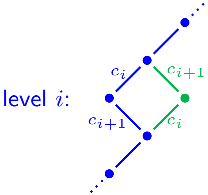

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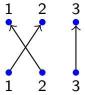
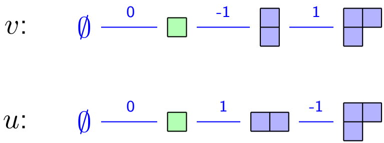
…on u is $\sqrt{1-(1/(c_{i+1}-c_i))^2}$
…on any other path is 0.

$s_1 =$ 

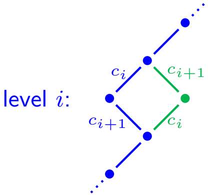
Back to S_3:



	v	u
v	$1/(-1-0)$	0
u	0	$1/(1-0)$

The rule for s_i:

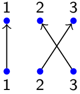
Suppose v goes with the path



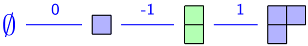
and u is almost the same, except at the ith step.

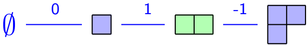
Let c_i be the content of the box added from $i-1$ to i.

Then the coefficient in $s_i \cdot v$
…on v is $1/(c_{i+1} - c_i)$
…on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$
…on any other path is 0.

$s_2 =$ 

Back to S_3:

v: 

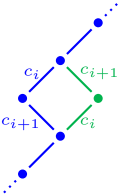
u: 

	v	u
v		
u		

The rule for s_i:

Suppose v goes with the path

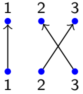
level i:



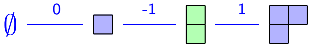
and u is almost the same, except at the ith step.

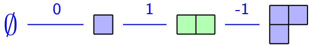
Let c_i be the content of the box added from $i-1$ to i.

Then the coefficient in $s_i \cdot v$
. . . on v is $1/(c_{i+1} - c_i)$
. . . on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$
. . . on any other path is 0.

$s_2 =$ 

Back to S_3:

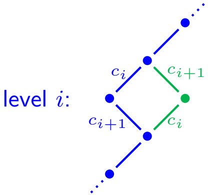
v: 

u: 

	v	u
v	$1/(1-(-1))$	
u		$1/(-1-1)$

The rule for s_i:

Suppose v goes with the path



and u is almost the same, except at the ith step.

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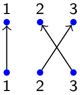
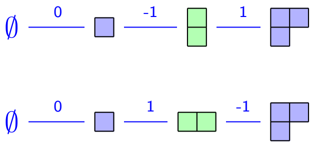
...on any other path is 0.

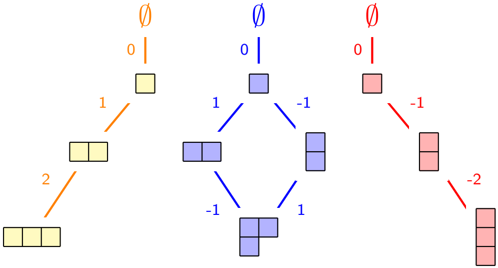
$s_2 =$ 

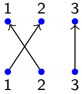
Back to S_3:

v: 

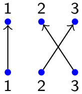
u:

	v	u
v	$1/(1-(-1))$	$\sqrt{1-1/4}$
u	$\sqrt{1-1/4}$	$1/(-1-1)$





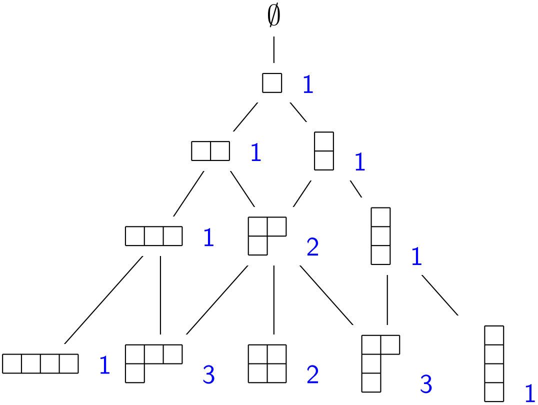
$$(1) \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (-1)$$



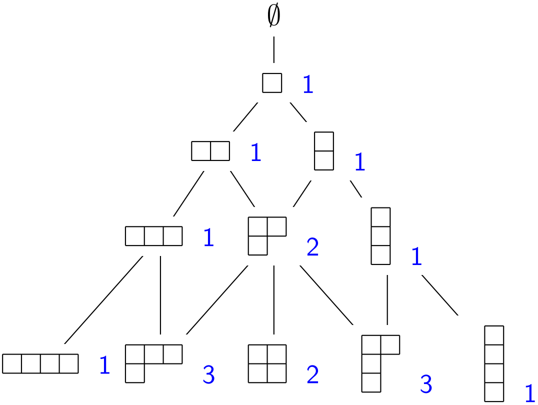
$$(1) \qquad \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1 \end{pmatrix} \qquad (-1)$$

"trivial" "alternating"

Counting tableaux and dimensions



Counting tableaux and dimensions



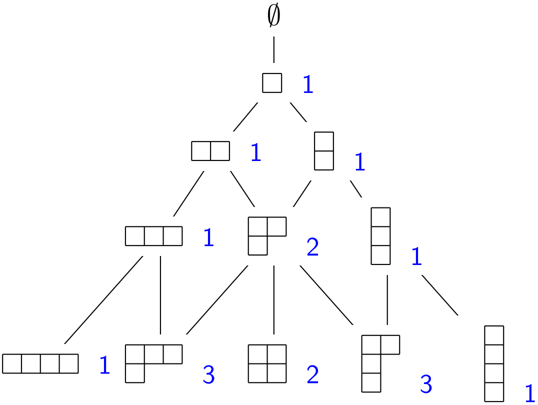
$1 = 1$

$1 + 1 = 2$

$1 + 2 + 1 = 4$

$1 + 3 + 2 + 3 + 1 = 10$

Counting tableaux and dimensions



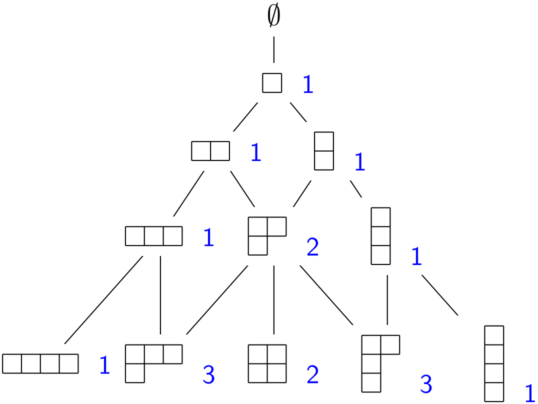
$$1^2 = 1$$

$$1^2 + 1^2 = 2$$

$$1^2 + 2^2 + 1^2 = 6$$

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 24$$

Counting tableaux and dimensions



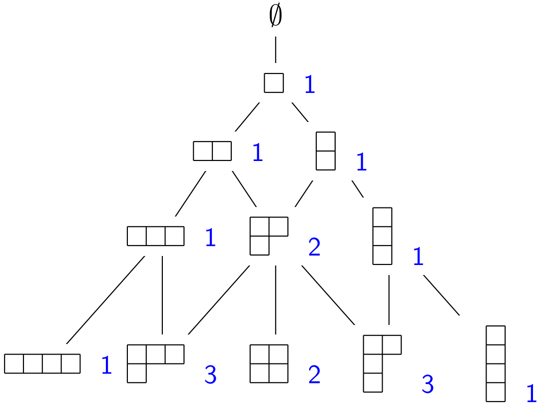
$1^2 = 1!$

$1^2 + 1^2 = 2!$

$1^2 + 2^2 + 1^2 = 3!$

$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 4!$

Counting tableaux and dimensions



$1^2 = |S_1|$

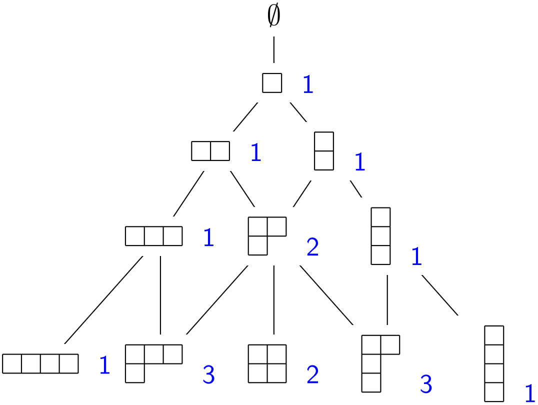
$1^2 + 1^2 = |S_2|$

$1^2 + 2^2 + 1^2 = |S_3|$

$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = |S_4|$

The matrix ring for an m-dimensional vec. sp. is m^2-dimensional!

Counting tableaux and dimensions



$1^2 = |S_1|$

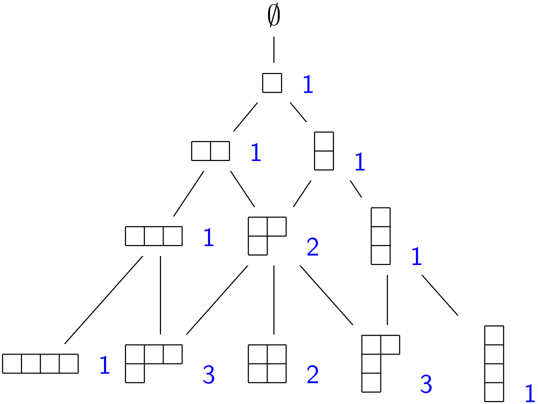
$1^2 + 1^2 = |S_2|$

$1^2 + 2^2 + 1^2 = |S_3|$

$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = |S_4|$

The matrix ring for an m-dimensional vec. sp. is m^2-dimensional!
Artin-Wedderburn theorem: "Nice" rings are isomorphic to the direct sum of matrix rings.

Counting tableaux and dimensions



$1^2 = |S_1|$

$1^2 + 1^2 = |S_2|$

$1^2 + 2^2 + 1^2 = |S_3|$

$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = |S_4|$

The matrix ring for an m-dimensional vec. sp. is m^2-dimensional!

Artin-Wedderburn theorem: "Nice" rings are isomorphic to the direct sum of matrix rings.

For example,

$\mathbb{R}S_3 \cong M_1(\mathbb{R}) \oplus M_2(\mathbb{R}) \oplus M_1(\mathbb{R}) \cong$