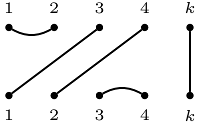
Representation theory of the two-boundary Temperley-Lieb algebra

Zajj Daugherty
(Joint work in progress with Arun Ram)

September 10, 2014

Temperley-Lieb algebras

The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices

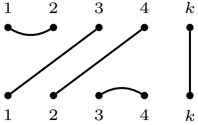


with multiplication given by stacking diagrams, subject to the relation

$$= q + q^{-1}$$

Temperley-Lieb algebras

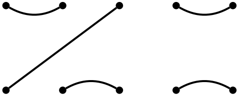
The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices



with multiplication given by stacking diagrams, subject to the relation

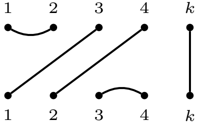
$$= q + q^{-1}$$

Multiplication:



Temperley-Lieb algebras

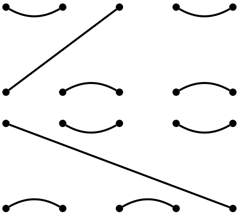
The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices



with multiplication given by stacking diagrams, subject to the relation

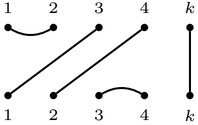
 $= q + q^{-1}$

Multiplication:



Temperley-Lieb algebras

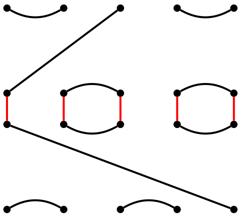
The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices



with multiplication given by stacking diagrams, subject to the relation

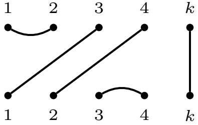
 $= q + q^{-1}$

Multiplication:



Temperley-Lieb algebras

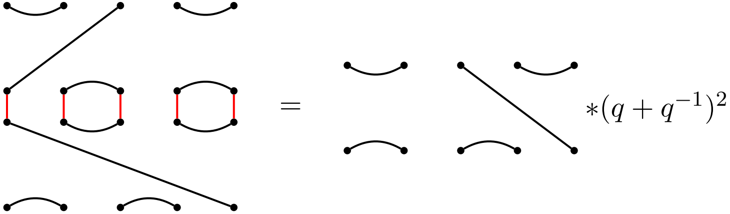
The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices



with multiplication given by stacking diagrams, subject to the relation

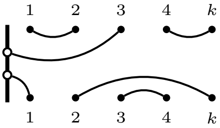
 $= q + q^{-1}$

Multiplication:



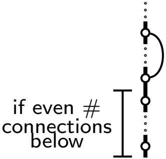
Temperley-Lieb algebras

The *one-boundary Temperley-Lieb algebra* $TL_k^{(1)}(q, z_0)$ is the algebra of one-walled non-crossing pairings on $2k$ vertices

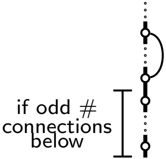


with multiplication given by stacking diagrams, subject to the relations

 $= q + q^{-1}$ and



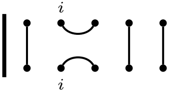
$= 1$ or

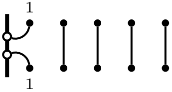


$= z_0$.

Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

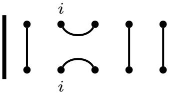
$e_i =$ 

and $e_0 =$ 

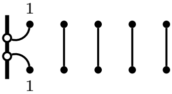
for $i = 1, \ldots, k-1$

Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

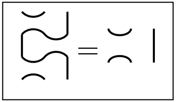
$e_i =$ 

and

$e_0 =$ 

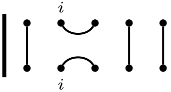
for $i = 1, \ldots, k-1$, with relations

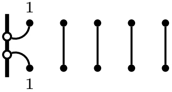
$e_i e_{i\pm1} e_i = e_i$ for $i \geq 1$



Odd/even relations

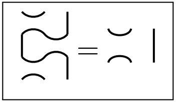
The algebra $TL_k^{(1)}(q, z_0)$ is generated by

$e_i =$ 

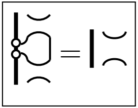
and
$e_0 =$ 

for $i = 1, \ldots, k-1$, with relations

$e_i e_{i\pm1} e_i = e_i$ for $i \geq 1$

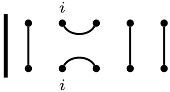


or

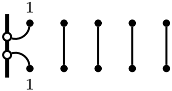


Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

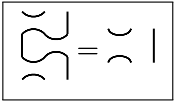
$e_i =$ 

and

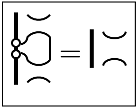
$e_0 =$ 

for $i = 1, \ldots, k-1$, with relations

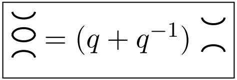
$e_i e_{i\pm1} e_i = e_i$ for $i \geq 1$



or

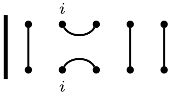
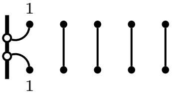


$e_i^2 = a e_i$



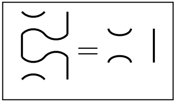
Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

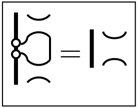
$e_i =$  and $e_0 =$ 

for $i = 1, \ldots, k-1$, with relations

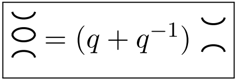
$e_i e_{i\pm 1} e_i = e_i$ for $i \geq 1$



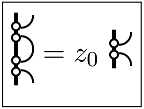
or



$e_i^2 = a e_i$

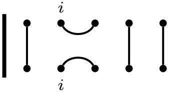
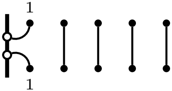


or



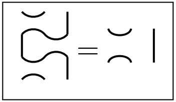
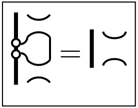
Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

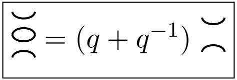
$e_i =$  and $e_0 =$ 

for $i = 1, \ldots, k-1$, with relations

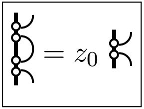
$e_i e_{i\pm1} e_i = e_i$ for $i \geq 1$

 or 

$e_i^2 = a e_i$



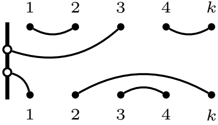
or



Side loops are resolved with a 1 or a z_0 depending on whether there are an even or odd number of connections below their lowest point.

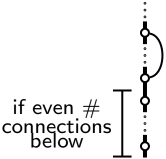
Temperley-Lieb algebras

The *one-boundary Temperley-Lieb algebra* $TL_k^{(1)}(q, z_0)$ is the algebra of one-walled non-crossing pairings on $2k$ vertices

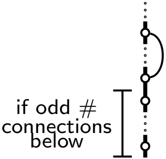


with multiplication given by stacking diagrams, subject to the relations

 $= q + q^{-1}$ and



$= 1$ or

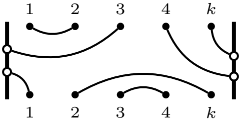


$= z_0.$

Our main object: two-boundary Temperley-Lieb algebra

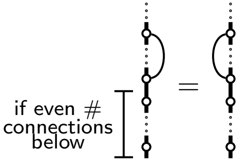
Nienhuis, De Gier, Batchelor (2004):

The *two-boundary Temperley-Lieb algebra* $TL_k^{(2)}(q, z_0, z_k) = \mathcal{T}_k$ is the algebra of two-walled non-crossing pairings on $2k$ vertices

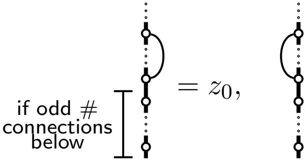


so that each wall always has an even number of connections, with multiplication given by stacking diagrams, subject to the relations

 $= q + q^{-1}$ and



$= \quad = 1$ or

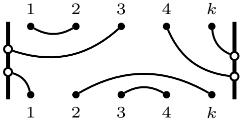


$= z_0, \quad = z_k.$

Our main object: two-boundary Temperley-Lieb algebra

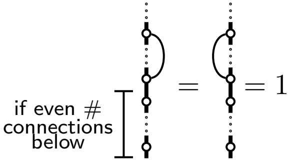
Nienhuis, De Gier, Batchelor (2004):

The *two-boundary Temperley-Lieb algebra* $TL_k^{(2)}(q, z_0, z_k) = \mathcal{T}_k$ is the algebra of two-walled non-crossing pairings on $2k$ vertices

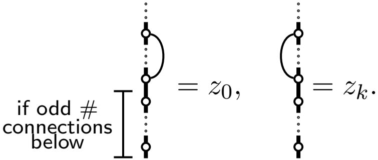


so that each wall always has an even number of connections, with multiplication given by stacking diagrams, subject to the relations

 $= q + q^{-1}$ and



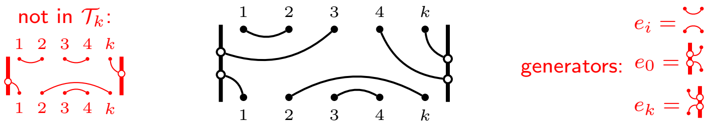
or



Our main object: two-boundary Temperley-Lieb algebra

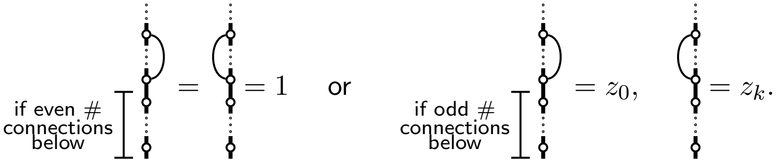
Nienhuis, De Gier, Batchelor (2004):

The *two-boundary Temperley-Lieb algebra* $TL_k^{(2)}(q, z_0, z_k) = \mathcal{T}_k$ is the algebra of two-walled non-crossing pairings on $2k$ vertices

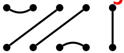


so that each wall always has an even number of connections, with multiplication given by stacking diagrams, subject to the relations

 $= q + q^{-1}$ and

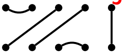


Our main object: two-boundary Temperley-Lieb algebra

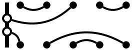


TL_k is finite-dimensional (nth Catalan number)

Our main object: two-boundary Temperley-Lieb algebra

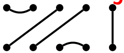


TL_k is finite-dimensional (nth Catalan number)

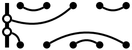


$TL_k^{(1)}$ is finite-dimensional

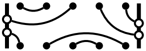
Our main object: two-boundary Temperley-Lieb algebra



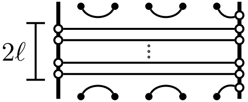
TL_k is finite-dimensional (nth Catalan number)



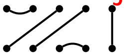
$TL_k^{(1)}$ is finite-dimensional



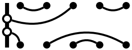
$TL_k^{(2)} = \mathcal{T}_k$ is infinite-dimensional!



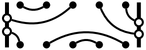
Our main object: two-boundary Temperley-Lieb algebra



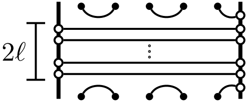
TL_k is finite-dimensional (nth Catalan number)



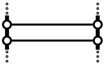
$TL_k^{(1)}$ is finite-dimensional



$TL_k^{(2)} = \mathcal{T}_k$ is infinite-dimensional!



de Gier, Nichols (2008): Explored representation theory of $\mathcal{T}_k$.

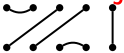
1. Take quotients giving  $= z$

   to get finite-dimensional algebras.

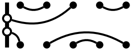
2. Establish connection to the affine Hecke algebras of type A and C to facilitate calculations.

3. Use diagrammatics and an action on $(\mathbb{C}^2)^{\otimes k}$ to help classify representations in quotient (most modules are 2^k dim'l; some split).

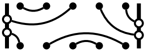
Our main object: two-boundary Temperley-Lieb algebra



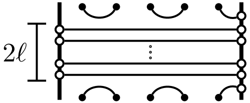
TL_k is finite-dimensional (nth Catalan number) SWD✓



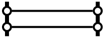
$TL_k^{(1)}$ is finite-dimensional SWD✓



$TL_k^{(2)} = \mathcal{T}_k$ is infinite-dimensional!



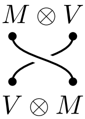
de Gier, Nichols (2008): Explored representation theory of $\mathcal{T}_k$.

1. Take quotients giving  $= z$ to get finite-dimensional algebras.
2. Establish connection to the affine Hecke algebras of type A and C (SWD✓) to facilitate calculations.
3. Use diagrammatics and an action on $(\mathbb{C}^2)^{\otimes k}$ to help classify representations in quotient (most modules are 2^k dim'l; some split).

Quantum groups and braids

Fix $q \in \mathbb{C}^*$. Let $U = U_q\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group associated to a reductive Lie algebra $\mathfrak{g}$. Let V, M be U-modules. Then $U \otimes U$ has invertible $R = \sum_R R_1 \otimes R_2$ that yields a map

$$\begin{aligned} \check{R}_{VM}: \quad V \otimes M &\longrightarrow M \otimes V \\ v \otimes m &\longmapsto \sum_R R_1 m \otimes R_2 v \end{aligned}$$

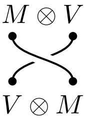


that (1) satisfies braid relations, and
(2) commutes with the action of $U_q\mathfrak{g}$.

Quantum groups and braids

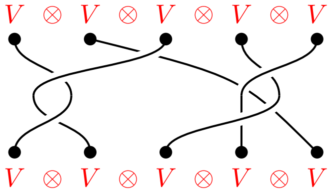
Fix $q \in \mathbb{C}^*$. Let $U = U_q\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group associated to a reductive Lie algebra $\mathfrak{g}$. Let V, M be U-modules. Then $U \otimes U$ has invertible $R = \sum_R R_1 \otimes R_2$ that yields a map

$$\begin{array}{rccc} \check{R}_{VM}: & V \otimes M & \longrightarrow & M \otimes V \\ & v \otimes m & \longmapsto & \sum_R R_1 m \otimes R_2 v \end{array}$$



that (1) satisfies braid relations, and
(2) commutes with the action of $U_q\mathfrak{g}$.

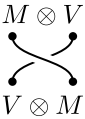
The braid group shares a commuting action with $U_q\mathfrak{g}$ on $V^{\otimes k}$:



Quantum groups and braids

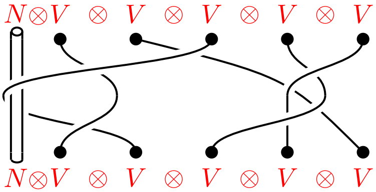
Fix $q \in \mathbb{C}^*$. Let $U = U_q\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group associated to a reductive Lie algebra $\mathfrak{g}$. Let V, M be U-modules. Then $U \otimes U$ has invertible $R = \sum_R R_1 \otimes R_2$ that yields a map

$$\begin{array}{rccc} \check{R}_{VM}\colon & V \otimes M & \longrightarrow & M \otimes V \\ & v \otimes m & \longmapsto & \sum_R R_1 m \otimes R_2 v \end{array}$$



that (1) satisfies braid relations, and
(2) commutes with the action of $U_q\mathfrak{g}$.

The one-boundary/affine braid group shares a commuting action with $U_q\mathfrak{g}$ on $N \otimes V^{\otimes k}$:



Around the pole:

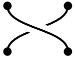
$$= \check{R}_{NV} \check{R}_{VN}$$

Quantum groups and braids

Fix $q \in \mathbb{C}^*$. Let $U = U_q\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group associated to a reductive Lie algebra $\mathfrak{g}$. Let V, M be U-modules. Then $U \otimes U$ has invertible $R = \sum_R R_1 \otimes R_2$ that yields a map

$$\check{R}_{VM}: \quad \begin{array}{ccc} V \otimes M & \longrightarrow & M \otimes V \\ v \otimes m & \longmapsto & \sum_R R_1 m \otimes R_2 v \end{array}$$

$M \otimes V$

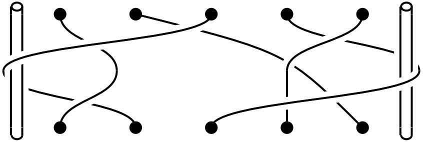


$V \otimes M$

that (1) satisfies braid relations, and
(2) commutes with the action of $U_q\mathfrak{g}$.

The two-boundary braid group shares a commuting action with $U_q\mathfrak{g}$ on $N \otimes V^{\otimes k} \otimes M$:

$N\otimes V \quad \otimes \quad V \quad \otimes \quad V \quad \otimes \quad V \quad \otimes \quad V\otimes M$



$N\otimes V \quad \otimes \quad V \quad \otimes \quad V \quad \otimes \quad V \quad \otimes \quad V\otimes M$

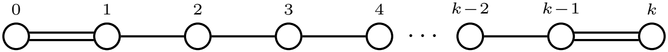
Around the pole:

$N\otimes V$

$N\otimes V$

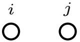
$= \check{R}_{NV}\check{R}_{VN}$

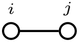
Affine type C Hecke algebra and two-boundary braids

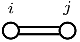


Fix constants t_0, t_k, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$\underbrace{T_iT_j\ldots}_{m_{i,j}\text{ factors}} = \underbrace{T_jT_i\ldots}_{m_{i,j}\text{ factors}} \qquad \text{where} \qquad m_{i,j} =$$

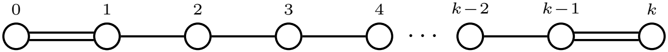
2 if 

3 if 

4 if 

and $T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1$.

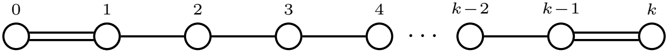
Affine type C Hecke algebra and two-boundary braids



Fix constants t_0, t_k, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$\underbrace{T_i T_j \ldots}_{m_{i,j} \text{ factors}} = \underbrace{T_j T_i \ldots}_{m_{i,j} \text{ factors}} \qquad \text{and} \qquad T_i^2 = (t_i^{1/2} - t_i^{-1/2}) T_i + 1.$$

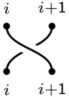
Affine type C Hecke algebra and two-boundary braids



Fix constants t_0, t_k, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

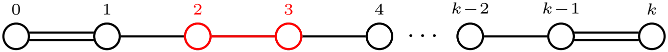
$$\underbrace{T_iT_j\ldots}_{m_{i,j}\text{ factors}} = \underbrace{T_jT_i\ldots}_{m_{i,j}\text{ factors}} \quad \text{and} \quad T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1.$$

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$  $T_0 =$  and $T_i =$ 

for $1 \le i \le k-1$.

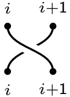
Affine type C Hecke algebra and two-boundary braids



Fix constants t_0, t_k, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

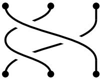
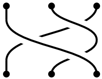
$$\underbrace{T_iT_j\ldots}_{m_{i,j}\text{ factors}} = \underbrace{T_jT_i\ldots}_{m_{i,j}\text{ factors}} \qquad \text{and} \qquad T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1.$$

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

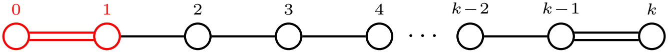
$T_k =$  $T_0 =$  and $T_i =$ 

for $1 \le i \le k-1$.

Relations:

$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1}$

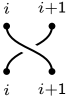
Affine type C Hecke algebra and two-boundary braids



Fix constants t_0, t_k, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

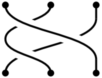
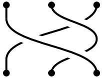
$$\underbrace{T_i T_j \ldots}_{m_{i,j} \text{ factors}} = \underbrace{T_j T_i \ldots}_{m_{i,j} \text{ factors}} \qquad \text{and} \qquad T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1.$$

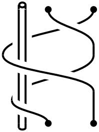
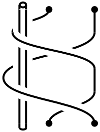
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$  $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$.

Relations:

$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$

$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$

Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let $U = U_q\mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$.
Let M, N, and V be finite-dimensional modules.

The two-boundary braid group $\mathcal{B}_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

(2) If $\mathfrak{g} = \mathfrak{gl}_n$, then (for good simple choices of M, N, and V), the affine Hecke algebra of type C, $\mathcal{H}_k$, acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

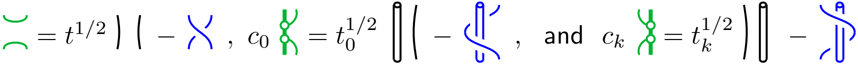
Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let $U = U_q\mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$.
Let M, N, and V be finite-dimensional modules.

The two-boundary braid group $\mathcal{B}_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

(2) If $\mathfrak{g} = \mathfrak{gl}_n$, then (for good simple choices of M, N, and V), the affine Hecke algebra of type C, $\mathcal{H}_k$, acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

Now using braid diagrammatics, [GN 08] says that by identifying



(where $c_i = t_i^{1/2}t^{-1/2} + t_i^{-1/2}t^{1/2}$),

then $\mathcal{T}_k$ is a quotient of $\mathcal{H}_k$ by...

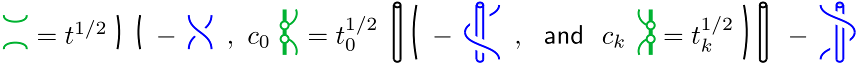
Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let $U = U_q\mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$.
Let M, N, and V be finite-dimensional modules.

The two-boundary braid group $\mathcal{B}_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

(2) If $\mathfrak{g} = \mathfrak{gl}_n$, then (for good simple choices of M, N, and V), the affine Hecke algebra of type C, $\mathcal{H}_k$, acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

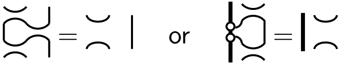
Now using braid diagrammatics, [GN 08] says that by identifying



(where $c_i = t_i^{1/2} t^{-1/2} + t_i^{-1/2} t^{1/2}$),

then $\mathcal{T}_k$ is a quotient of $\mathcal{H}_k$ by

$e_i e_{i\pm1} e_i$ for $1 \leq i \leq k-1$:



(and reverses)

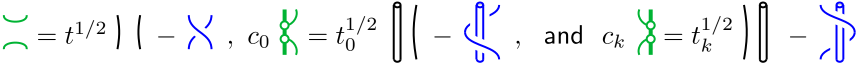
Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let $U = U_q\mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$.
Let M, N, and V be finite-dimensional modules.

The two-boundary braid group $\mathcal{B}_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

(2) If $\mathfrak{g} = \mathfrak{gl}_n$, then (for good simple choices of M, N, and V), the affine Hecke algebra of type C, $\mathcal{H}_k$, acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of U.

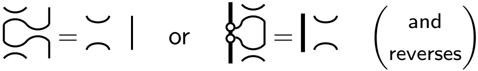
Now using braid diagrammatics, [GN 08] says that by identifying



(where $c_i = t_i^{1/2} t^{-1/2} + t_i^{-1/2} t^{1/2}$),

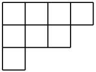
then $\mathcal{T}_k$ is a quotient of $\mathcal{H}_k$ by

$e_i e_{i\pm1} e_i$ for $1 \le i \le k-1$:

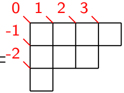


(3) When $\mathfrak{g} = \mathfrak{gl}_2$, $\mathcal{T}_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ (for good choices).

Consider the fin-dim'l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

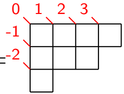
$\lambda =$ 

Consider the fin-dim'l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

$\lambda =$ 

The content of a box is its diagonal number.

Consider the fin-dim'l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

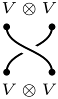
$\lambda =$ 

The content of a box is its diagonal number.

Fix $V = L(\square)$. The generators of $\mathcal{H}_k$ acting on $N \otimes V^{\otimes k} \otimes M$ look like

$T_k =$ 

$T_0 =$ 

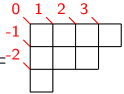
and $T_i =$ 

The eigenvalues of these operators (of which there should be two, since

$$(T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = (T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \;)$$

are controlled by contents of addable boxes.

Consider the fin-dim'l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

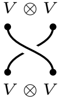
$\lambda =$ 

The content of a box is its diagonal number.

Fix $V = L(\square)$. The generators of $\mathcal{H}_k$ acting on $N \otimes V^{\otimes k} \otimes M$ look like

$T_k =$ 

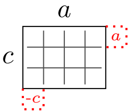
$T_0 =$ 

and $T_i =$ 

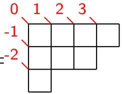
The eigenvalues of these operators (of which there should be two, since

$$(T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = (T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \;)$$

are controlled by contents of addable boxes. So let M and N be indexed by rectangular partitions, which have two addable boxes:

$(a^c) =$ 

Consider the fin-dim'l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

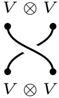
$\lambda =$ 

The content of a box is its diagonal number.

Fix $V = L(\square)$. The generators of $\mathcal{H}_k$ acting on $N \otimes V^{\otimes k} \otimes M$ look like

$T_k =$ 

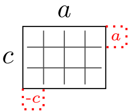
$T_0 =$ 

and $T_i =$ 

The eigenvalues of these operators (of which there should be two, since

$$(T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = (T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\ )$$

are controlled by contents of addable boxes. So let M and N be indexed by rectangular partitions, which have two addable boxes:

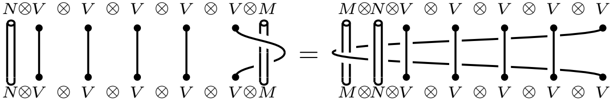
$(a^c) =$ 

$\mathcal{H}_k$ has a commuting action with $U_q\mathfrak{gl}_n$ on the space

$$L((b^d)) \otimes \big(L(\square)\big)^{\otimes k} \otimes L((a^c)) \quad \text{with } c, d < n$$

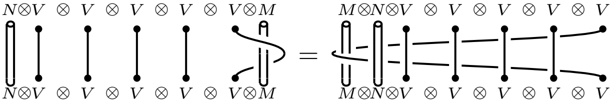
Exploring tensor space structure

Move the right pole to the left:

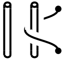
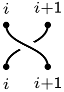


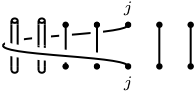
Exploring tensor space structure

Move the right pole to the left:



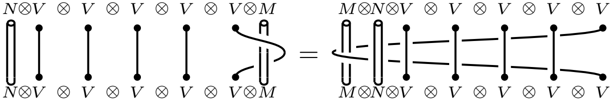
New favorite generators:

$T_0 =$  , $T_i =$ 

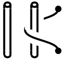
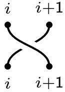
and $Y_j =$ 

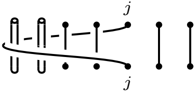
Exploring tensor space structure

Move the right pole to the left:



New favorite generators:

$T_0 =$  , $T_i =$ 

and $Y_j =$ 

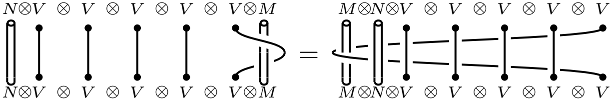
Then

$$M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \qquad \text{(multiplicity one!)}$$

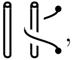
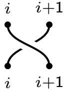
where Λ is the following set of partitions:

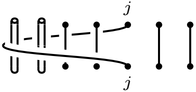
Exploring tensor space structure

Move the right pole to the left:



New favorite generators:

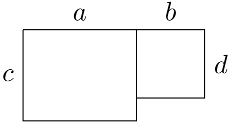
$T_0 =$  , $T_i =$ 

and $Y_j =$ 

Then

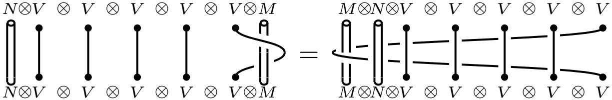
$$M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \qquad \text{(multiplicity one!)}$$

where Λ is the following set of partitions:

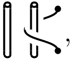
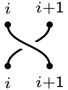


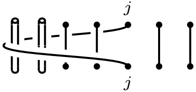
Exploring tensor space structure

Move the right pole to the left:



New favorite generators:

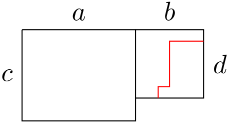
$T_0 =$ , $T_i =$ 

and $Y_j =$ 

Then

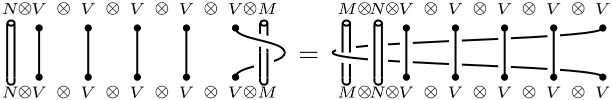
$$M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda),$$ (multiplicity one!)

where Λ is the following set of partitions:

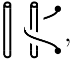
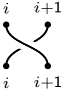


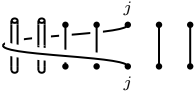
Exploring tensor space structure

Move the right pole to the left:



New favorite generators:

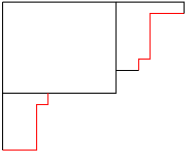
$T_0 =$  , $T_i =$ 

and $Y_j =$ 

Then

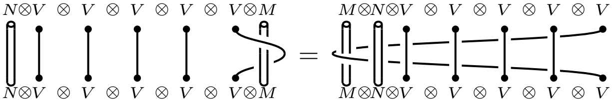
$$M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda),$$ (multiplicity one!)

where Λ is the following set of partitions:

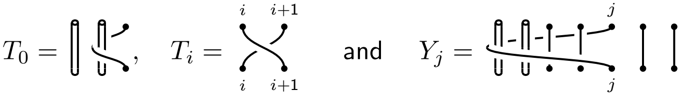


Exploring tensor space structure

Move the right pole to the left:



New favorite generators:

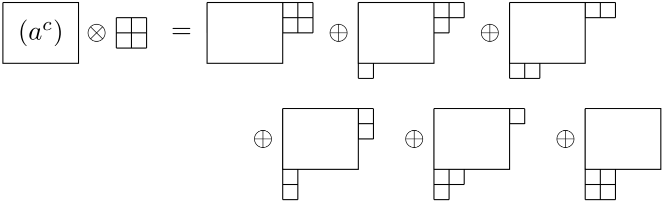


Then

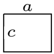
$$M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda),$$

(multiplicity one!)

where Λ is the following set of partitions...

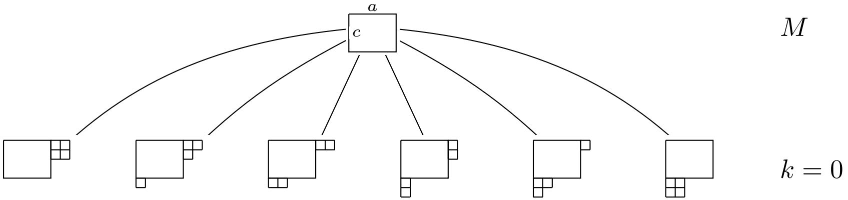


Exploring tensor space structure

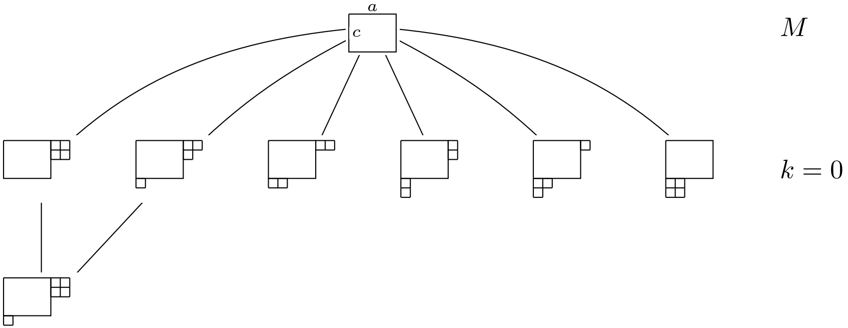


M

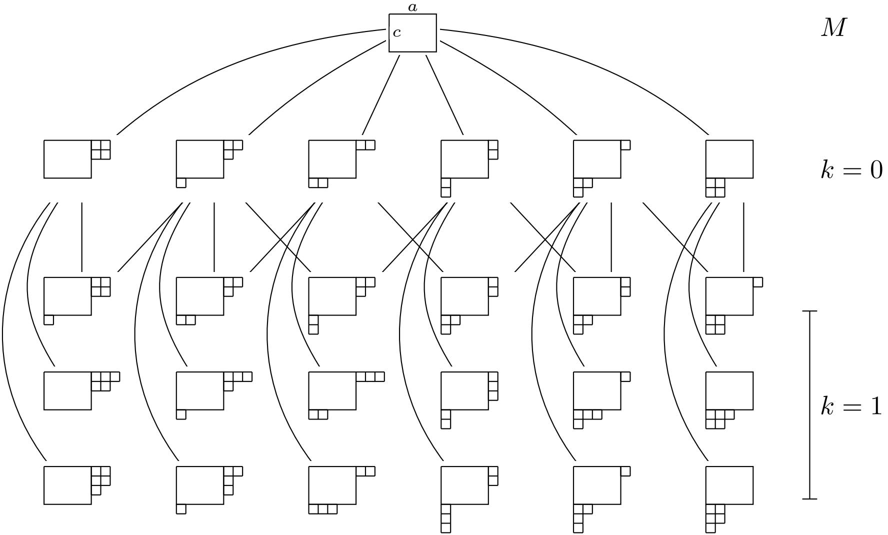
Exploring tensor space structure



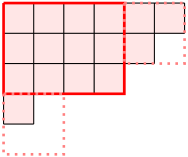
Exploring tensor space structure

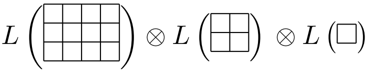


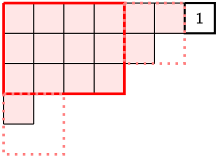
Exploring tensor space structure



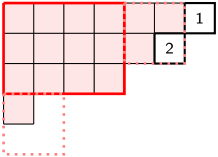
$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right)$$

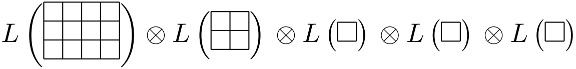


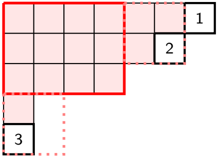



1

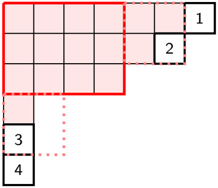
$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right)$$



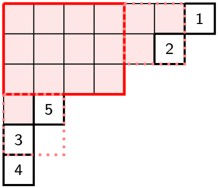



1
2
3

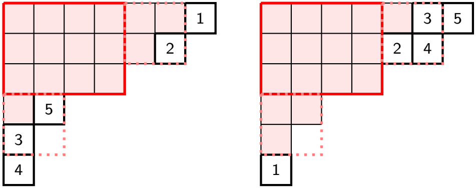
$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right)$$



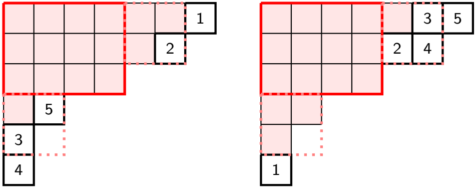
$$L\left(\yng(4,4,4)\right) \otimes L\left(\yng(2,2)\right) \otimes L\left(\yng(1)\right) \otimes L\left(\yng(1)\right) \otimes L\left(\yng(1)\right) \otimes L\left(\yng(1)\right) \otimes L\left(\yng(1)\right)$$



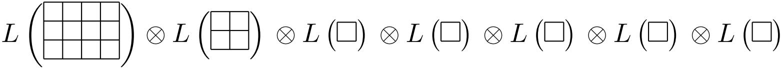
$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right)$$

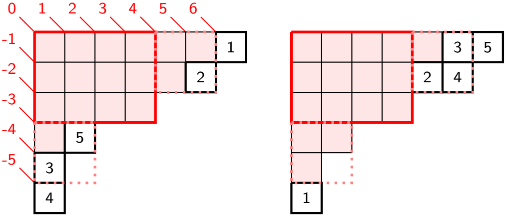


$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right)$$

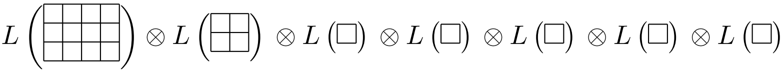


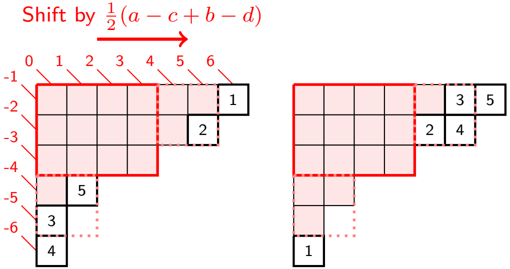
$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ.



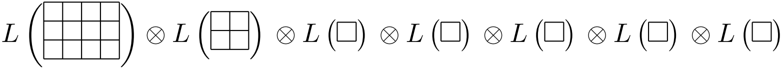


$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.

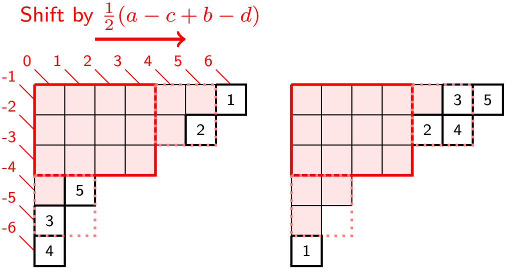




$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.

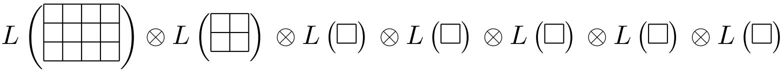


$$\begin{aligned} Y_1 &\mapsto t^{5.5} \\ Y_2 &\mapsto t^{3.5} \\ Y_3 &\mapsto t^{-4.5} \\ Y_4 &\mapsto t^{-5.5} \\ Y_5 &\mapsto t^{-2.5} \end{aligned}$$

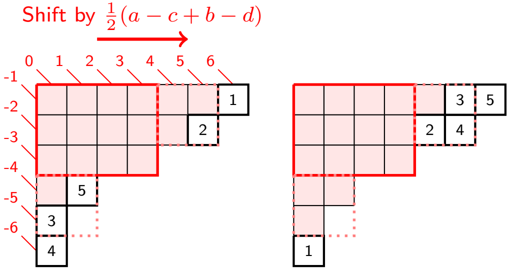


$$\begin{aligned} Y_1 &\mapsto t^{-5.5} \\ Y_2 &\mapsto t^{2.5} \\ Y_3 &\mapsto t^{4.5} \\ Y_4 &\mapsto t^{3.5} \\ Y_5 &\mapsto t^{5.5} \end{aligned}$$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.

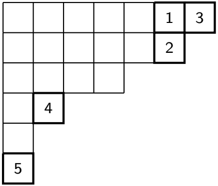


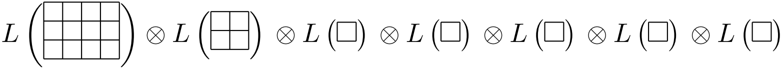
$$Y_1 \mapsto t^{5.5}$$
$$Y_2 \mapsto t^{3.5}$$
$$Y_3 \mapsto t^{-4.5}$$
$$Y_4 \mapsto t^{-5.5}$$
$$Y_5 \mapsto t^{-2.5}$$



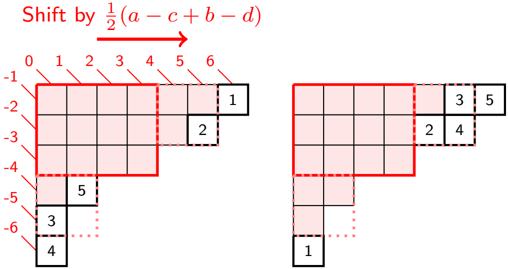
$$Y_1 \mapsto t^{-5.5}$$
$$Y_2 \mapsto t^{2.5}$$
$$Y_3 \mapsto t^{4.5}$$
$$Y_4 \mapsto t^{3.5}$$
$$Y_5 \mapsto t^{5.5}$$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.



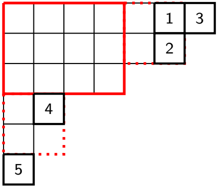


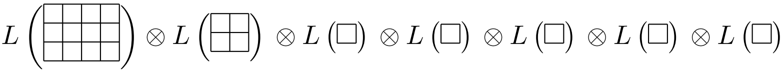
$$\begin{aligned} Y_1 &\mapsto t^{5.5} \\ Y_2 &\mapsto t^{3.5} \\ Y_3 &\mapsto t^{-4.5} \\ Y_4 &\mapsto t^{-5.5} \\ Y_5 &\mapsto t^{-2.5} \end{aligned}$$



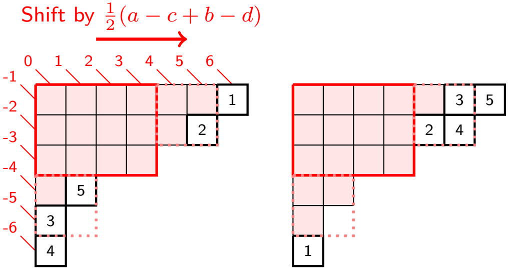
$$\begin{aligned} Y_1 &\mapsto t^{-5.5} \\ Y_2 &\mapsto t^{2.5} \\ Y_3 &\mapsto t^{4.5} \\ Y_4 &\mapsto t^{3.5} \\ Y_5 &\mapsto t^{5.5} \end{aligned}$$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.



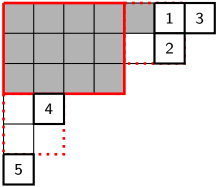


$Y_1 \mapsto t^{5.5}$
$Y_2 \mapsto t^{3.5}$
$Y_3 \mapsto t^{-4.5}$
$Y_4 \mapsto t^{-5.5}$
$Y_5 \mapsto t^{-2.5}$

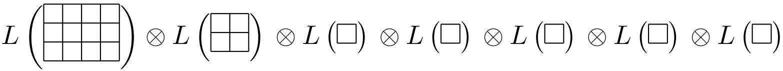


$Y_1 \mapsto t^{-5.5}$
$Y_2 \mapsto t^{2.5}$
$Y_3 \mapsto t^{4.5}$
$Y_4 \mapsto t^{3.5}$
$Y_5 \mapsto t^{5.5}$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.

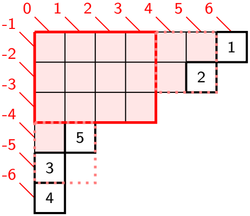


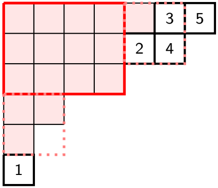
■ = boxes that must appear in the partition at level 0.



Shift by $\frac{1}{2}(a-c+b-d)$

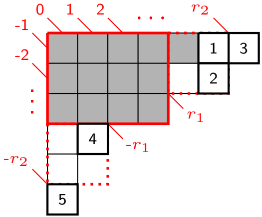
$Y_1 \mapsto t^{5.5}$
$Y_2 \mapsto t^{3.5}$
$Y_3 \mapsto t^{-4.5}$
$Y_4 \mapsto t^{-5.5}$
$Y_5 \mapsto t^{-2.5}$



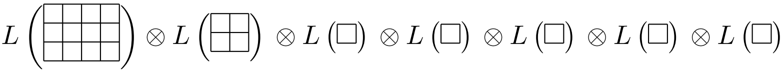


$Y_1 \mapsto t^{-5.5}$
$Y_2 \mapsto t^{2.5}$
$Y_3 \mapsto t^{4.5}$
$Y_4 \mapsto t^{3.5}$
$Y_5 \mapsto t^{5.5}$

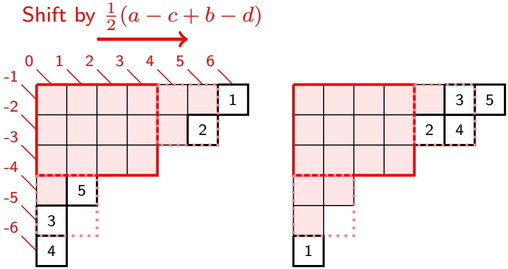
$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.



 = boxes that must appear in the partition at level 0.

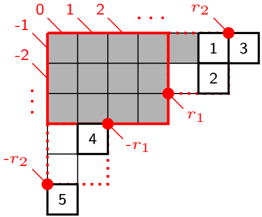


$$Y_1 \mapsto t^{5.5}$$
$$Y_2 \mapsto t^{3.5}$$
$$Y_3 \mapsto t^{-4.5}$$
$$Y_4 \mapsto t^{-5.5}$$
$$Y_5 \mapsto t^{-2.5}$$

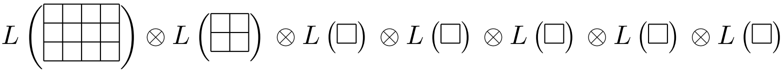


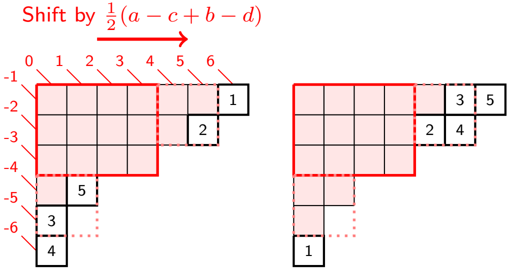
$$Y_1 \mapsto t^{-5.5}$$
$$Y_2 \mapsto t^{2.5}$$
$$Y_3 \mapsto t^{4.5}$$
$$Y_4 \mapsto t^{3.5}$$
$$Y_5 \mapsto t^{5.5}$$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.



 = boxes that must appear in the partition at level 0.

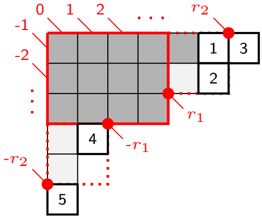




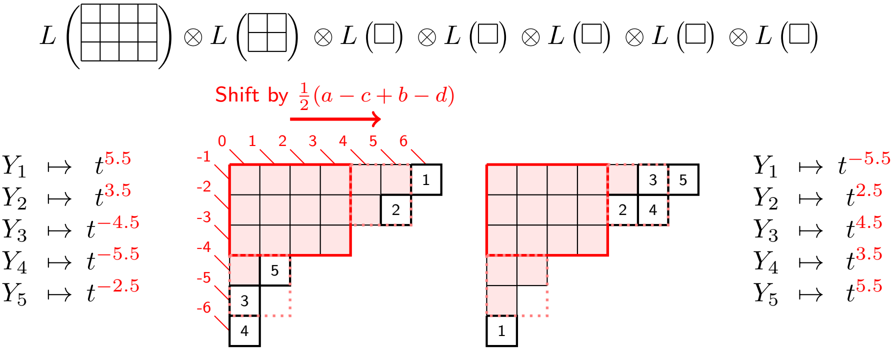
$$Y_1 \mapsto t^{5.5} \qquad Y_2 \mapsto t^{3.5} \qquad Y_3 \mapsto t^{-4.5} \qquad Y_4 \mapsto t^{-5.5} \qquad Y_5 \mapsto t^{-2.5}$$

$$Y_1 \mapsto t^{-5.5} \qquad Y_2 \mapsto t^{2.5} \qquad Y_3 \mapsto t^{4.5} \qquad Y_4 \mapsto t^{3.5} \qquad Y_5 \mapsto t^{5.5}$$

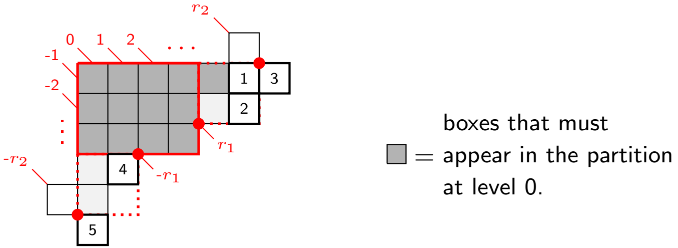
$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.

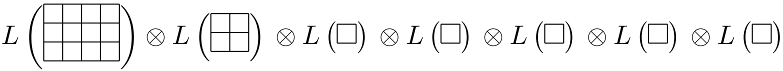


■ $=$ boxes that must appear in the partition at level 0.

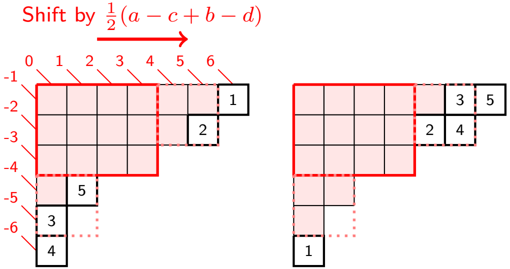


$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.



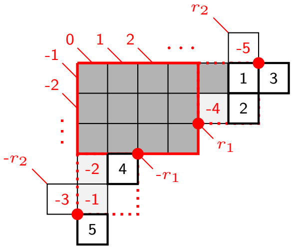


$$\begin{aligned} Y_1 &\mapsto t^{5.5} \\ Y_2 &\mapsto t^{3.5} \\ Y_3 &\mapsto t^{-4.5} \\ Y_4 &\mapsto t^{-5.5} \\ Y_5 &\mapsto t^{-2.5} \end{aligned}$$

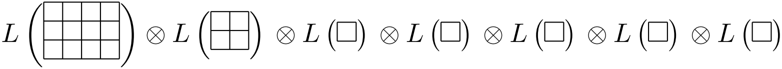


$$\begin{aligned} Y_1 &\mapsto t^{-5.5} \\ Y_2 &\mapsto t^{2.5} \\ Y_3 &\mapsto t^{4.5} \\ Y_4 &\mapsto t^{3.5} \\ Y_5 &\mapsto t^{5.5} \end{aligned}$$

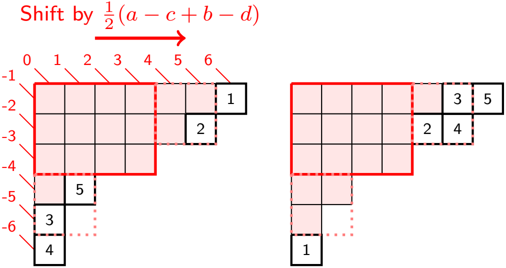
$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.



 = boxes that must appear in the partition at level 0.

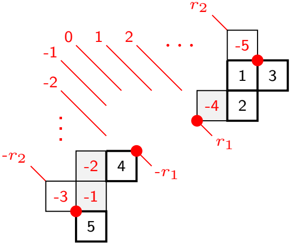


$$\begin{aligned} Y_1 &\mapsto t^{5.5} \\ Y_2 &\mapsto t^{3.5} \\ Y_3 &\mapsto t^{-4.5} \\ Y_4 &\mapsto t^{-5.5} \\ Y_5 &\mapsto t^{-2.5} \end{aligned}$$



$$\begin{aligned} Y_1 &\mapsto t^{-5.5} \\ Y_2 &\mapsto t^{2.5} \\ Y_3 &\mapsto t^{4.5} \\ Y_4 &\mapsto t^{3.5} \\ Y_5 &\mapsto t^{5.5} \end{aligned}$$

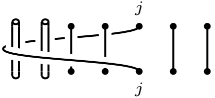
$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions λ, with basis labeled by tableaux from some partition μ in $(a^c) \otimes (b^d)$ to λ. Rep are calibrated, i.e. Y_j's act by constants controlled by content.



 = boxes that must appear in the partition at level 0.

Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part).

$Y_j =$ 

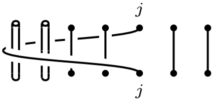
Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part).

The Weyl group $\mathcal{W}$ of type C (the group of signed permutations) acts on $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ by permuting the subscripts, with $Y_{-i} = Y_i^{-1}$. Then the center of $\mathcal{H}_k$ is symmetric Laurent polynomials

$$Z(\mathcal{H}_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{\mathcal{W}}.$$

$Y_j =$



Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part).
The Weyl group $\mathcal{W}$ of type C (the group of signed permutations) acts on $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ by permuting the subscripts, with $Y_{-i} = Y_i^{-1}$. Then the center of $\mathcal{H}_k$ is symmetric Laurent polynomials

$$Z(\mathcal{H}_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{\mathcal{W}}.$$

We can encode central characters as maps

$$\gamma : \{Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}\} \to \mathbb{C}^{\times}$$

with equivalence under $\mathcal{W}$ action;

Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part).
The Weyl group $\mathcal{W}$ of type C (the group of signed permutations) acts on $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ by permuting the subscripts, with $Y_{-i} = Y_i^{-1}$. Then the center of $\mathcal{H}_k$ is symmetric Laurent polynomials

$$Z(\mathcal{H}_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{\mathcal{W}}.$$

We can encode central characters as maps

$$\gamma : \{Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}\} \to \mathbb{C}^{\times}$$

with equivalence under $\mathcal{W}$ action; i.e. representative k-tuples

$$\gamma = (\gamma_1, \ldots, \gamma_k) \quad \text{with} \quad \gamma(Y_i^{\pm 1}) = (\gamma_i)^{\pm 1}$$

Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part).
The Weyl group $\mathcal{W}$ of type C (the group of signed permutations) acts on $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ by permuting the subscripts, with $Y_{-i} = Y_i^{-1}$. Then the center of $\mathcal{H}_k$ is symmetric Laurent polynomials

$$Z(\mathcal{H}_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{\mathcal{W}}.$$

We can encode central characters as maps

$$\gamma : \{Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}\} \to \mathbb{C}^{\times}$$

with equivalence under $\mathcal{W}$ action; i.e. representative k-tuples

$$\gamma = (\gamma_1, \ldots, \gamma_k) \quad \text{with} \quad \gamma(Y_i^{\pm 1}) = (\gamma_i)^{\pm 1}$$

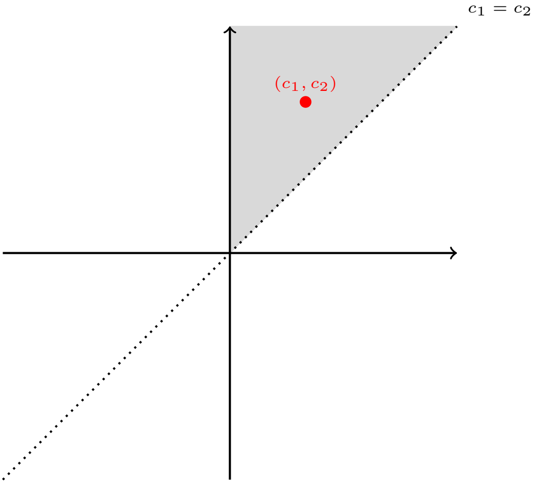
$$\mathbf{c} = (c_1, \ldots, c_k) \quad \text{with} \quad \gamma(Y_i^{\pm 1}) = t^{\pm c_i}$$

(when $\mathbf{c}$ is real, favorite representatives satisfy $0 \leq c_1 \leq \cdots \leq c_k$.)

Central characters as points

Restrict to real points.

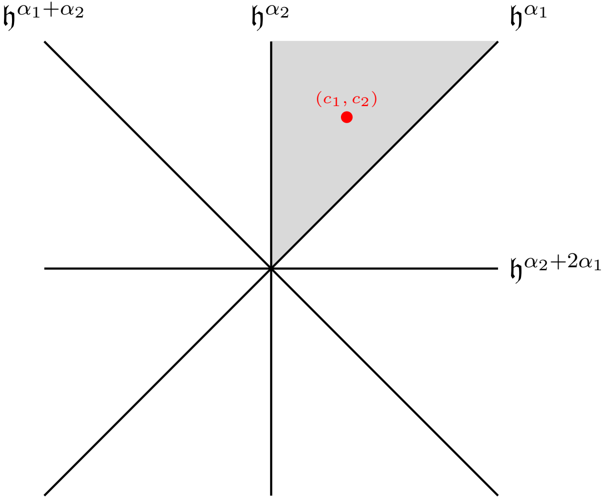
Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.
When $k = 2$:



Central characters as points

Restrict to real points.

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.
When $k = 2$:

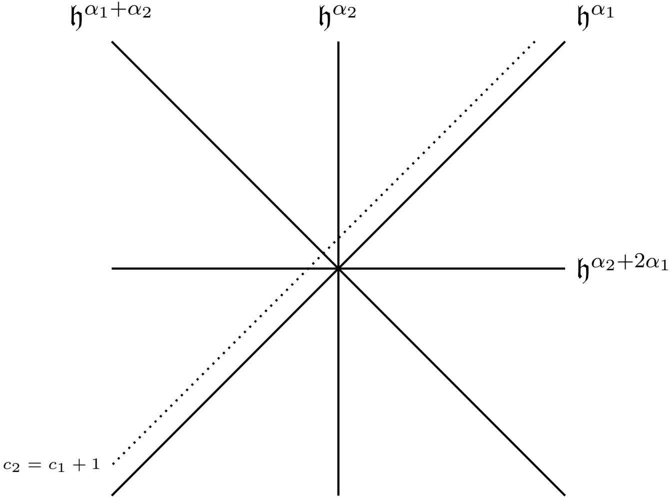


Central characters as points

Restrict to real points.

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.

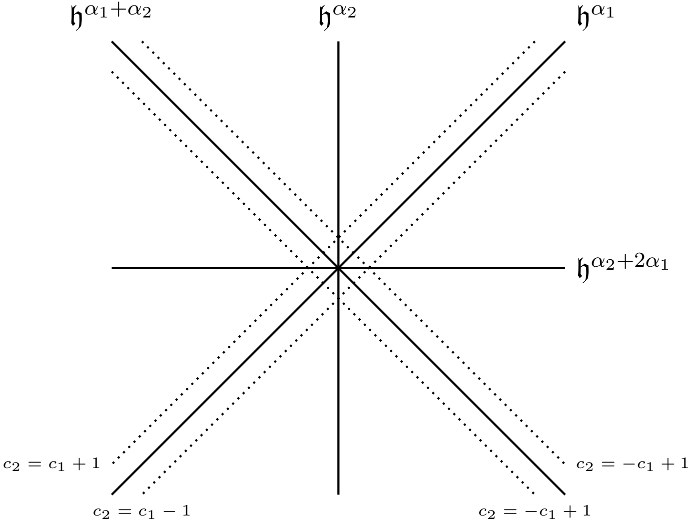
When $k = 2$:



Central characters as points

Restrict to real points.

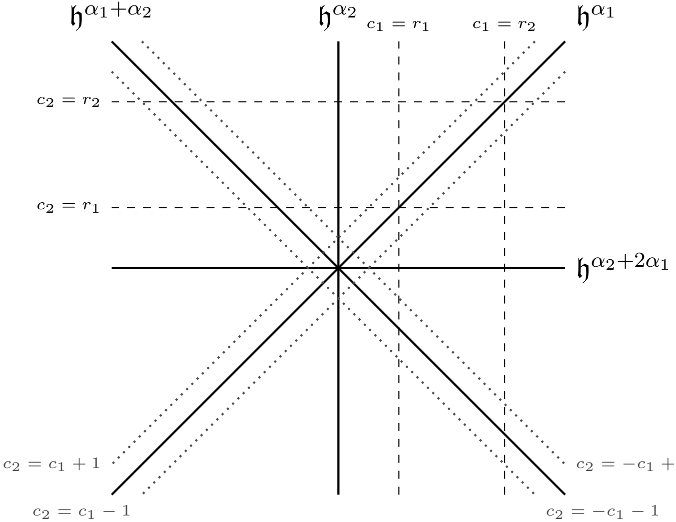
Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.
When $k = 2$:



Central characters as points

Restrict to real points.

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.
When $k = 2$:

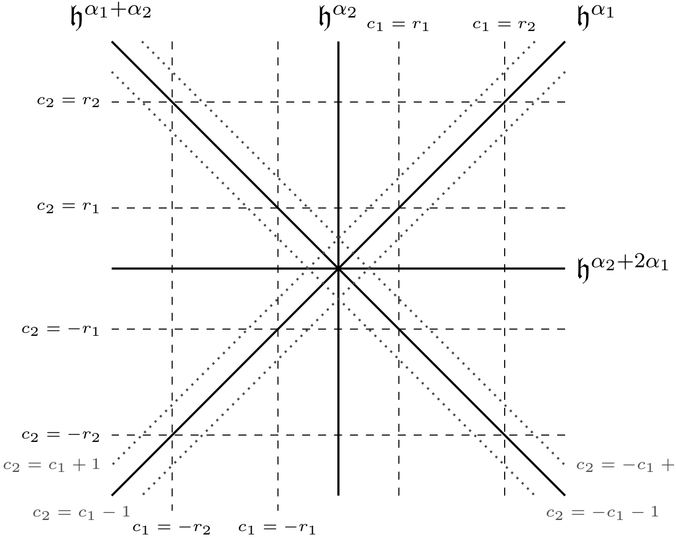


The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points

Restrict to real points.

Fav equivalence class reps: $0 \le c_1 \le \cdots \le c_k$.
When $k = 2$:

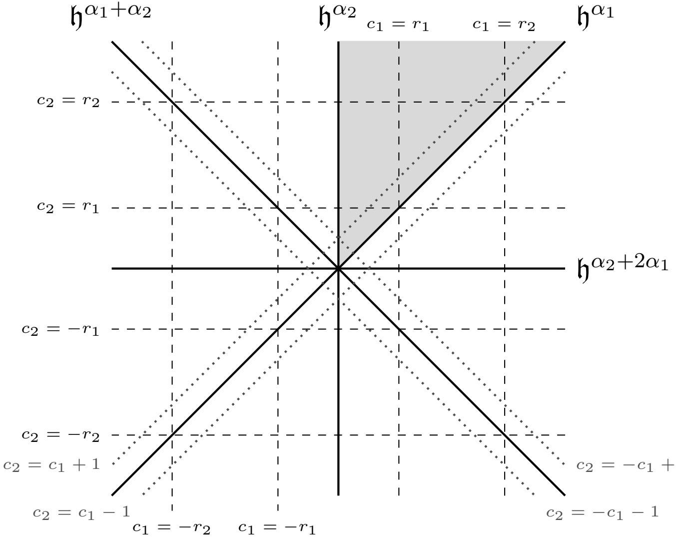


The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points

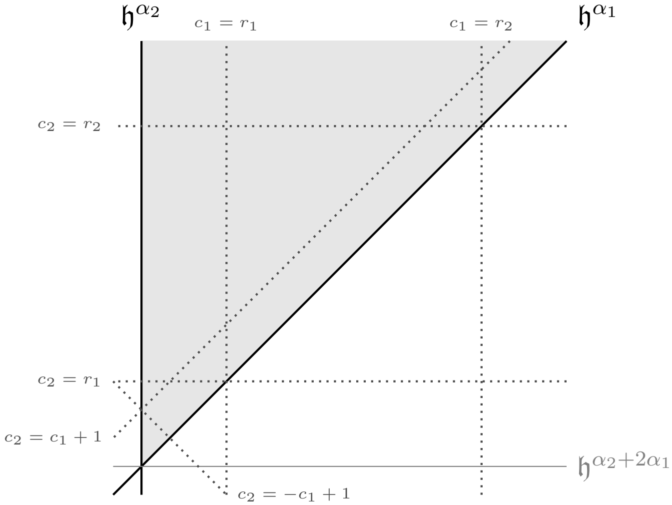
Restrict to real points.

Fav equivalence class reps: $0 \le c_1 \le \cdots \le c_k$.
When $k = 2$:



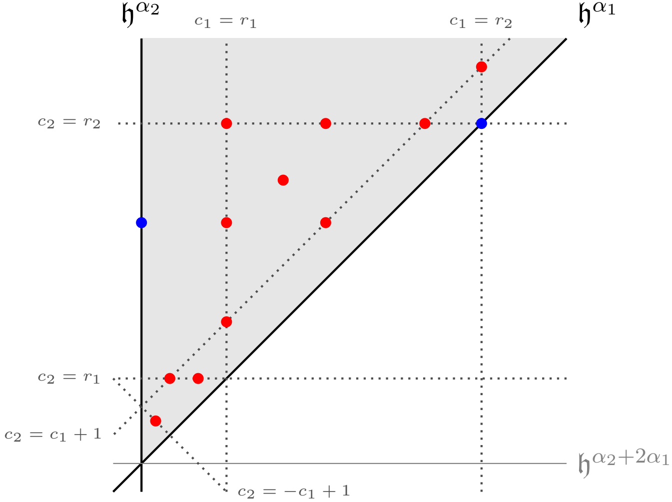
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points; Calibrated reps as "skew local regions"



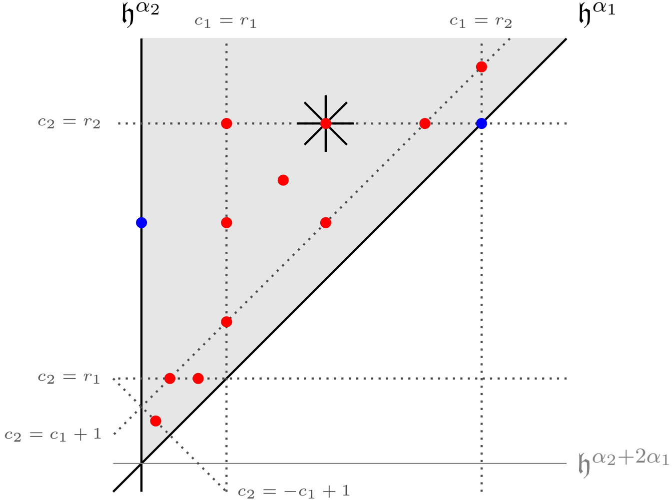
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points;
Calibrated reps as "skew local regions"



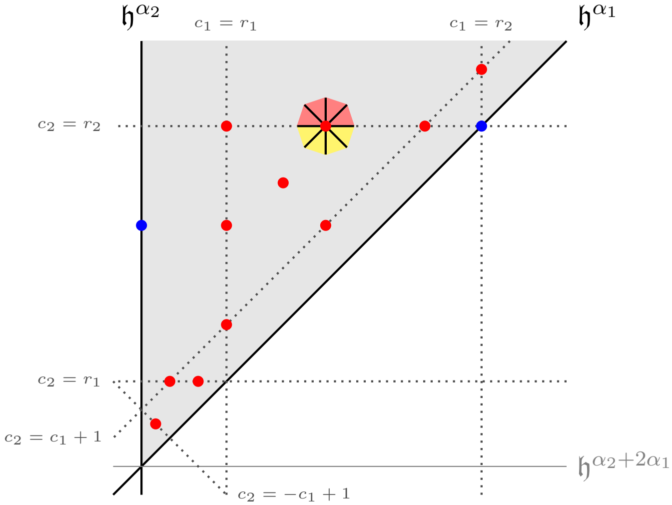
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points;
Calibrated reps as "skew local regions"



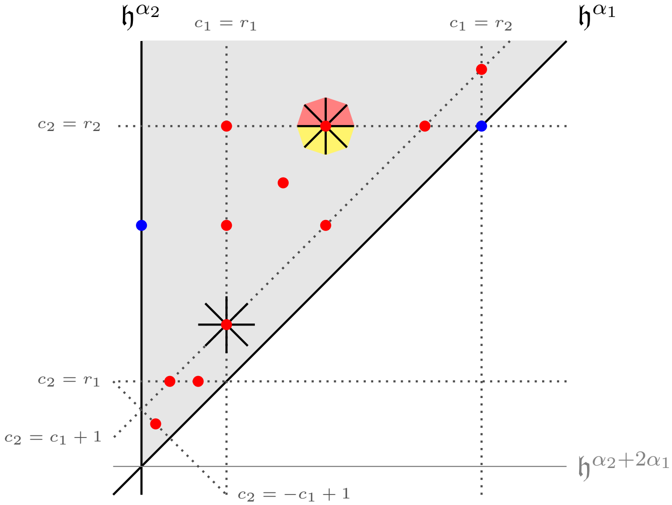
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points; Calibrated reps as “skew local regions”



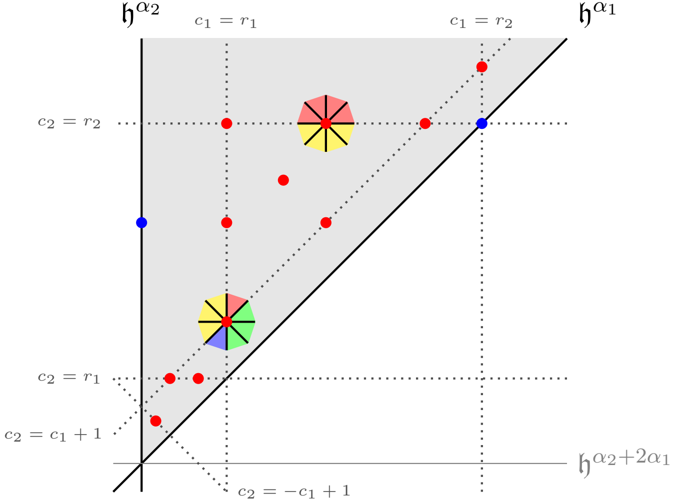
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points;
Calibrated reps as "skew local regions"



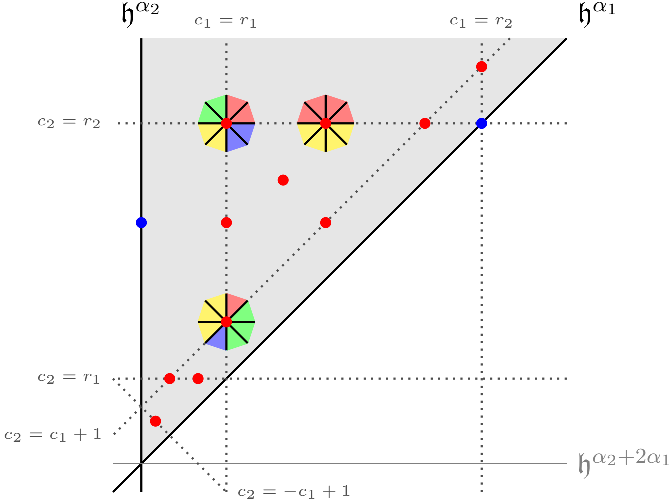
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points; Calibrated reps as "skew local regions"



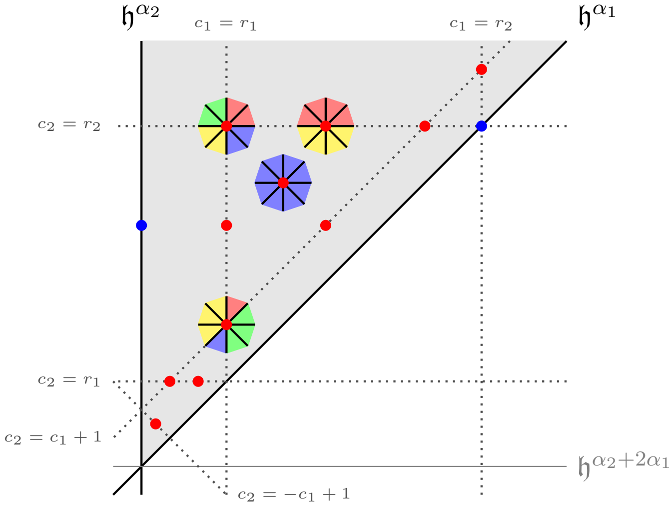
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points; Calibrated reps as “skew local regions”



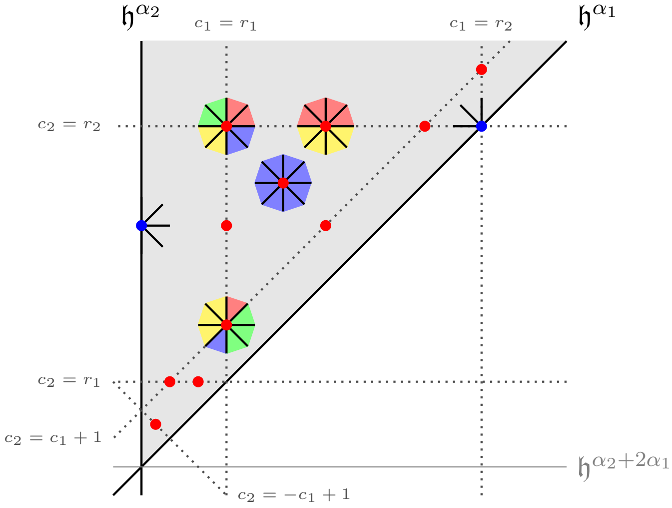
The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points; Calibrated reps as "skew local regions"

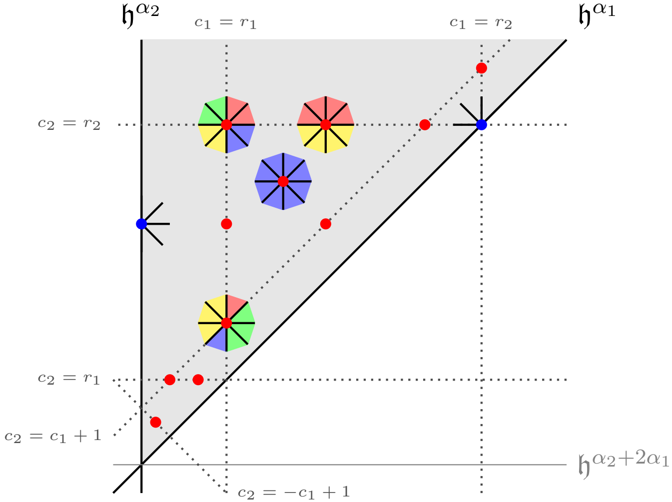


The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

Central characters as points; Calibrated reps as “skew local regions”

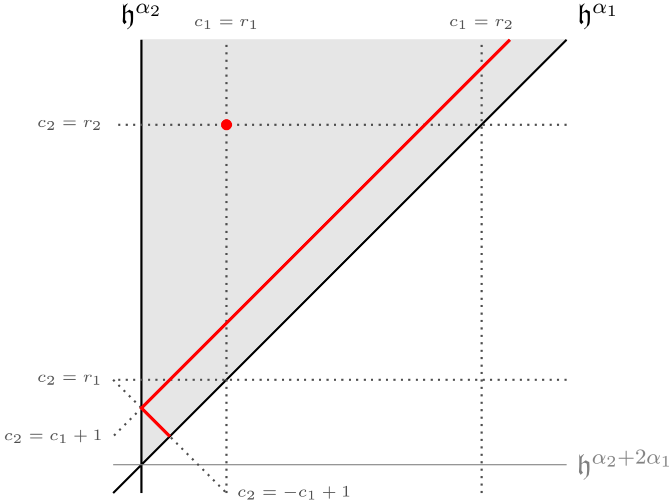


The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$



Thm. (D.-Ram)

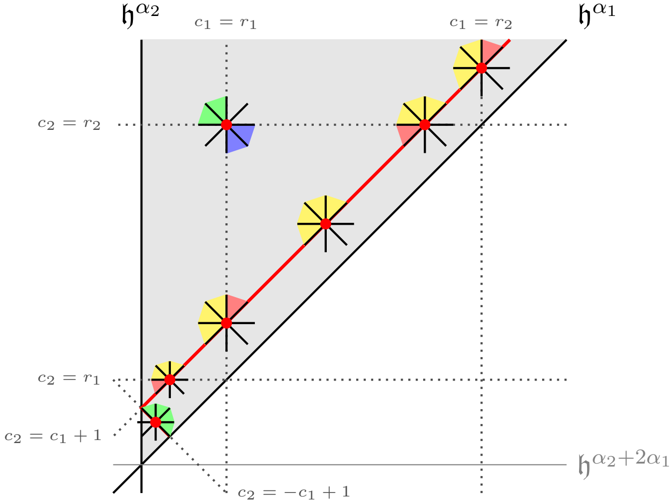
(1) Calibrated representations of $\mathcal{H}_k$ are indexed by skew local regions at regular (interior red) points.



Thm. (D.-Ram)

(1) Calibrated representations of $\mathcal{H}_k$ are indexed by skew local regions at regular (interior red) points.

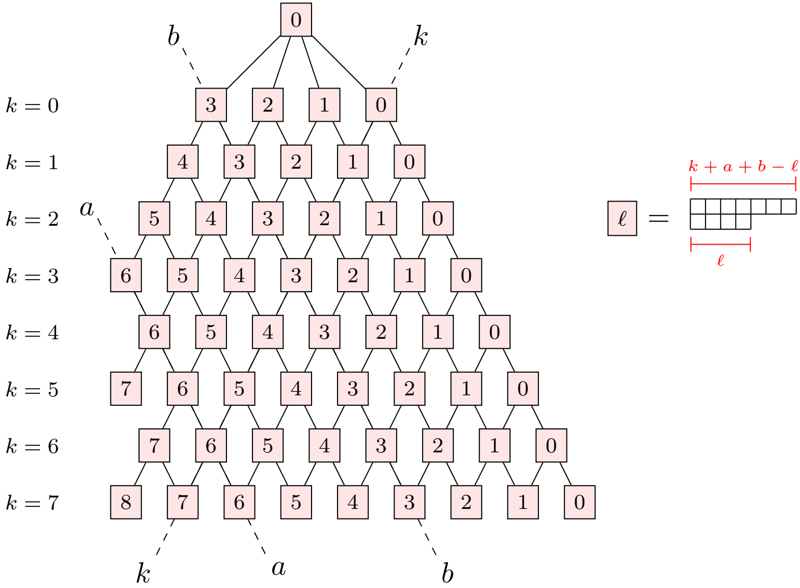
(2) The representations of $\mathcal{H}_k$ which factor through the TL quotient are (see above).



Thm. (D.-Ram)

(1) Calibrated representations of $\mathcal{H}_k$ are indexed by skew local regions at regular (interior red) points.

(2) The representations of $\mathcal{H}_k$ which factor through the TL quotient are (see above).



b
k
k = 0
k = 1
k = 2
k = 3
k = 4
k = 5
k = 6
k = 7
a
k
a
b
k + a + b − ℓ
ℓ =
ℓ