Central elements as parameters for centralizer algebras

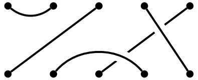
Zajj Daugherty
(Joint with Arun Ram and Rahbar Virk)

September 3, 2014

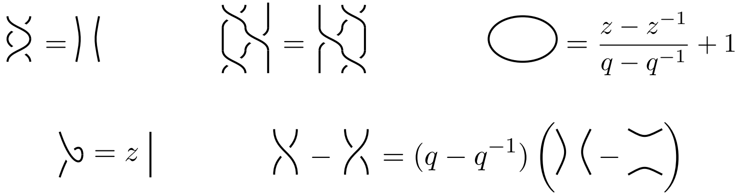
The BMW algebra

Fix $z, q \in \mathbb{C}^*$.

The *Birman-Murakami-Wenzl (BMW) algebra* $BMW_k(q, z)$ is the algebra of tangles on k strands:



with multiplication given by stacking diagrams, subject to the relations



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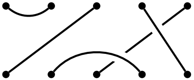

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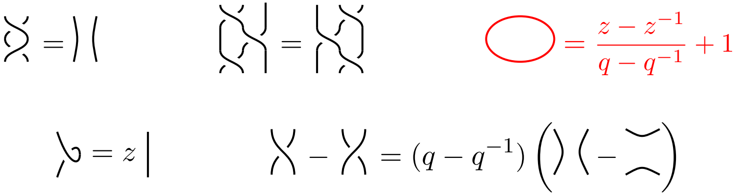

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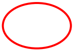
$\cdots$  $= \frac{z - z^{-1}}{q - q^{-1}} + 1 = Z_0$ $\cdots$

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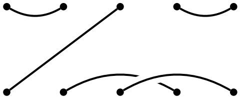

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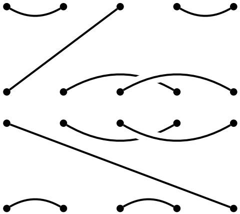

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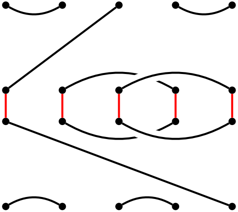

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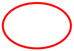
$\cdots$  $= \dfrac{z - z^{-1}}{q - q^{-1}} + 1 = Z_0$ $\cdots$

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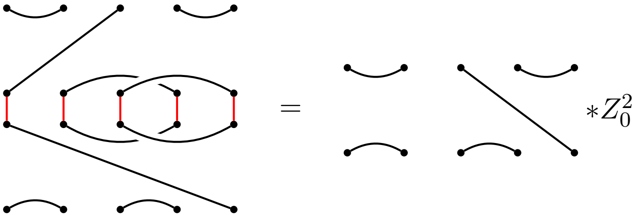

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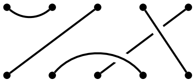

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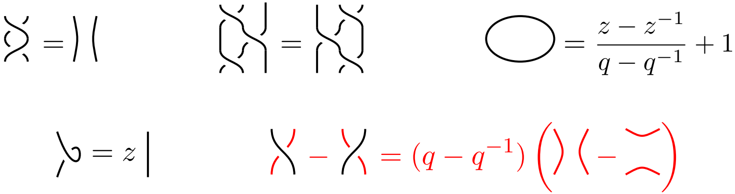

Introduced by Birman-Wenzl (1989) and Murakami (1986), giving a diagram algebra approach to the Kauffman polynomial of a link.

The affine BMW algebra

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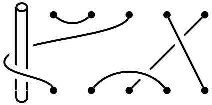

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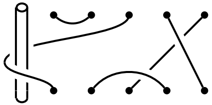

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(central)

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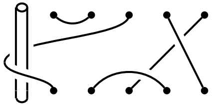

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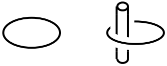

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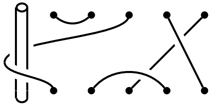

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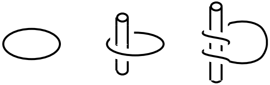

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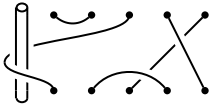

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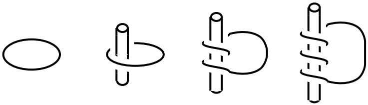

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Z_0 $\quad Z_1$ $\quad Z_2$ $\quad Z_3$ $\quad \cdots$

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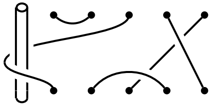

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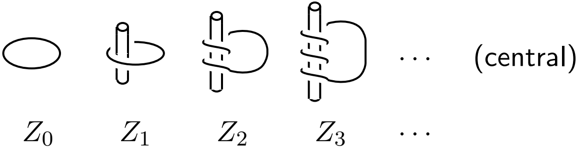

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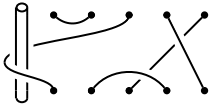

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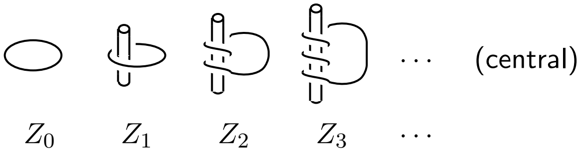

Admissibility conditions: For which choices of $\mathbf{Z} = \{Z_\ell \mid \ell \in \mathbb{Z}\}$ is $W_k(q, \mathbf{Z})$ nontrivial? Studied at length by Ariki-Mathas-Rui, Wilcox-Yu, Goodman-Mosley, etc..

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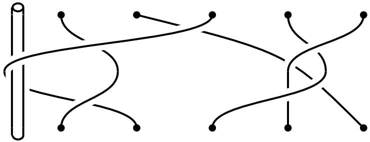
The affine braid group B_k is the group of braids on k strands in a space with a puncture (drawn as a pole).



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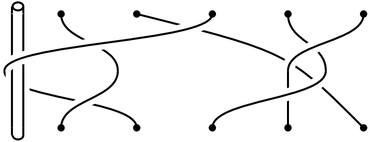
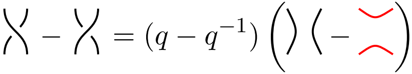


Recall, the affine BMW algebra has the relation



So $W_k(q, \mathbf{Z})$ is the quotient of the group algebra of B_k.
(Orellana-Ram)

Quantum groups and braids

Fix $q \in \mathbb{C}^*$. Let $U = U_q\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group associated to a reductive Lie algebra $\mathfrak{g}$. Let V, M be U-modules. Then $U \otimes U$ has invertible $R = \sum_R R_1 \otimes R_2$ that yields a map

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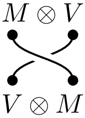

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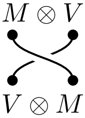

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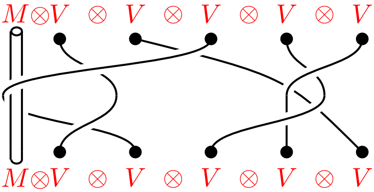

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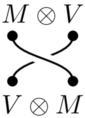

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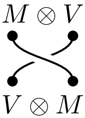

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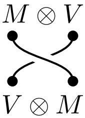

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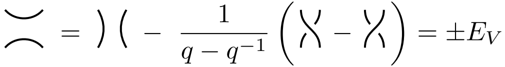
Start with B_k action; what do the parameters Z_ℓ turn out to be?

With $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_n$ and V the natural representation, then the action of the braid group gives



where E_V is given by

$$E_V : V \otimes V^* \xrightarrow{(v^{-1}\otimes 1)\check{R}_{VV^*}} V^* \otimes V \xrightarrow{\mathrm{ev}} \mathbf{1} \xrightarrow{\mathrm{coev}} V \otimes V^*,$$

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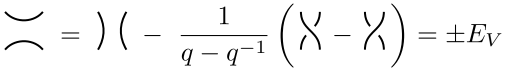

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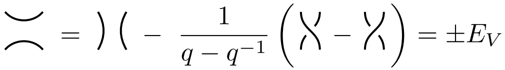

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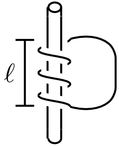

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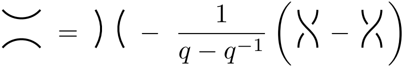

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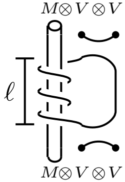

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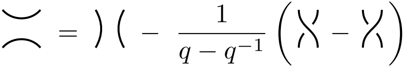

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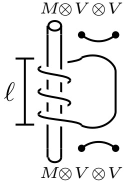

$$= (1 \otimes E_V)\big((\ \cdot\ \check{R}_{MV}\check{R}_{VM}) \otimes 1\big)^{\ell}(1 \otimes E_V)$$

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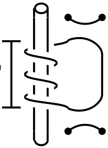


$$= (1 \otimes E_V)\left((\ \cdot \ \check{R}_{MV}\check{R}_{VM}) \otimes 1\right)^{\ell} (1 \otimes E_V)$$

$$= \underbrace{\left((1 \otimes \mathrm{qtr}_V)\left((\ \cdot \ \check{R}_{MV}\check{R}_{VM})^{\ell}\right)\right)}_{\text{A central element of } U_q\mathfrak{g}} \otimes E_V$$

(Drinfel'd 1990)

With $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_n$ and V the natural representation, then the action of the braid group gives

 $= \;)\,(\; - \dfrac{1}{q - q^{-1}}\Big($  $-$  $\Big) = \pm E_V$

where E_V is given by

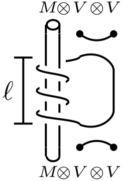
$$E_V : V \otimes V^* \xrightarrow{(v^{-1}\otimes 1)\check{R}_{VV^*}} V^* \otimes V \xrightarrow{\text{ev}} \mathbf{1} \xrightarrow{\text{coev}} V \otimes V^*,$$

and v is a ribbon element in $U_q\mathfrak{g}$. (don't worry: $V \cong V^*$)

Wrapping around the pole: 

$= \cdot\ \check{R}_{MV}\check{R}_{VM}$

And so



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(Drinfel'd 1990)

Let

$$Z_\ell = \pm(1 \otimes \mathrm{qtr}_V)\left((z\check{R}_{21}\check{R})^\ell\right) \in Z(U_q\mathfrak{g}).$$

Wonderful things about $\{Z_\ell \mid \ell \in \mathbb{Z}\}$:

(1) These central elements
   (a) are higher Casimir elements (defined by Reshetikhin-Takhtajan-Faddeev (1990)),
   (b) which correspond to nice symmetric functions via Harish-Chandra.

(2) Admissibility conditions are satisfied by design.

(3) Recursion relations of Beliakova-Blanchet (1998) appear.

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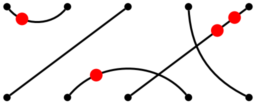
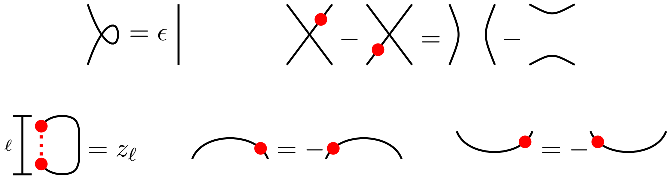
Let's do it again!

The degenerate affine BMW algebra $\mathcal{W}_k$

Fix $\epsilon = \pm 1$. The degenerate affine BMW algebra $\mathcal{W}_k(\epsilon, z_0, z_1, \dots)$ is the algebra with basis given by decorated Brauer diagrams



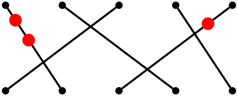
with multiplication given by stacking and resolving, subject to the relations



This algebra was introduced by Nazarov (1996), extending the Brauer algebra by Murphy elements.

The degenerate affine braid algebra $\mathcal{B}_k$

The degenerate affine braid algebra $\mathcal{B}_k$ is the algebra of decorated permutations



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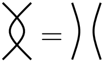
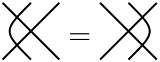
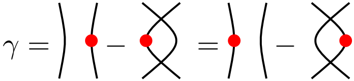


and





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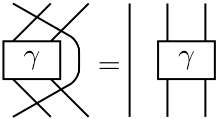

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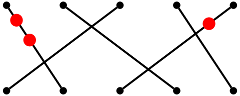

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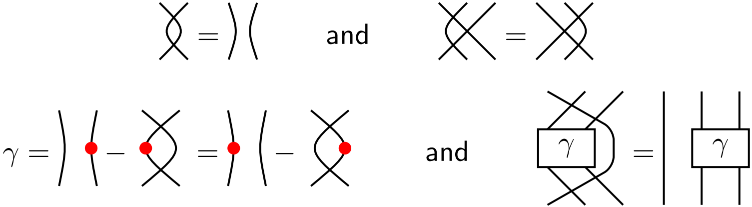
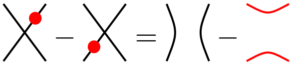


Thm. (D.-Ram-Virk) $\mathcal{B}_k$ acts on $M \otimes V^{\otimes k}$, for $U\mathfrak{g}$-modules M, V, and commutes with the action of $U\mathfrak{g}$.

Action: Permutations act as expected;

γ acts via the Casimir, analogously to $\check{R}_{MV}$.

Recall, the degenerate affine BMW algebra has the relation



Thm. (D.-Ram-Virk) The algebra $\mathcal{W}_k$ is a quotient of $\mathcal{B}_k$. Further, when $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_n$, and $V = \mathbb{C}^n$ the action of $\mathcal{B}_k$ gives an action of $\mathcal{W}_k$ for good choices of parameters.

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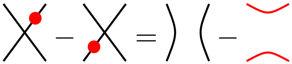

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Again, starting with the action of the braid algebra and using Drinfel'd, we get

$$z_\ell = \epsilon(1 \otimes \mathrm{tr}_V)\left((\gamma + \cdot)^\ell\right),$$

a well-behaved central element of $U\mathfrak{g}$.

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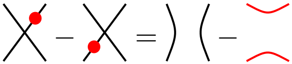

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