

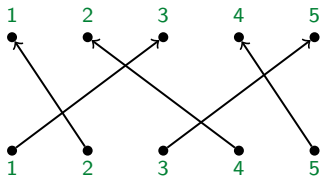
# Representations of the two-boundary Temperley-Lieb algebras

Zajj Daugherty  
(Joint work in progress with Arun Ram)

March 13, 2015

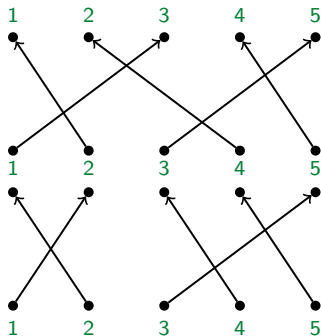
# Motivating example: Schur-Weyl Duality

The **symmetric group**  $S_k$  (permutations) as diagrams:



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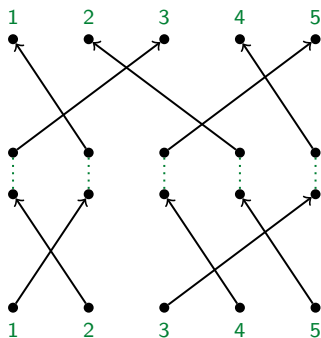
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(with multiplication given by concatenation)

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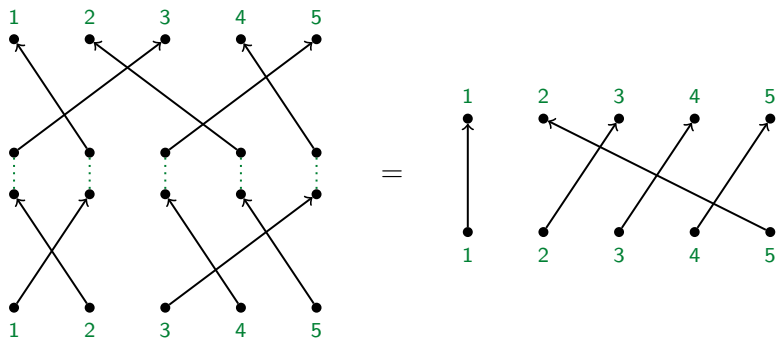
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## Motivating example: Schur-Weyl Duality

$\mathrm{GL}_n(\mathbb{C})$  acts on  $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$  diagonally.

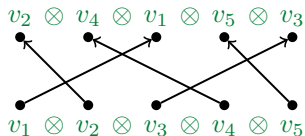
$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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$S_k$  also acts on  $(\mathbb{C}^n)^{\otimes k}$  by place permutation.

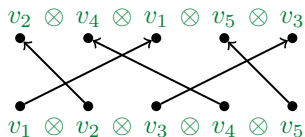


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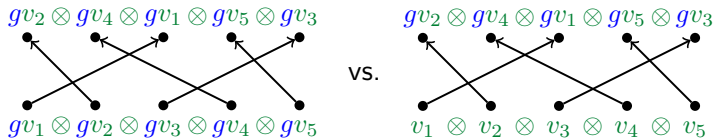
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These actions commute!





## Motivating example: Schur-Weyl Duality

Schur (1901):

$$\underbrace{\text{End}_{\text{GL}_n} \left( (\mathbb{C}^n)^{\otimes k} \right)}_{\text{(all linear maps that commute with } \text{GL}_n)} = \underbrace{\pi(\mathbb{C}S_k)}_{\text{(img of } S_k \text{ action)}} \quad \text{and} \quad \text{End}_{S_k} \left( (\mathbb{C}^n)^{\otimes k} \right) = \underbrace{\rho(\mathbb{C}\text{GL}_n)}_{\text{(img of } \text{GL}_n \text{ action)}}.$$

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Powerful consequence: a duality between representations

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \text{GL}_n\text{-}S_k \text{ bimodule,}$$

where  $G^\lambda$  are distinct irreducible  $\text{GL}_n$ -modules  
 $S^\lambda$  are distinct irreducible  $S_k$ -modules

## More centralizer algebras

Brauer (1937)

Orthogonal and symplectic groups  
(and Lie algebras) acting on  
 $(\mathbb{C}^n)^{\otimes k}$  diagonally centralize  
the **Brauer algebra**:

$$\delta_{b,c} \sum_{i=1}^n v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d$$

with  $\bigcirc = n$

(Diagrams encode maps  $V^{\otimes k} \rightarrow V^{\otimes k}$  that commute with the action of some classical algebra.)

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Temperley-Lieb (1971)

$GL_2$  and  $SL_2$  (and  $\mathfrak{gl}_2$  and  $\mathfrak{sl}_2$ ) acting on  $(\mathbb{C}^2)^{\otimes k}$  diagonally centralize the **Temperley-Lieb algebra**:

$$\delta_{c,d} \sum_{i=1}^2 v_a \otimes v_i \otimes v_i \otimes v_b \otimes v_e$$

with  $\bigcirc = 2$

(Diagrams encode maps  $V^{\otimes k} \rightarrow V^{\otimes k}$  that commute with the action of some classical algebra.)


## Quantum groups and braids

Fix  $q \in \mathbb{C}$ , and let  $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$  be the Drinfeld-Jimbo quantum group associated to Lie algebra  $\mathfrak{g}$ .

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$\mathcal{U} \otimes \mathcal{U}$  has an invertible element  $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$  that yields a map

$$\check{\mathcal{R}}_{VW}: V \otimes W \longrightarrow W \otimes V$$


that (1) satisfies braid relations, and


(2) commutes with the  $\mathcal{U}$ -action on  $V \otimes W$

for any  $\mathcal{U}$ -module  $V$ .

## Quantum groups and braids

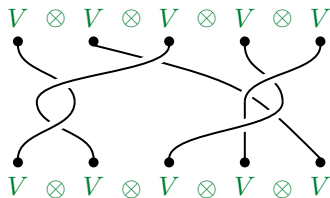
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The braid group shares a commuting action with  $\mathcal{U}$  on  $V^{\otimes k}$ :



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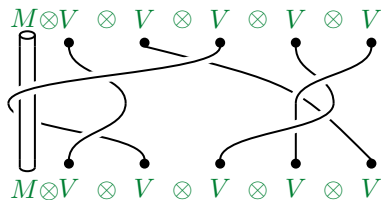
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The **one-pole/affine** braid group shares a commuting action with  $\mathcal{U}$  on  $M \otimes V^{\otimes k}$ :



Around the pole:




$$= \check{\mathcal{R}}_{MV} \check{\mathcal{R}}_{VM}$$



## Quantum groups and braids

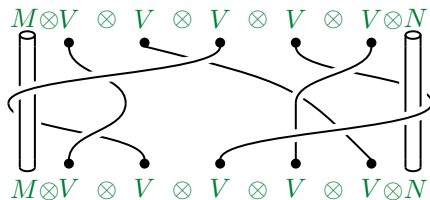
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The **two-pole** braid group shares a commuting action with  $\mathcal{U}$  on  $M \otimes V^{\otimes k} \otimes N$ :



Around the pole:

$$\begin{array}{c} M \otimes V \\ \text{pole} \\ M \otimes V \end{array} = \check{\mathcal{R}}_{MV} \check{\mathcal{R}}_{VM}$$

Universal

Type B, C, D

Type A

Small Type A

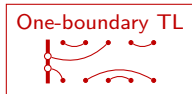
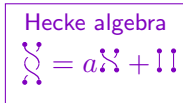
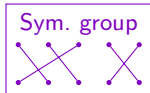
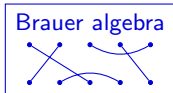
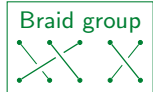
(orthog. & simpl.)

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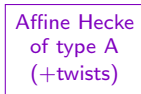
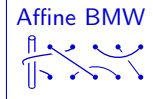
( $GL_2$  &  $SL_2$ )

Lie grp/alg

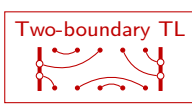
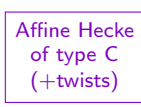
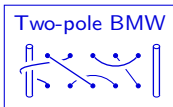
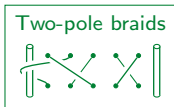
Quantum groups



$V = \square$   
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$M \otimes (V \otimes^k)$



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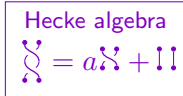
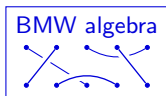
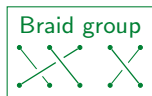
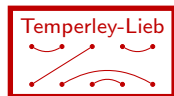
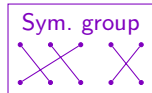
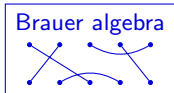
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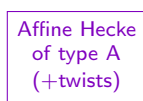
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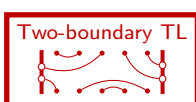
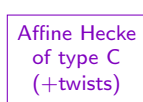
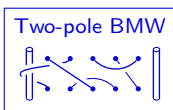
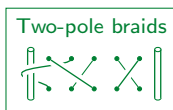


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Quantum groups



$M \otimes (V \otimes^k)$



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## Two-boundary Temperley-Lieb algebras

[MNGB04] Fix  $z, z_0, z_k \in \mathbb{C}$ . The *two-boundary Temperley-Lieb algebra*  $TL_k$  is a diagram algebra generated over  $\mathbb{C}$  by diagrams

$$e_0 = \begin{array}{c} 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 1 \end{array}, \quad e_k = \begin{array}{c} k \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ k \end{array}, \quad \text{and} \quad e_i = \begin{array}{c} i \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ i \end{array}$$

for  $i = 1, \dots, k-1$ , with relations  $e_i e_j = e_j e_i$  for  $|i-j| > 1$ ,

$$e_i e_{i\pm 1} e_i = e_i$$

for  $1 \leq i \leq k-1$ ,

$$e_i^2 = z_i e_i.$$

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Diagrammatic relations for  $e_i e_{i\pm 1} e_i = e_i$ :

- Box 1: A crossing of two strands is equal to a single strand.
- Box 2: A loop on a strand is equal to a single strand.
- Box 3: A loop on a strand with a crossing is equal to a single strand.

$$e_i^2 = z_i e_i.$$

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Diagrammatic relations for  $e_i e_{i\pm 1} e_i = e_i$ :

- Box 1: A diagram with three strands and two crossings (a braid) is equal to a single strand with two crossings.
- Box 2: A diagram with three strands and two crossings (a braid) is equal to a single strand with two crossings.
- Box 3: A diagram with three strands and two crossings (a braid) is equal to a single strand with two crossings.

$$e_i^2 = z_i e_i.$$

Diagrammatic relations for  $e_i^2 = z_i e_i$ :

- Box 1: A diagram with two strands and two crossings is equal to  $z$  times a single strand with two crossings.
- Box 2: A diagram with two strands and two crossings is equal to  $z_0$  times a single strand with two crossings.
- Box 3: A diagram with two strands and two crossings is equal to  $z_k$  times a single strand with two crossings.

## Two-boundary Temperley-Lieb algebras

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for  $i = 1, \dots, k-1$ , with relations  $e_i e_j = e_j e_i$  for  $|i-j| > 1$ ,

$$e_i e_{i\pm 1} e_i = e_i$$

for  $1 \leq i \leq k-1$ ,

Diagrammatic relations for  $e_i e_{i+1} e_i = e_i$ . The first box shows a strand with a side loop on the left and a cap on the right, equal to a single strand with a cap. The second box shows a strand with a side loop on the right and a cap on the left, equal to a single strand with a cap. The third box shows a strand with a side loop on the left and a cap on the left, equal to a single strand with a cap.

$$e_i^2 = z_i e_i.$$

Diagrammatic relations for  $e_i^2 = z_i e_i$ . The first box shows two strands with side loops on the left, equal to a single strand with a cap. The second box shows two strands with side loops on the right, equal to  $z_0$  times a single strand with a cap. The third box shows two strands with side loops on the left, equal to  $z_k$  times a single strand with a cap.

(Side loops are resolved with a 1 or a  $z_i$  depending on whether there are an even or odd number of connections below their lowest point.)



Diagram multiplication:

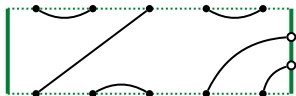


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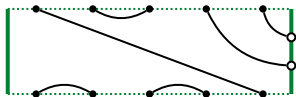
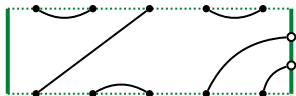


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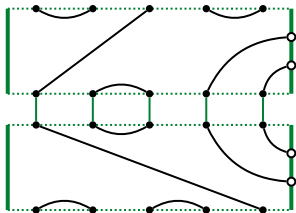
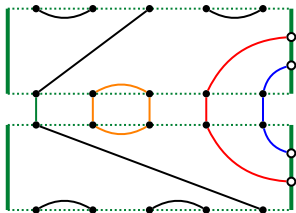
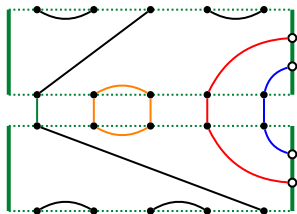


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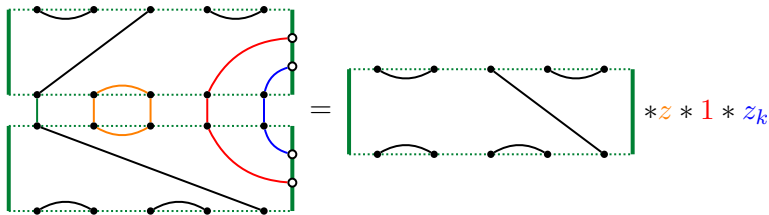


=



$*z * 1 * z_k$

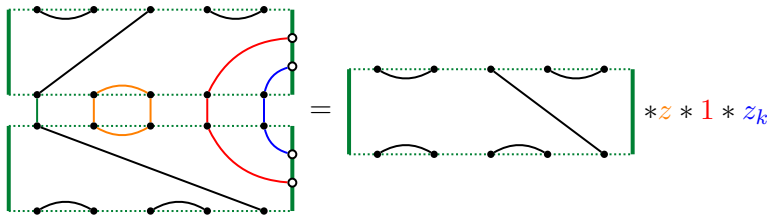
## Diagram multiplication:



In short,  $TL_k$  has basis given by non-crossing diagrams with

- (1)  $k$  connections to the top and to the bottom,
- (2) an even number of connections to the right and to the left, and
- (3) no edges with both ends on the left or both ends on the right.

## Diagram multiplication:



In short,  $TL_k$  has basis given by non-crossing diagrams with

- (1)  $k$  connections to the top and to the bottom,
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- (3) no edges with both ends on the left or both ends on the right.

However,

$$2\ell \left[ \begin{array}{c} \text{Diagram with } 2\ell \text{ horizontal lines and arcs} \end{array} \right] \in TL_k$$

The diagram shows a vertical stack of  $2\ell$  horizontal lines. Each line has a small arc at its top and bottom ends. The lines are connected to the right side of the diagram, forming a vertical stack of  $2\ell$  vertices. The entire diagram is enclosed in a green dashed box.

So unlike the classical T-L algebras,  $TL_k$  is not finite dimensional!

Universal

Type B, C, D

Type A

Small Type A

(orthog. & simpl.)

(gen. & sp. linear)

( $GL_2$  &  $SL_2$ )

Lie grp/alg

Quantum groups

Brauer algebra



Sym. group



Temperley-Lieb



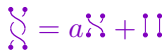
Braid group



BMW algebra



Hecke algebra



Affine braids

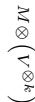


Affine BMW



Affine Hecke of type A (+twists)

One-boundary TL



Two-pole braids



Two-pole BMW



Affine Hecke of type C (+twists)

Two-boundary TL





Universal

Type B, C, D

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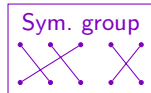
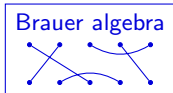
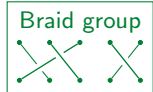
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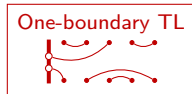
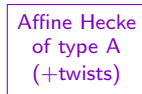
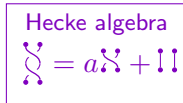
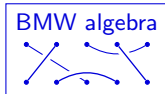
( $GL_2$  &  $SL_2$ )

Lie grp/alg

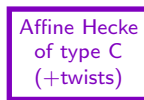
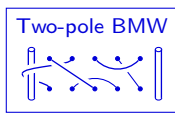
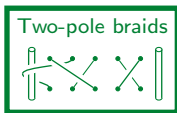
Quantum groups



$V = \square$   
 $V \otimes \dots \otimes V$   
 $V \otimes \dots \otimes V$



$M \otimes (V \otimes_k \dots \otimes V)$



$M \otimes (V \otimes_k \dots \otimes V) \otimes N$

The two-boundary (two-pole) braid group  $\mathcal{B}_k$  is generated by

$$T_k = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad T_0 = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \overset{i}{\bullet} \quad \overset{i+1}{\bullet} \\ \diagdown \quad \diagup \\ \underset{i}{\bullet} \quad \underset{i+1}{\bullet} \end{array} \quad \text{for } 1 \leq i \leq k-1,$$

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$$T_i T_{i+1} T_i = \text{diagram} = \text{diagram} = T_{i+1} T_i T_{i+1},$$

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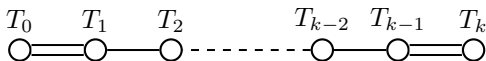
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subject to relations



i.e.

$$T_i T_{i+1} T_i = \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ i \quad i+1 \end{array} = \begin{array}{c} \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ i \quad i+1 \end{array} = T_{i+1} T_i T_{i+1},$$

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(2) Fix constants  $t_0, t_k, t = t_1 = t_2 = \dots = t_{k-1} \in \mathbb{C}$ .

The affine type C Hecke algebra  $\mathcal{H}_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by the relations  $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$ .



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so that  $e_j^2 = z_j e_j$  (for good  $z_j$ ).

The **two-boundary Temperley-Lieb algebra** is the quotient of  $\mathcal{H}_k$  by the relations  $e_i e_{i\pm 1} e_i = e_i$  for  $i = 1, \dots, k-1$ .

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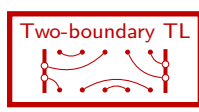
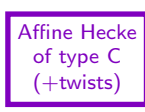
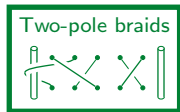
Universal

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Qu grp



## Representation theory of $\mathcal{H}_k$

The representations of  $\mathcal{H}_k$  are indexed by pairs  $(\mathbf{c}, J)$ , where

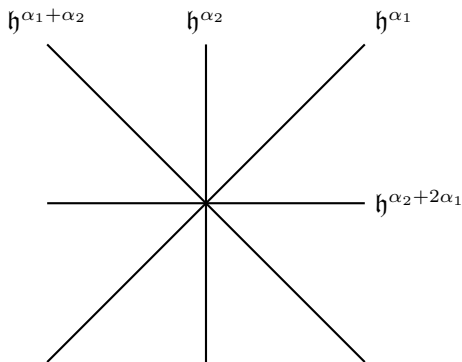
$\mathbf{c}$  is a point in the fundamental chamber of  
the (finite) type C hyperplane system, and  
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Example:  $k = 2$

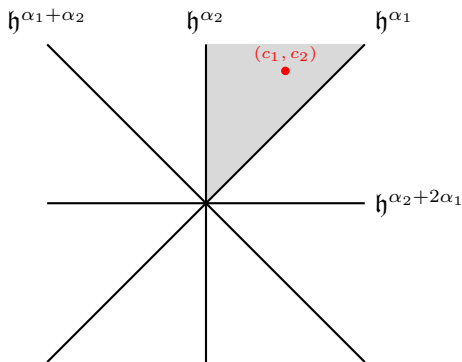


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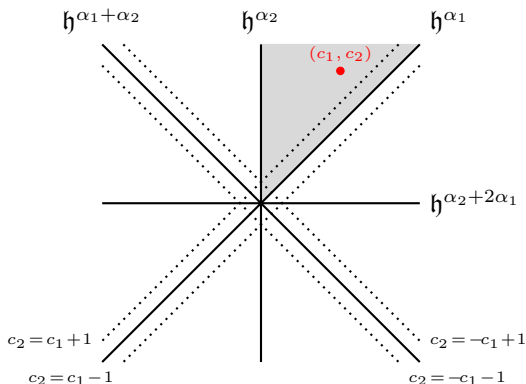


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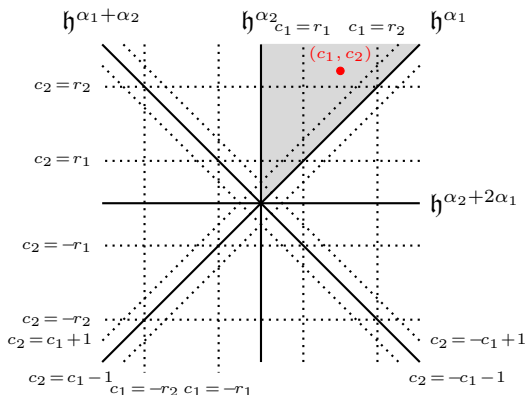


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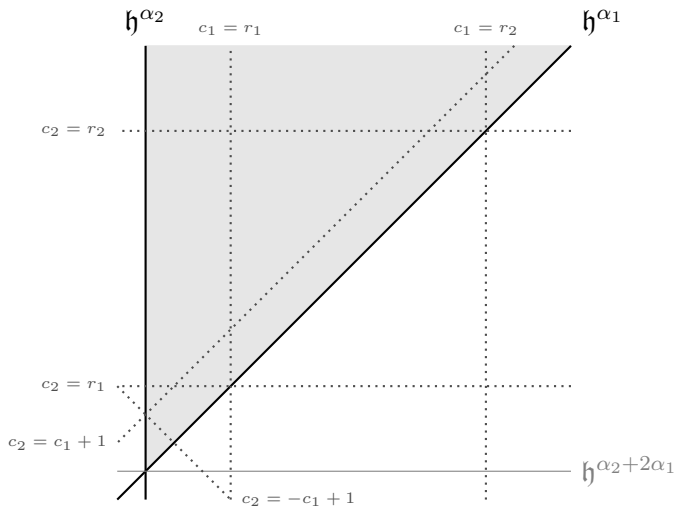
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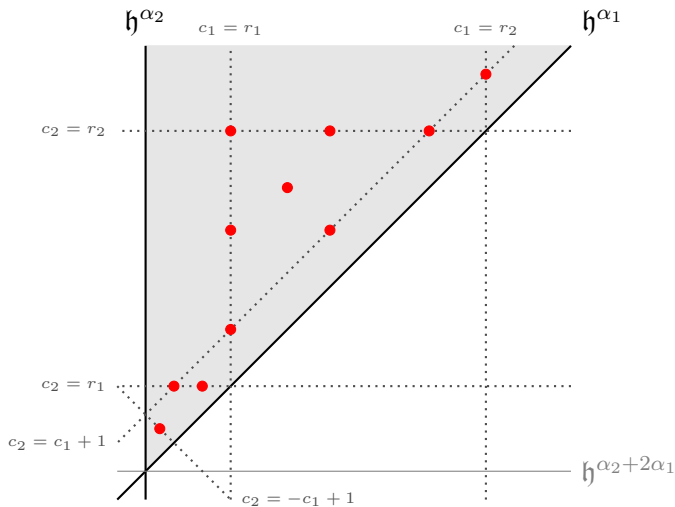
The  $r_i$ s depend on  $\mathcal{H}_k$ 's parameters  $t_0$  and  $t_k$ :  $r_1 = \log_t(t_0/t_k)$ ,  $r_2 = \log_t(t_0 t_k)$

# Representation theory of $\mathcal{H}_k$



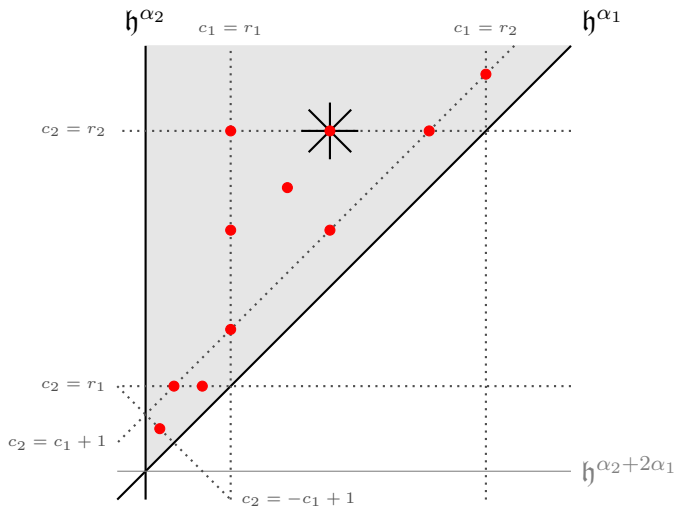
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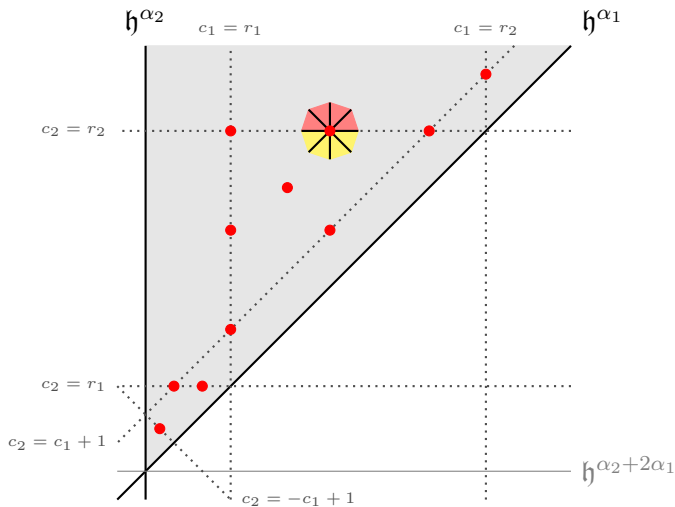
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# Representation theory of $\mathcal{H}_k$



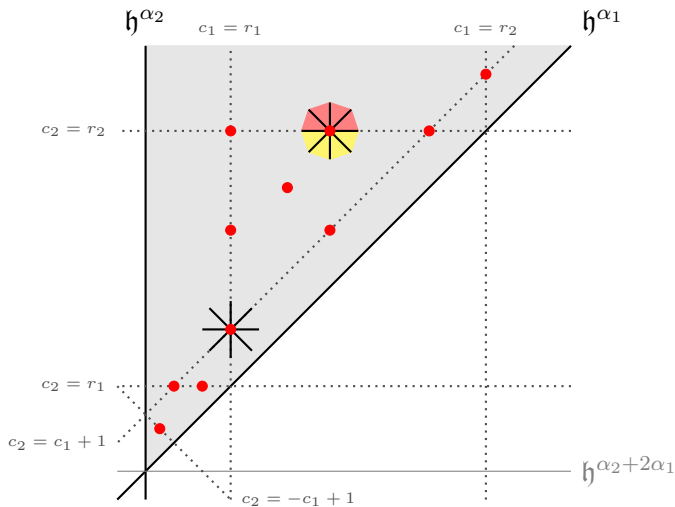
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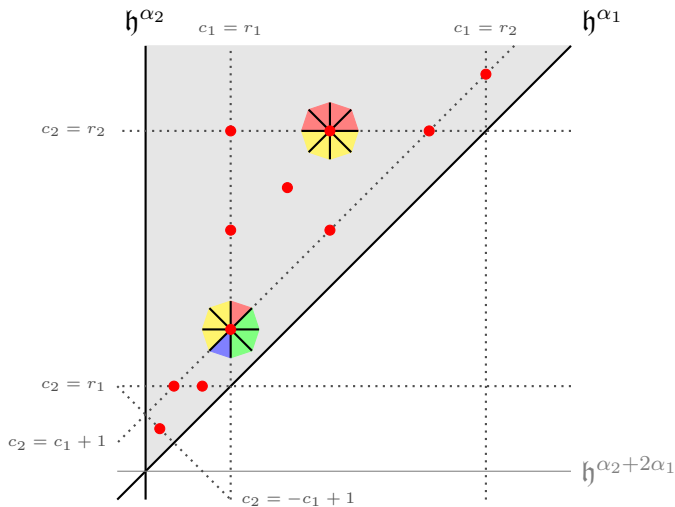
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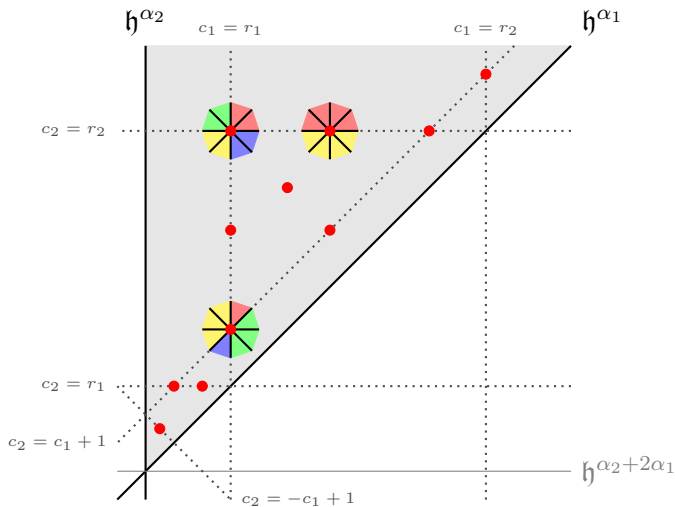
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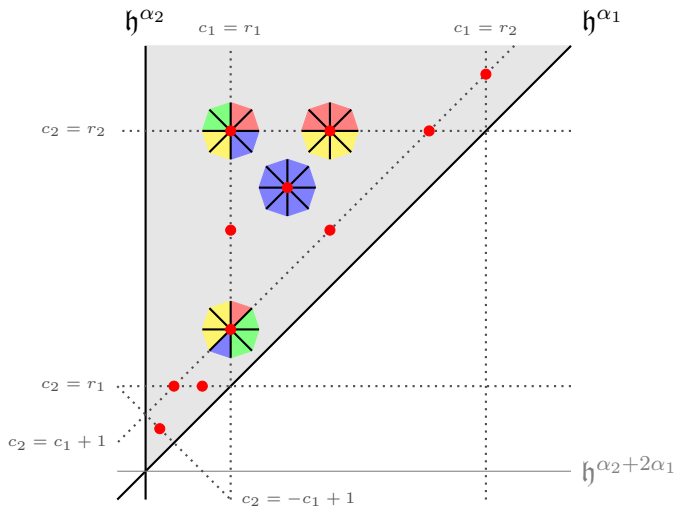


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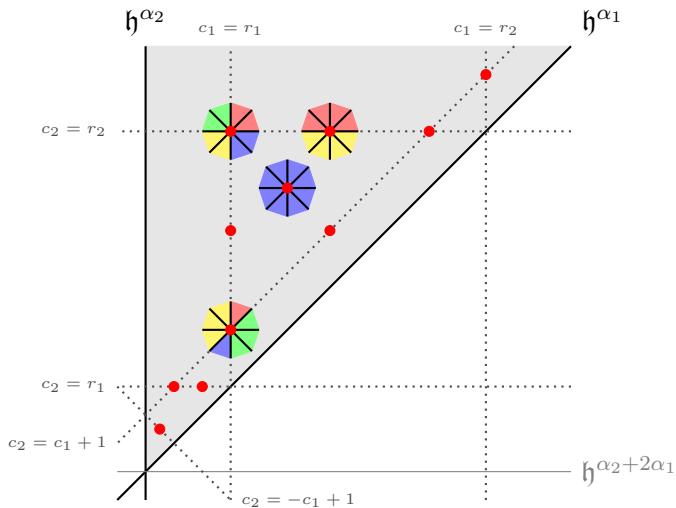


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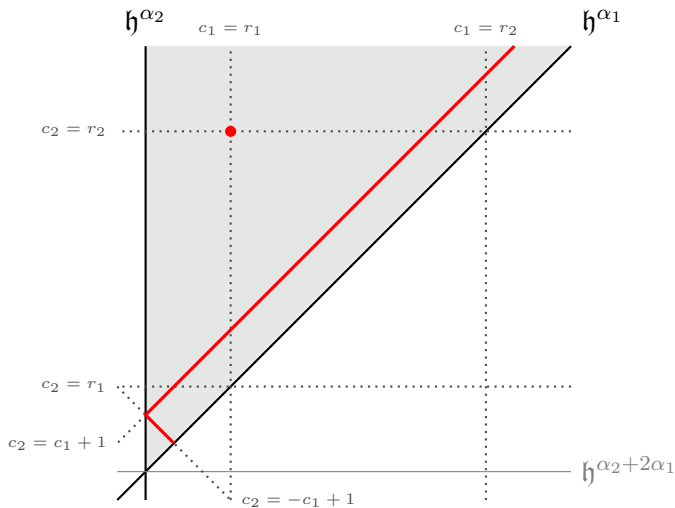


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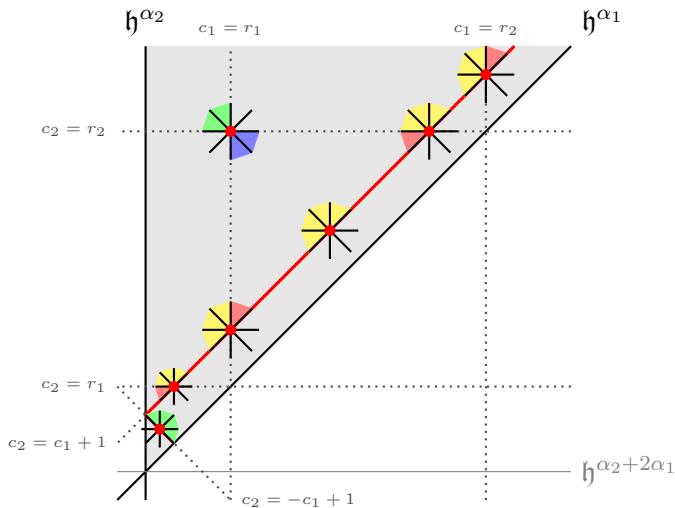
Thm. (D.-Ram)

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