Centralizers of the infinite symmetric group

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Finite-dimensional permutation modules

Classical Schur-Weyl duality established an amazing duality between the representations the general linear group $\mathrm{GL}_n(\mathbb{C})$ and the symmetric group S_k via their commuting actions on

$$V^{\otimes k} = V \otimes \cdots \otimes V, \qquad \text{where } V = \mathbb{C}^n.$$

In short, the action of $\mathbb{C}\mathrm{GL}_n$ produces the entire centralizer of the action of $\mathbb{C}S_k$ in $\mathrm{End}(V^{\otimes k})$, and vice versa. This double-centralizer relationship produces a multiplicity-free decomposition

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda, \quad \text{where } \begin{array}{l} G^\lambda \text{ are distinct simple } GL_n\text{-modules, and} \\ S^\lambda \text{ are distinct simple } S_k\text{-modules,} \end{array}$$

as a (GL_n, S_k)-bimodule, which gives the duality

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{simple } \mathrm{GL}_n \text{ modules in } V^{\otimes k} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{simple } S_k \text{ modules in } V^{\otimes k} \end{array} \right\}.$$

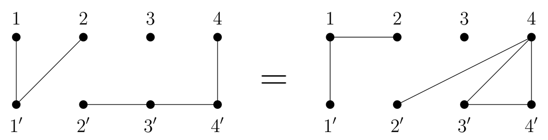
This is now only one of many examples of this phenomena, our favorite of which arises from restricting to the permutation matrices inside of GL_n; then the analog to S_k is the partition algebra, defined as follows.

The partition algebra.

Fix $k \in \mathbb{Z}_{>0}$, and denote $[k] = \{1, \dots, k\}$ and $[k'] = \{1', \dots, k'\}$, To each set partition of

$$[k] \cup [k'] = \{1, \dots, k, 1', \dots, k'\},$$

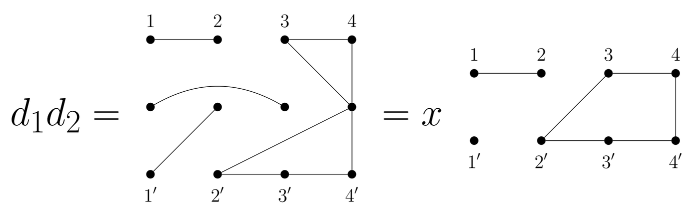
we associate a diagram, which is the equivalence class of graphs on vertices $[k] \cup [k']$ whose connected components determine the parts. For example, as diagrams,



$$= \{\{1, 2, 1'\}, \{3\}, \{4, 2', 3', 4'\}\}.$$

Let D_k be the set of k-diagrams.

Fix an indeterminate x. There is a multiplication on diagrams given by stacking them, gluing the middle vertices, and resolving the resulting connections; resolve any floating components by replacing each with a factor of x. For example, if d_1 and d_2 are the first two diagrams in Table 1, then



The partition algebra $P_k(x)$ is spanned over $\mathbb{C}[x]$ by the diagrams in D_k, with this multiplication.

Duality between S_n and $P_k(n)$.

Symmetric group action. The action of S_n on permutation module $V = \mathbb{C}\{v_1, \dots, v_n\}$ is given by permuting subscripts; this action extends diagonally to the k-fold tensor product $V^{\otimes k}$:

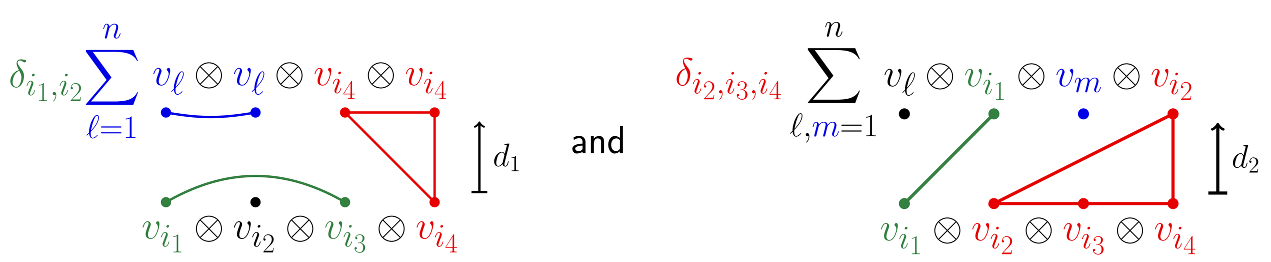
$$\sigma \cdot v_{i_1} \otimes \cdots \otimes v_{i_k} = v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)}. \tag{1}$$

Partition algebra action. For each k-diagram d and each pair of k-tuples $(i_1, \dots, i_k), (i_{1'}, \dots, i_{k'}) \in [n]^k$, we define

$$d^{(i_1, \dots, i_k)}_{(i_{1'}, \dots, i_{k'})} = \begin{cases} 1 & \text{if } i_\ell = i_m \text{ whenever vertices } \ell, m \in [k] \cup [k'] \text{ are connected in } d, \\ 0 & \text{otherwise.} \end{cases}$$

Then $d \in D_k$ acts on $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k}$ by $d \cdot v_{\mathbf{i}} = \sum_{\mathbf{j} \in [n]^k} d^{\mathbf{j}}_{\mathbf{i}} v_{\mathbf{j}}$.

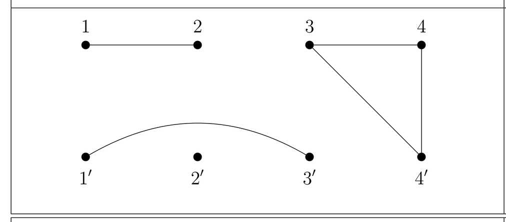
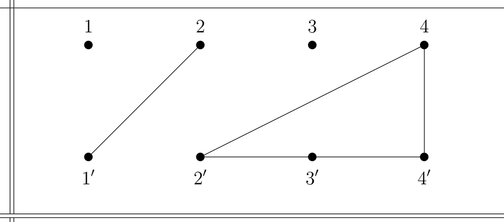
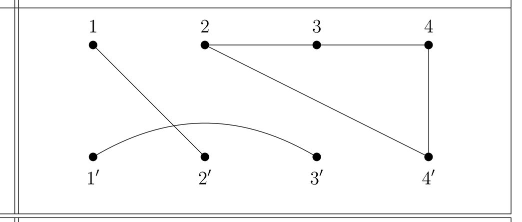
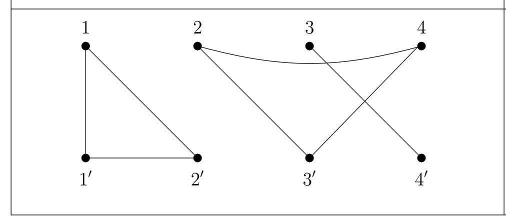
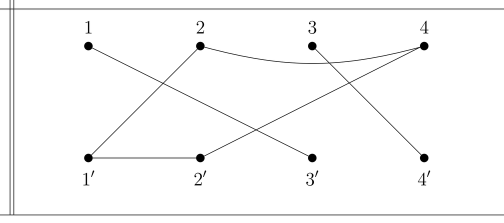
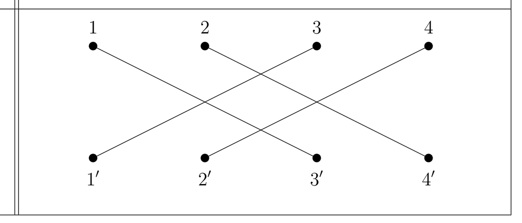
Pictorially, each diagram encodes an (upward moving) map on a simple tensor:



(2)

Duality theorem (V. Jones, 1994). The actions of S_n and $P_k(n)$ on $V^{\otimes k}$ commute with each other. Further, all maps in $\mathrm{End}(V^{\otimes k})$ that commute with the action of S_n come from the action of $P_k(n)$, and vice versa.

Table 1: Examples of diagrams with relevant properties

d_1 : Partition algebra diagram with no restrictions	d_2 : Bottom-propagating diagram	d_3 : Top-propagating diagram
		
d_4 : (non-uniform) Block permutation	d_5 : Uniform block permutation	d_6 : Permutation diagram
		

Motivations for moving to S_∞, and the resulting issues

Representation-theoretic stability.

Representation theorists are interested in the stability of the representation theory of of chains of groups, most salient being the chain of finite symmetric groups

$$S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow \cdots \xrightarrow{n \to \infty} S_\infty. \tag{3}$$

In [Bowman et al., 2014], the authors use the duality between S_n and $P_k(n)$ to study stability in decomposition numbers of representations of symmetric groups (the Kronecker product). Both [Church et al., 2012] and and [Sam and Snowden, 2013] use category-theoretic methods; the latter additionally make the connection to Schur-Weyl duality.

One way to think about all three papers is using the fact that, for all but a few values of n,

the representation theory of the partition algebra $P_k(n)$ is independent of n.

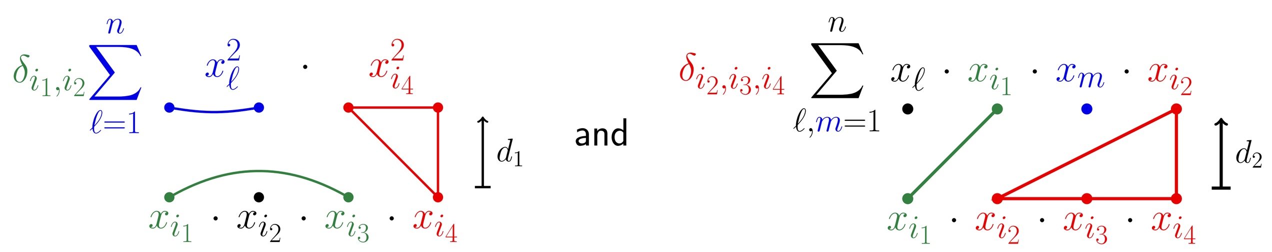
So since all the representation theory of S_n can be passed via Schur-Weyl duality to $P_k(n)$ for some k, it must also be stable. However, depending on what you're hoping to do with S_∞ itself, the 'right' analog to $P_k(n)$ varies.

Symmetric functions.

The tensor space $V^{\otimes k}$ studied to the left is isomorphic as a vector space to the homogeneous degree-k polynomial functions in n non-commuting variables $x_1, \dots, x_n$, with

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mapsto x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The symmetric functions within this space are exactly the vectors fixed by the S_n-action, and so form a natural $P_k(n)$-module.



However, in the limit as the number of variables goes to infinity, so again does the parameter for $P_k(n)$, which is intractable. Further, $V^{\otimes k}$ itself approaches the countable-dimensional vector space, which has no non-trivial symmetric elements (i.e. does not contain NCSym).

References

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[Church et al., 2012] Church, T., Ellenberg, J. S., and Farb, B. (2012).
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Infinite-dimensional permutation modules

Let S_∞ be the limit of the chain of finite symmetric groups in (3). To set up the study of analogs to the partition algebra, we explored three main examples of the following goal.

Choose
(1) a vector space V containing a countable linearly independent subset $\{v_i\}_{i \in \mathbb{N}}$, and
(2) an algebra of endomorphisms of V that are determined by their images on $\{v_i\}_{i \in \mathbb{N}}$,
such that with the action of S_∞ on the subscripts of $\{v_i\}_{i \in \mathbb{N}}$, the algebra of endomorphisms in (2) contains the action of $\mathbb{C}S_\infty$.

Countable-dimensional vector spaces

Let V be a countable-dimensional vector space with basis $\{v_i\}_{i \in \mathbb{N}}$, so that

$$V^{\otimes k} = \mathbb{C}\{v_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^n} = \left\{ \sum_{\mathbf{i} \in \mathbb{N}} a_{\mathbf{i}} v_{\mathbf{i}} \,\middle|\, a_i = 0 \text{ for all but finitely many } i \right\},$$

with $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k}$. As observed in [Sam and Snowden, 2013], the centralizer of the diagonal action of S_∞ on $V^{\otimes k}$ is a subalgebra of partition algebra. However, the diagrams with isolated blocks on the top (like d_1 and d_2 in Table 1) no longer have images in $V^{\otimes k}$ (as shown in (2)). Instead, the centralizer of the action of S_∞ in this case is the top-propagating partition algebra, spanned over $\mathbb{C}$ by diagrams with no blocks containing only top vertices. For example, d_1 and d_2 in Table 1 are *not* top-propagating, but the rest *are* top-propagating. Notice that the product of two top-propagating diagrams will never have an isolated component, so no parameter x (or n) is needed.

p-power summable sequences

A Banach space is a normed vector space where every Cauchy sequence converges. A Schauder basis for a Banach space V is a linearly independent set $\{v_1, v_2, \dots\}$ such that every element $v \in V$ can be written uniquely as series $v = a_1 v_1 + a_2 v_2 + \cdots$, with $a_i \in \mathbb{C}$.

Let $V = L^p(\mathbb{N}, \mu^p)$ be the space of bounded sequences with weighted counting measure μ^p satisfying $\sum_i \mu_i^p < \infty$ (so that vector $v_1 + v_2 + \cdots$ is in V). Then the analog to the k-fold tensor product is

$$\overline{V^{\otimes k}} = L^p(\mathbb{N}^k, (\mu^p)^{\times k}) = \left\{ v = \sum_{\mathbf{i} \in \mathbb{N}^k} a_{\mathbf{i}} v_{\mathbf{i}} \in \mathbb{C}^{\mathbb{N}^k} \,\middle|\, \|v\|^p = \sum_{\mathbf{i} \in \mathbb{N}^k} |a_{\mathbf{i}}|^p \mu_{\mathbf{i}}^p < \infty \right\},$$

where $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k}$ and $\mu_{\mathbf{i}} = \mu_{i_1} \cdots \mu_{i_k} = \|v_{\mathbf{i}}\|$. Then the set of endomorphisms which are determined by their images on the Shauder basis $\{v_{\mathbf{i}} \mid \mathbf{i} \in \mathbb{N}^k\}$ are the bounded operators $\mathcal{B}(\overline{V^{\otimes k}})$.

Note that the permutation action of S_∞ are, indeed, bounded.

Then the centralizer of the action of $\mathbb{C}S_\infty$ inside of $\mathcal{B}(\overline{V^{\otimes k}})$ is the algebra of uniform block permutations.

Bounded sequences

Let $V = \ell^\infty$ be the Banach space of bounded sequences; then the analog to the k-fold tensor product is

$$\ell^\infty(\mathbb{N}^k) = \left\{ v = (a_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^k} \in \mathbb{C}^{\mathbb{N}^k} \,\middle|\, \|v\|_\infty = \sup_{\mathbf{i} \in \mathbb{N}^k} |a_{\mathbf{i}}| < \infty \right\}$$

where $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k} = (\delta_{\mathbf{ij}})_{\mathbf{j} \in \mathbb{N}^k}$. Unfortunately, $\ell^\infty(\mathbb{N}^k)$ has no countable Schauder bases. However, we can restrict our consideration to operators that are determined by their actions on $\{v_{\mathbf{i}} \mid \mathbf{i} \in \mathbb{N}^k\}$.To this end, let

$$\mathcal{B}_{\mathsf{Mat}}(\ell^\infty(\mathbb{N}^k)) = \left\{ A = (A^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i},\mathbf{j} \in \mathbb{N}^k} \in \mathbb{C}^{\mathbb{N}^k \times \mathbb{N}^k} \,\middle|\, \|A\|_{\mathsf{Mat}} = \sup_{\mathbf{j} \in \mathbb{N}^k} \left\{ \sum_{\mathbf{i} \in \mathbb{N}^k} |A^{\mathbf{j}}_{\mathbf{i}}| \right\} < \infty \right\}.$$

Again, this contains the action of $\mathbb{C}S_\infty$ as desired.

Then the centralizer of the action of $\mathbb{C}S_\infty$ inside of $\mathcal{B}_{\mathsf{Mat}}(\ell^\infty(\mathbb{N}^k))$ is the bottom-propagating partition algebra.