Combinatorics of affine Hecke algebras of type C.

Zajj Daugherty
(joint with Arun Ram)

January 31, 2014

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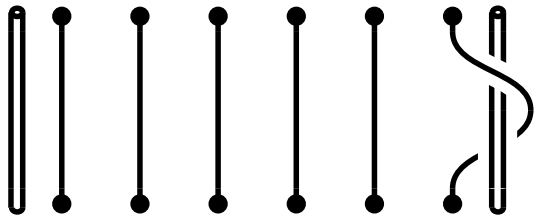
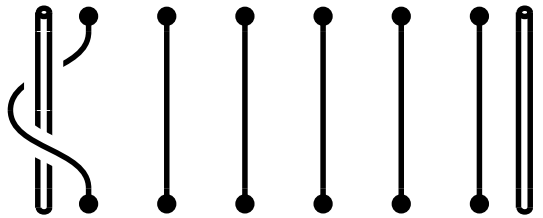
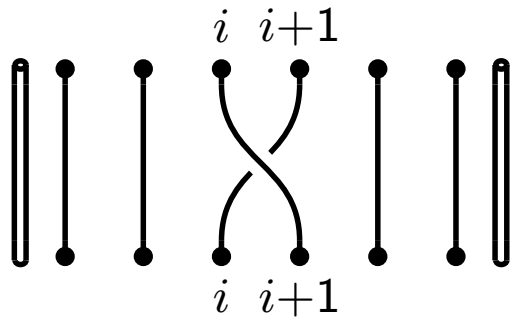
Pictorially, the generators of $\mathcal{B}_k$ are identified with the diagrams

$T_k =$  , $T_0 =$  ,

and

$T_i =$ 

for $i = 1, \ldots, k-1$.

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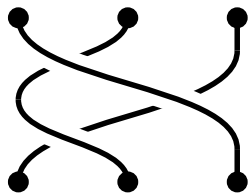
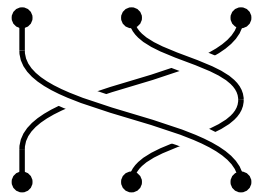
$$T_iT_{i+1}T_i \;=\;$$  $$\;=\;$$  $$\;=\; T_{i+1}T_iT_{i+1}$$

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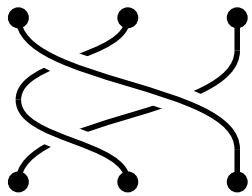
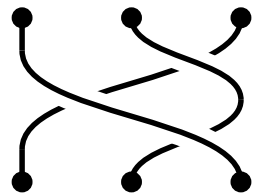
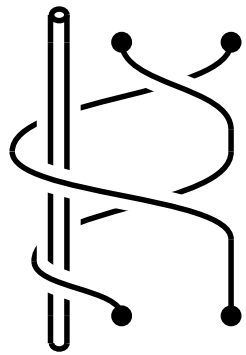
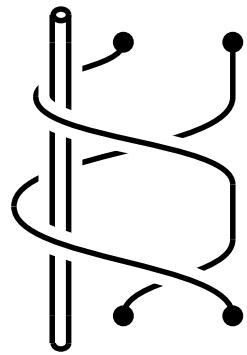
$T_iT_{i+1}T_i$ =  =  = $T_{i+1}T_iT_{i+1}$

$T_0T_1T_0T_1$ =  =  = $T_1T_0T_1T_0$

(similar picture for $T_kT_{k-1}T_kT_{k-1} = T_{k-1}T_kT_{k-1}T_k$)

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1. Fix $t, t_0, t_k \in \mathbb{C}^\times$. The affine Hecke algebras of type C H_k is the quotient of $\mathbb{C}\mathcal{B}_k$ by

$$(*) \quad 0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$

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2. Let A, B, C be finite dim'l $U_q\mathfrak{g}$-modules. Then $\mathbb{C}\mathcal{B}_k$ acts on

$$B \otimes \underbrace{C \otimes \cdots \otimes C}_{k \text{ factors}} \otimes A$$

Under good (to be defined) conditions, this action factors through the quotient $(*)$.

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Goal today:

Tell you 3 descriptions of irreducible representations of H_k.

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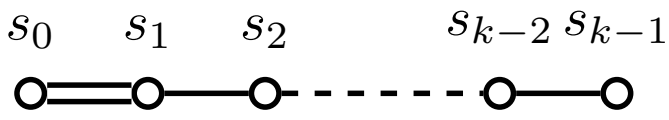
W is a group of signed permutations generated by transpositions $s_0, s_1, \ldots, s_{k-1}$ with relations



$$s_j^2 = 1, \qquad s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for all j and for $i \neq 0$.

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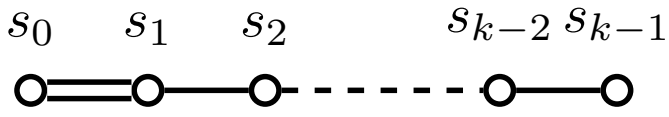

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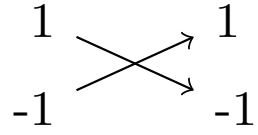
s_0 swaps 

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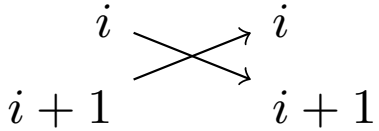
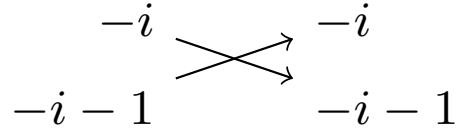
s_i swaps 

and 

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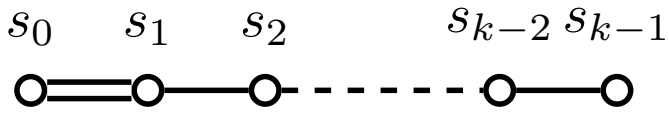

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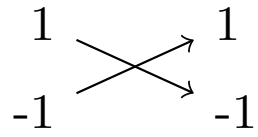
s_0 swaps 

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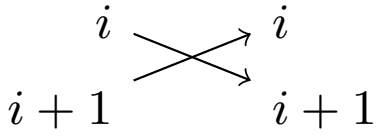
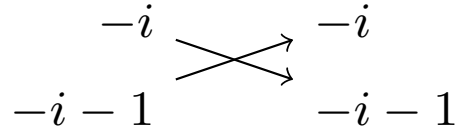
s_i swaps 

and 

W acts on the subscripts of the Y_i's with $Y_{-i} = Y_i^{-1}$.

Central characters

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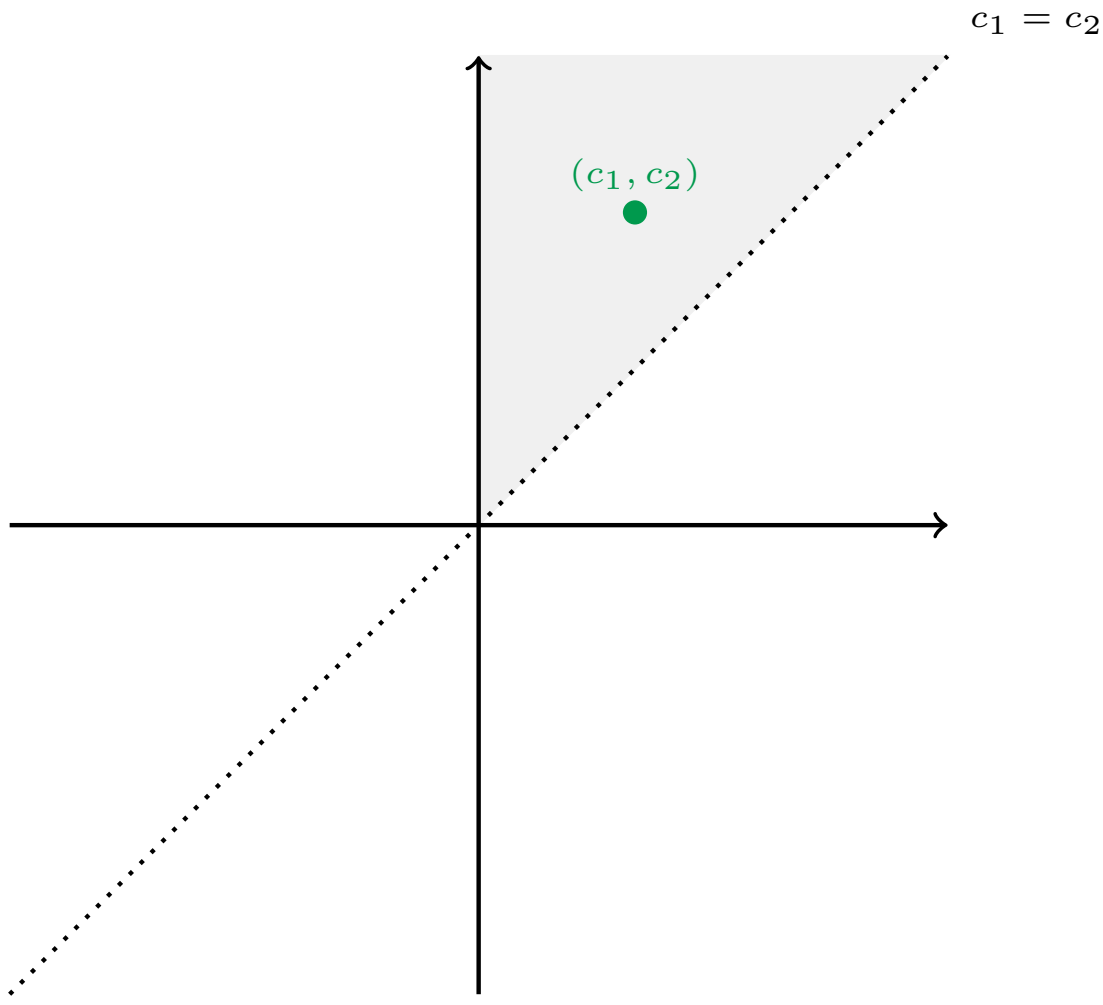
Description 1: Central characters are indexed by points in k dimensions.

Central characters as points

Restrict to real points.

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.
When $k = 2$:



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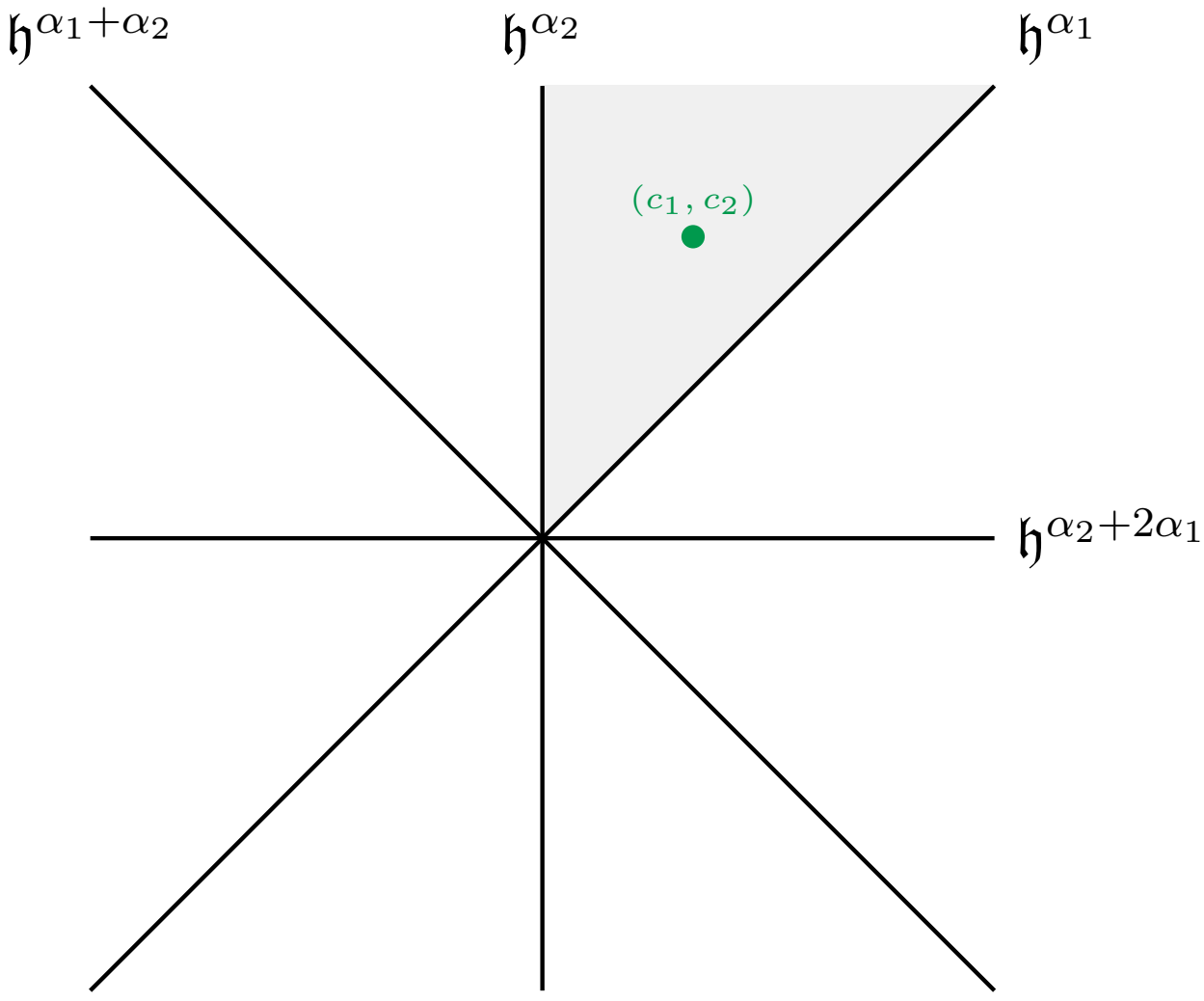

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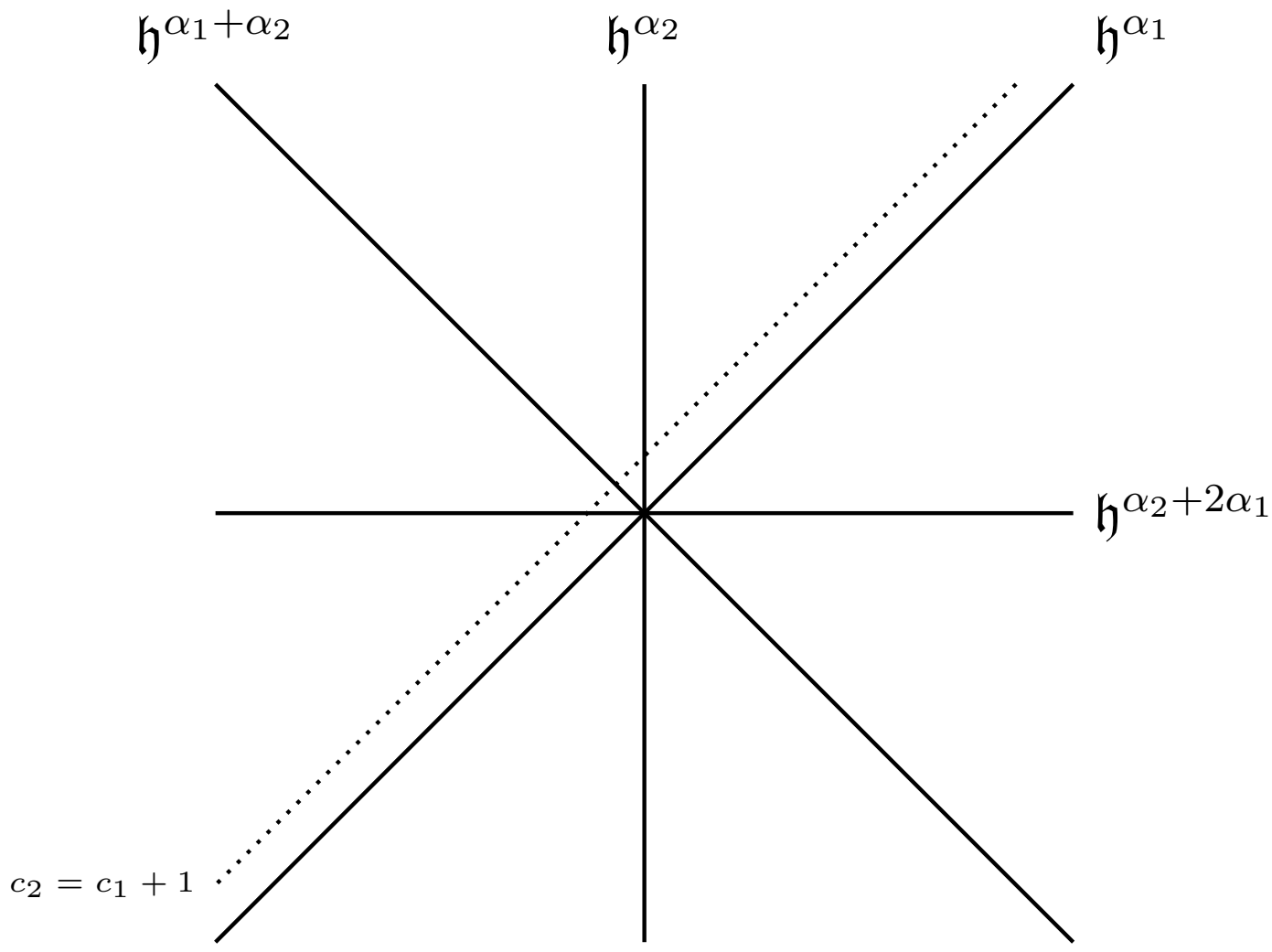

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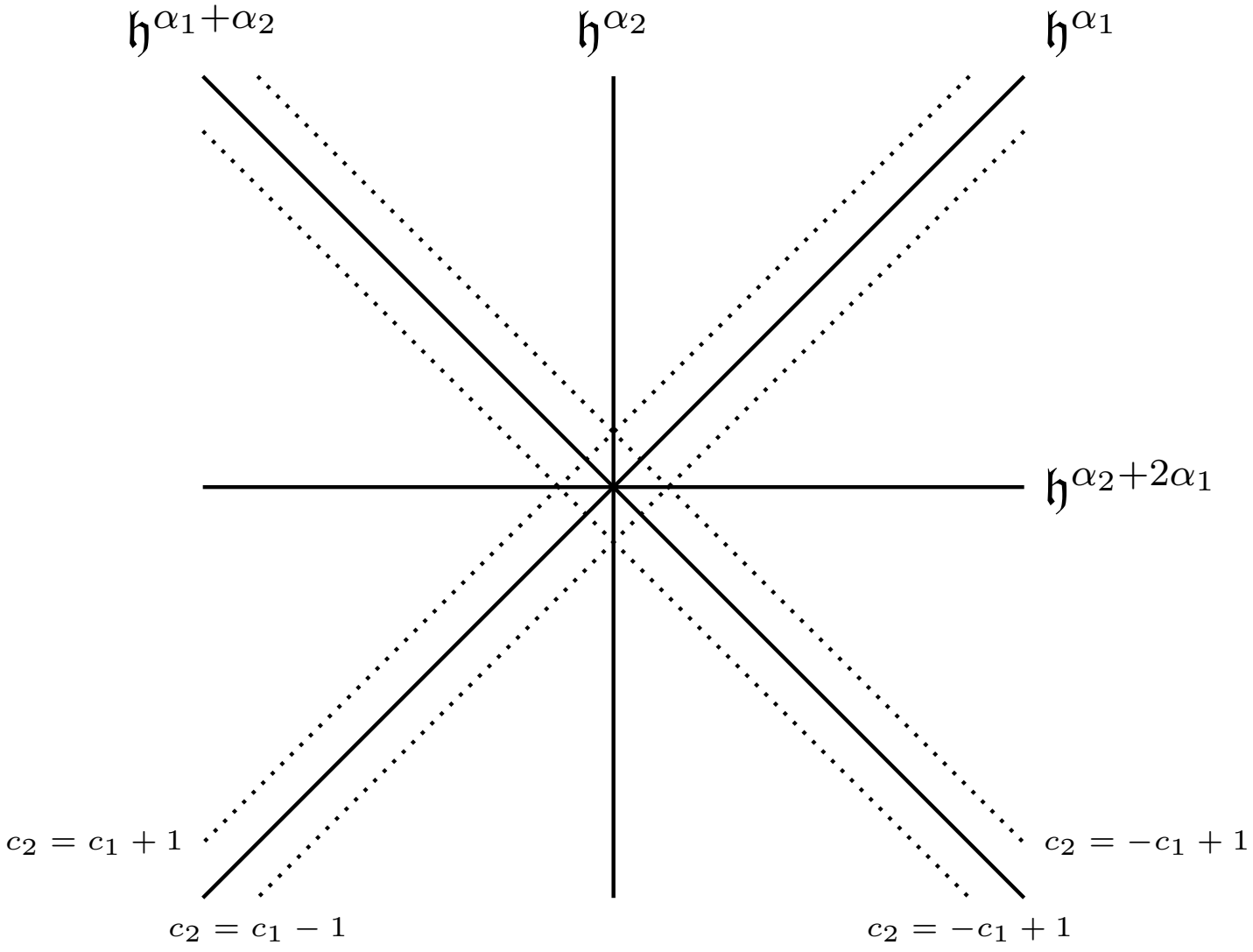

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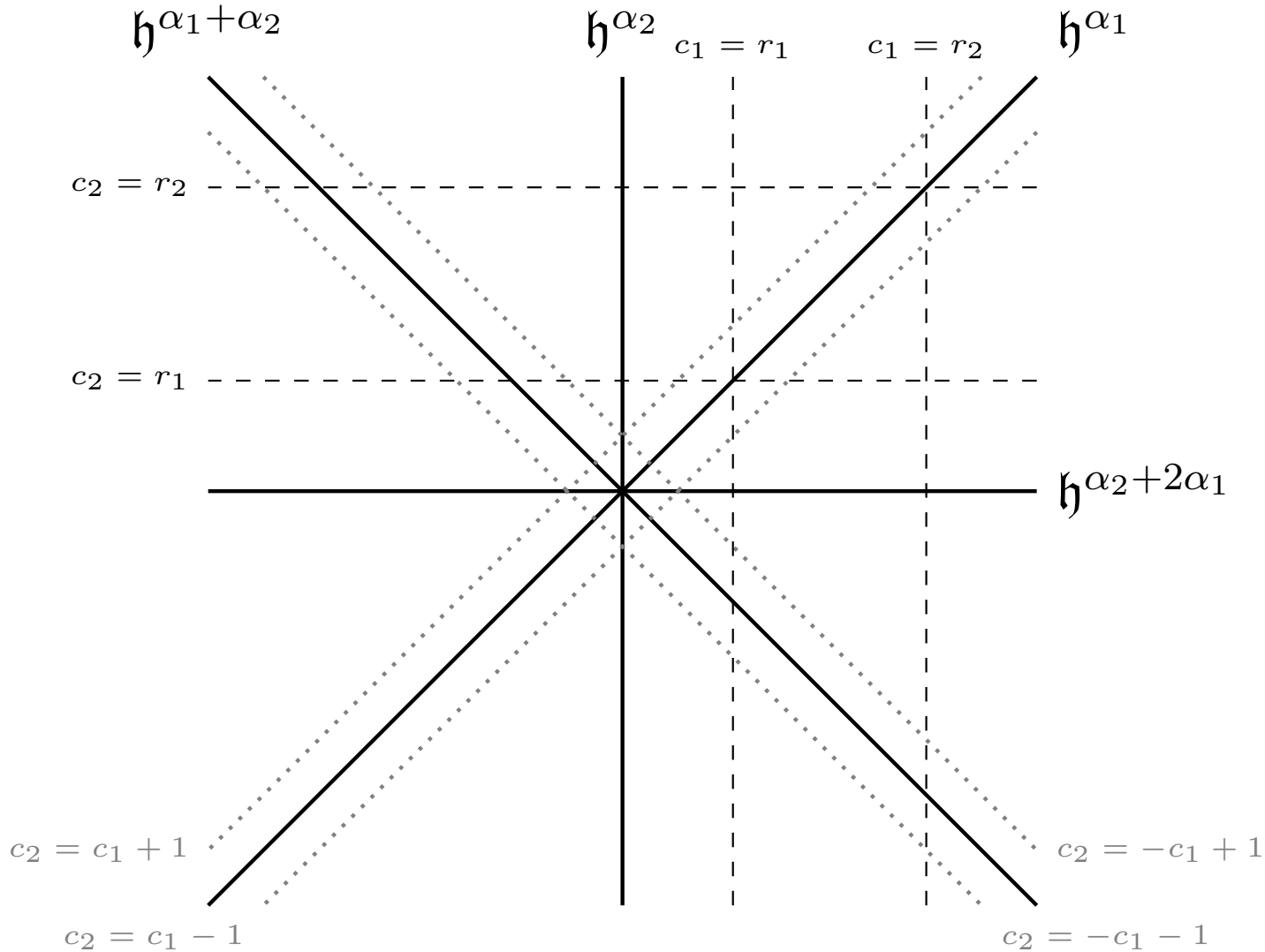

The r_is depend on H_k's parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

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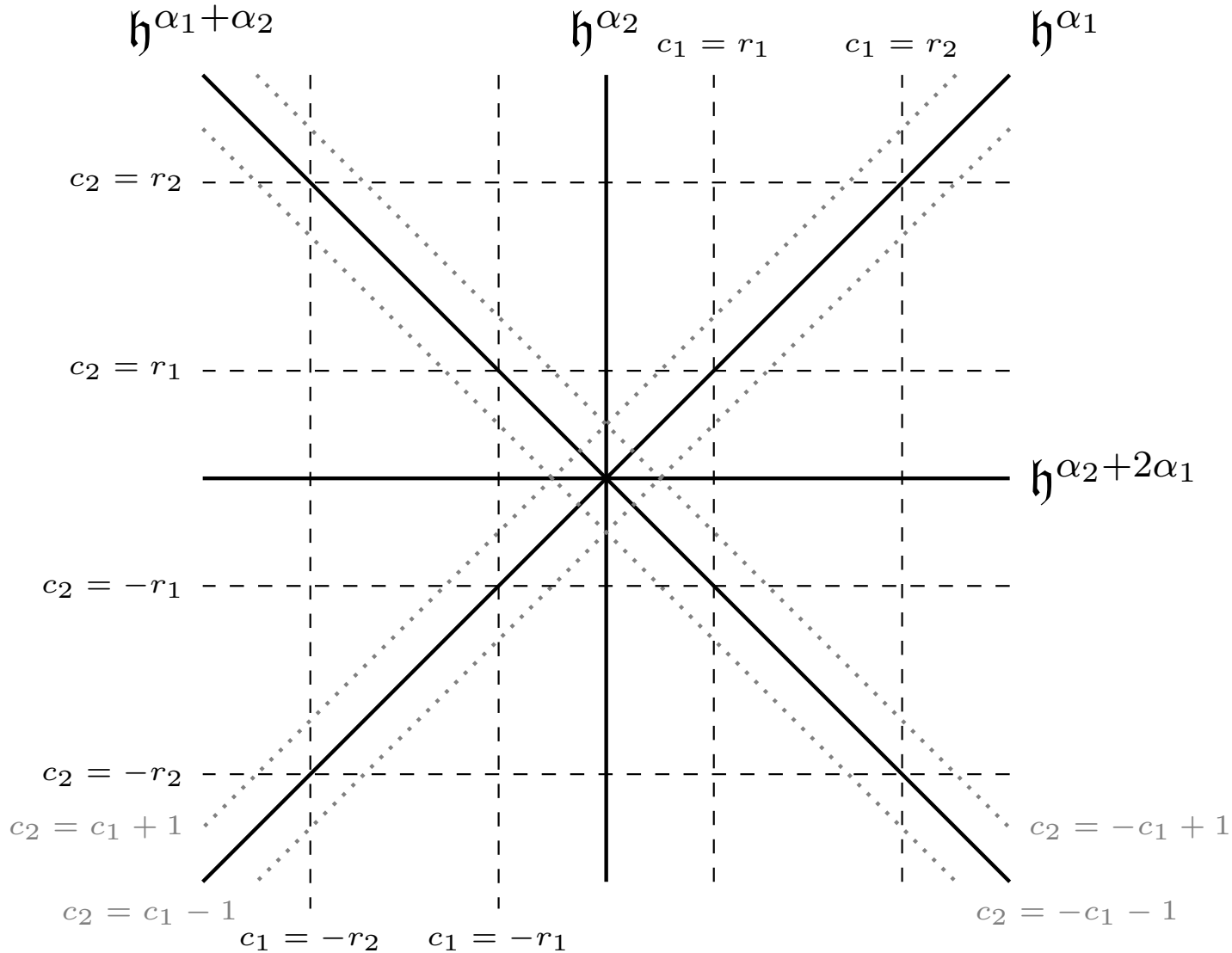

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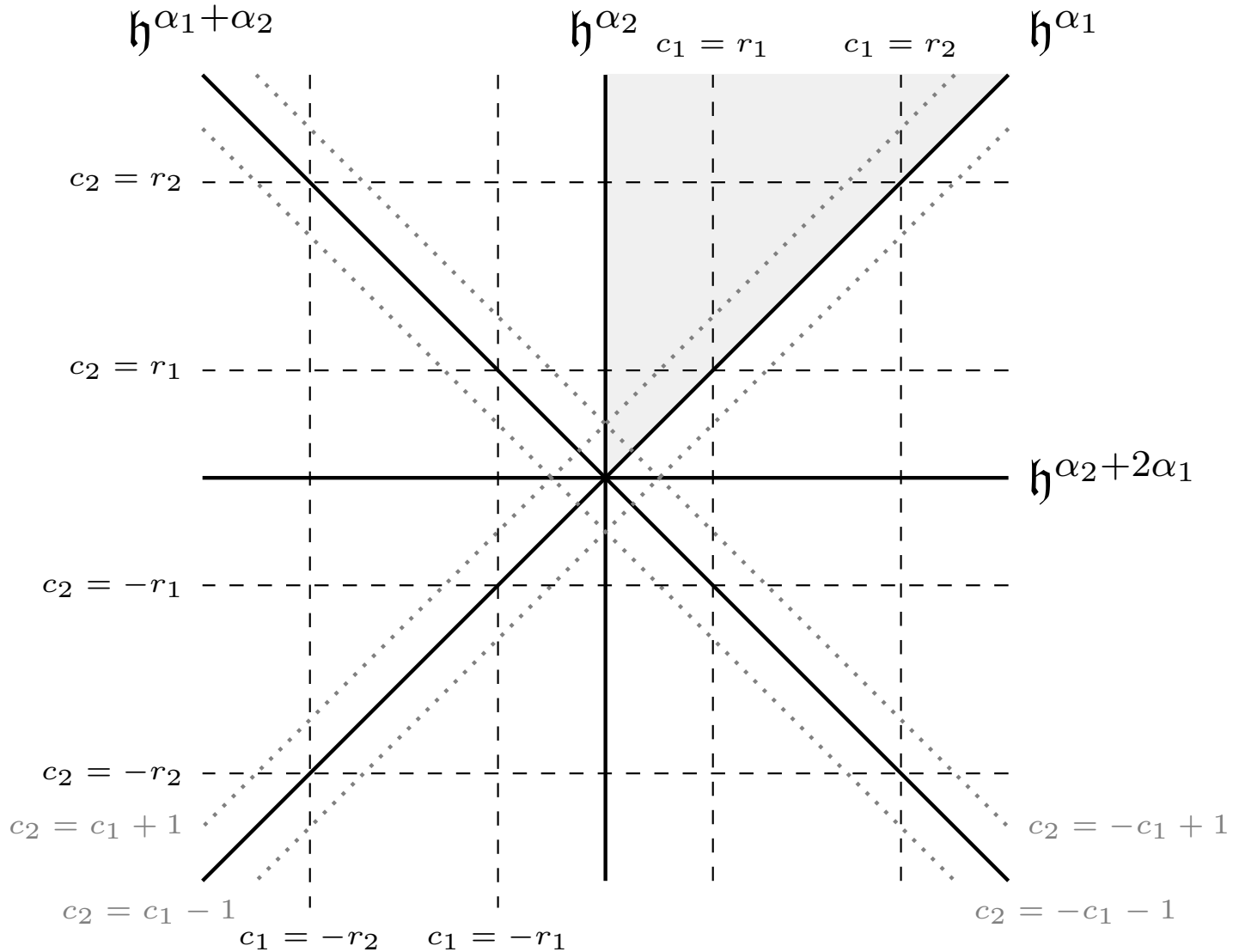

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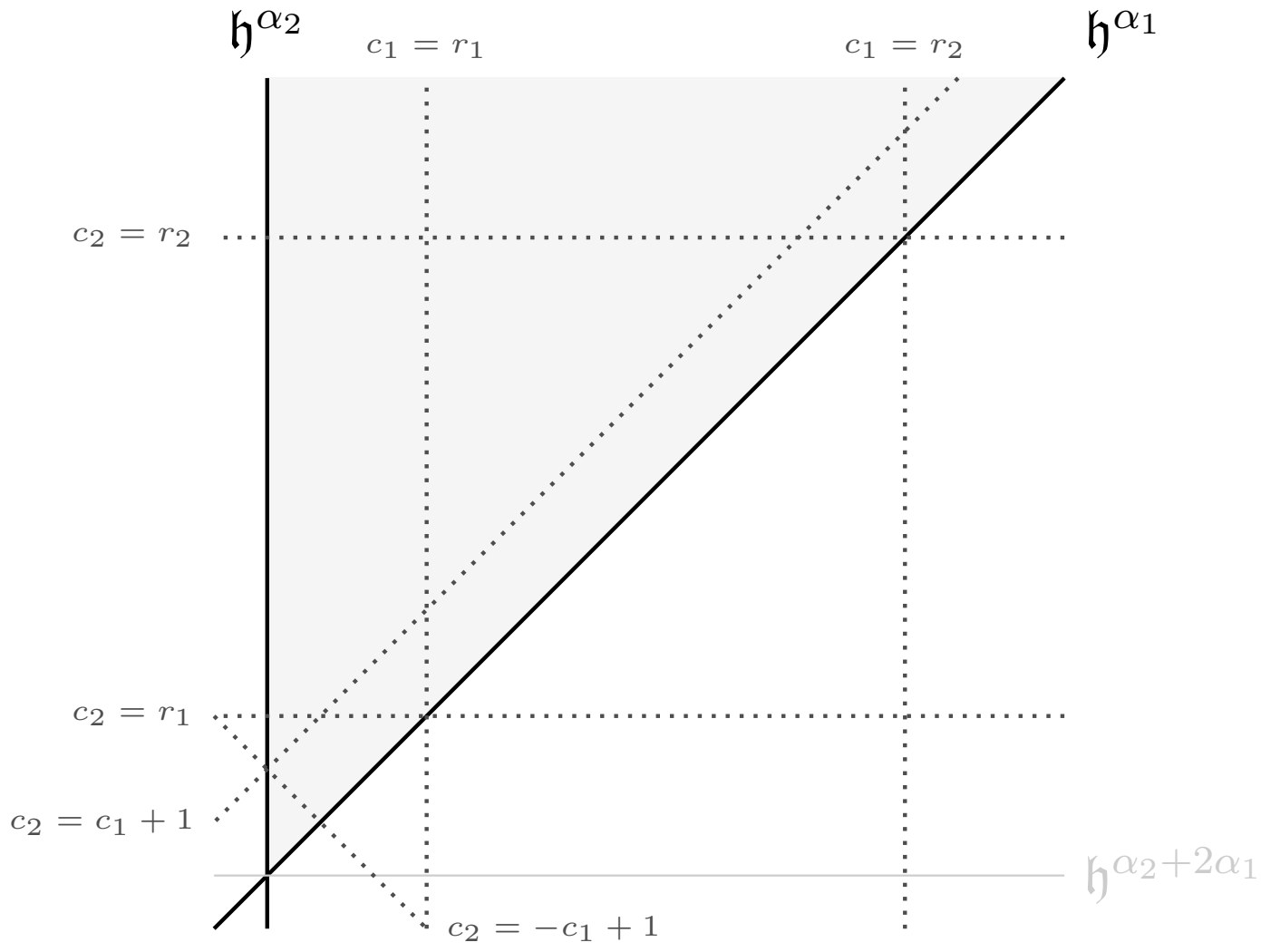

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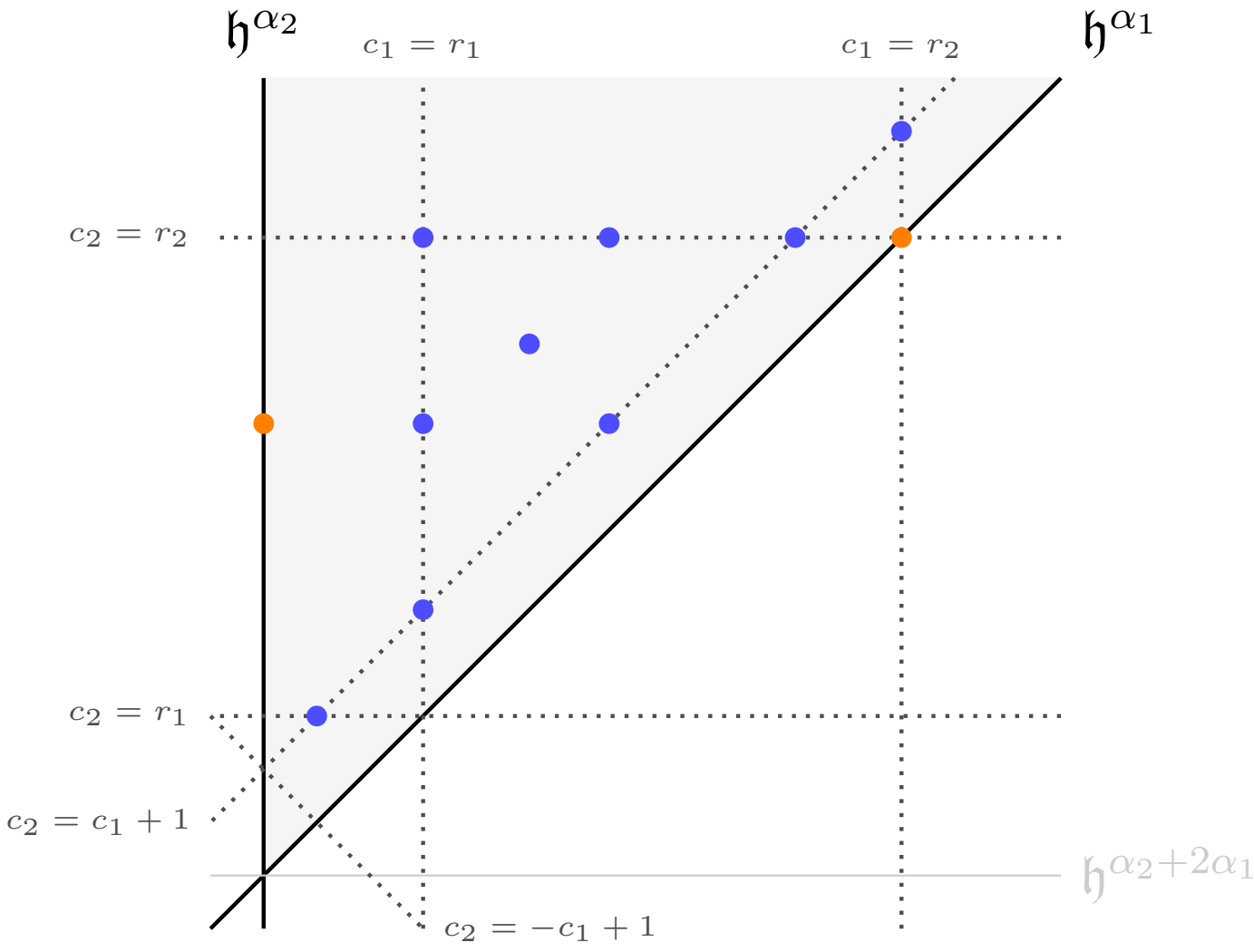

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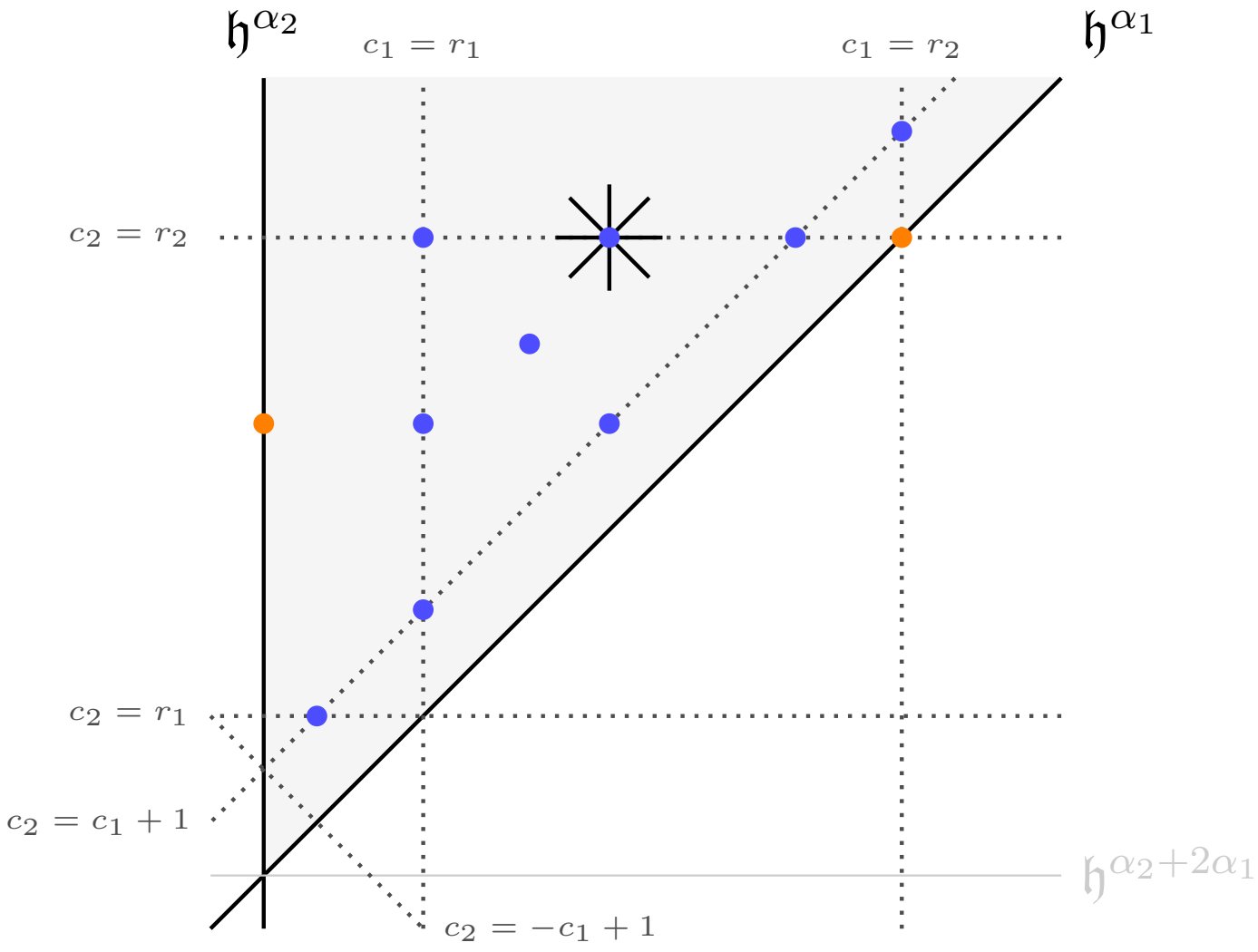

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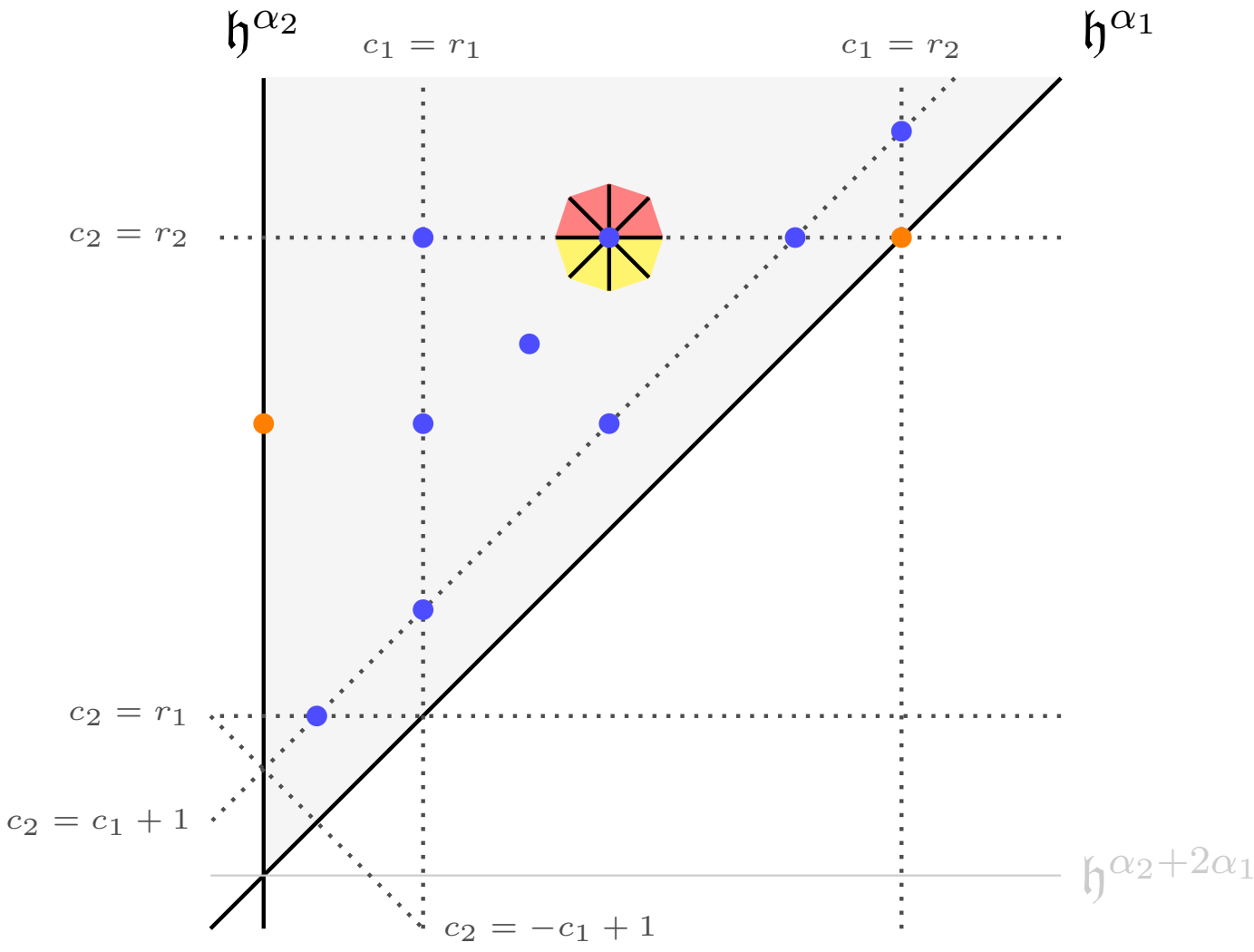

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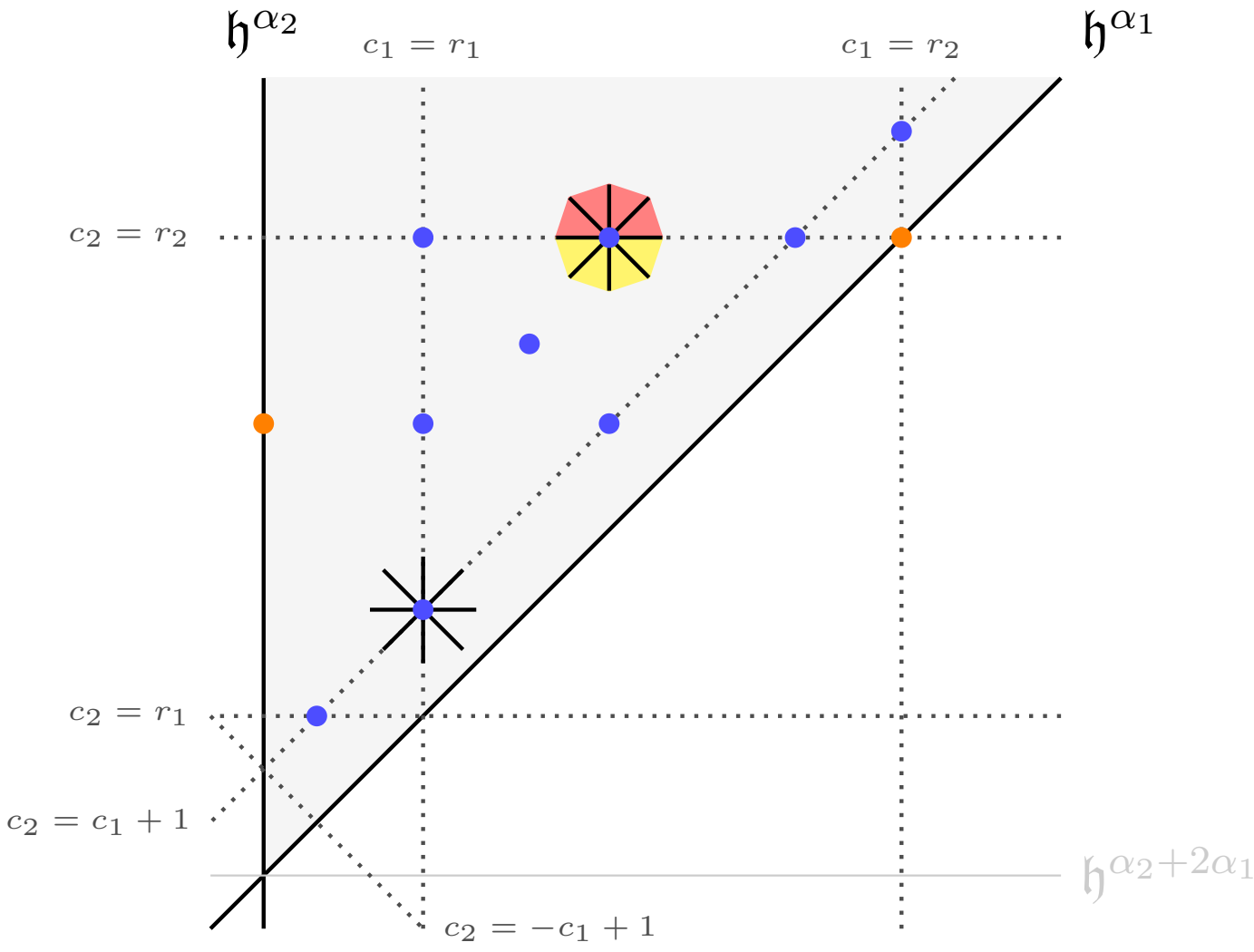

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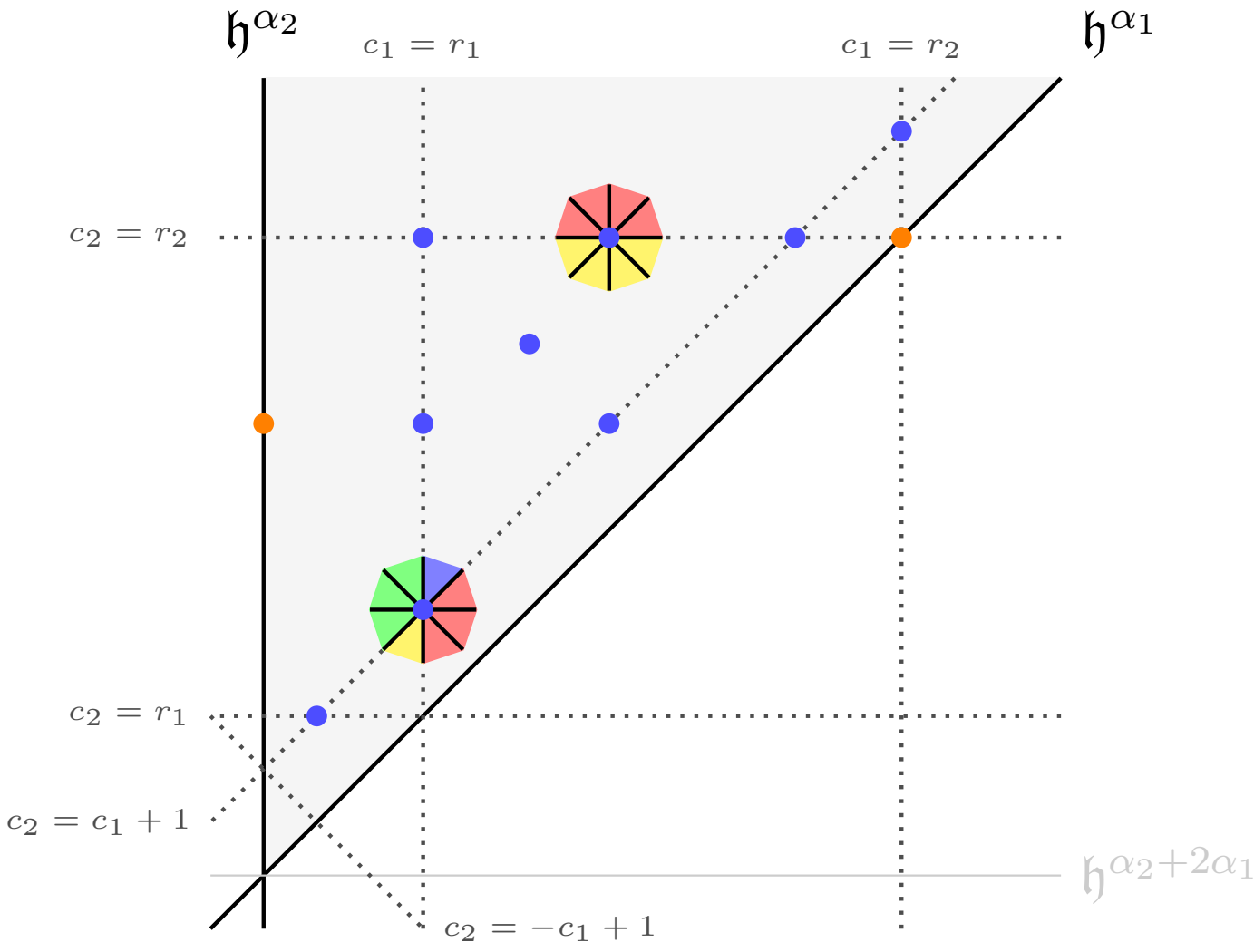

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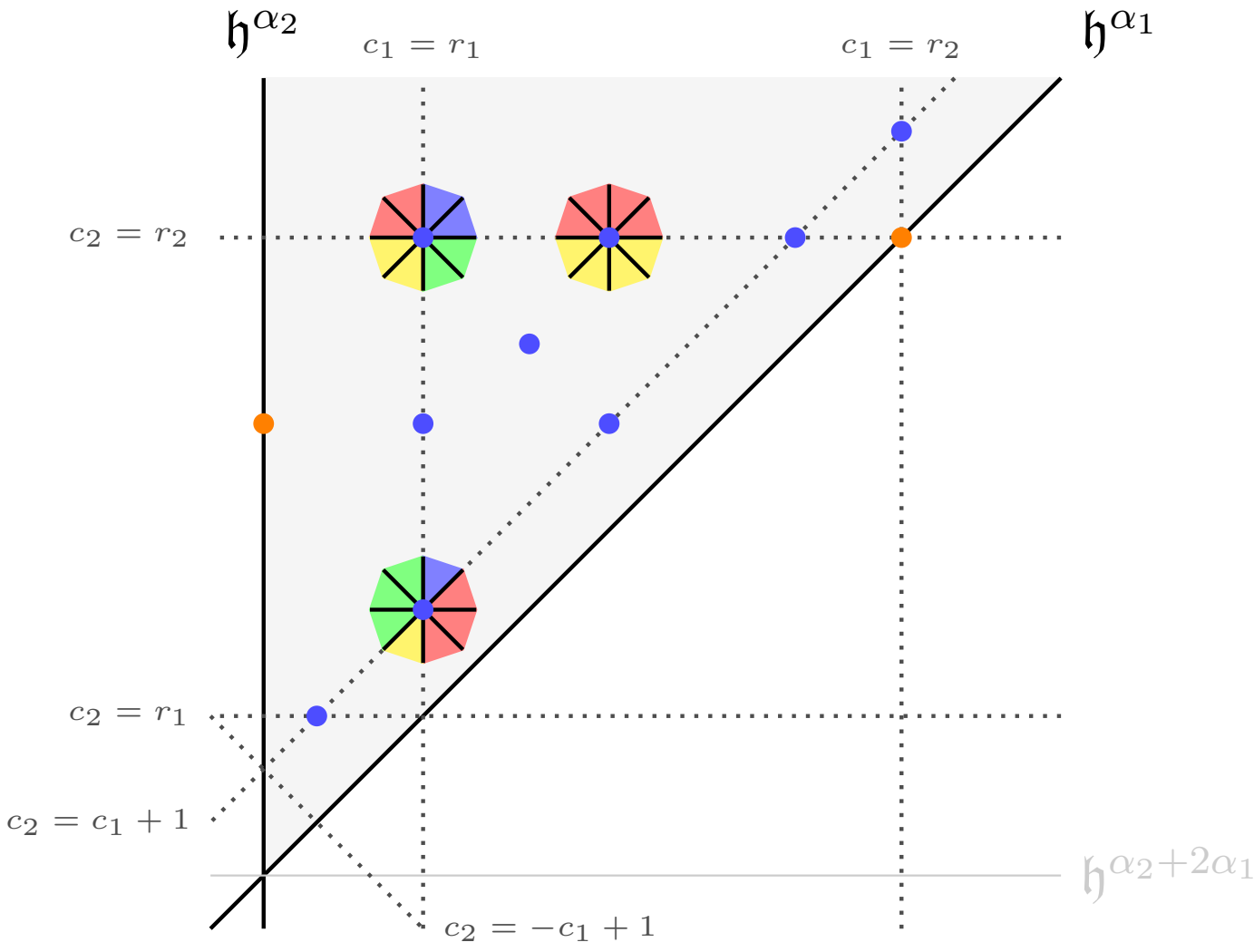

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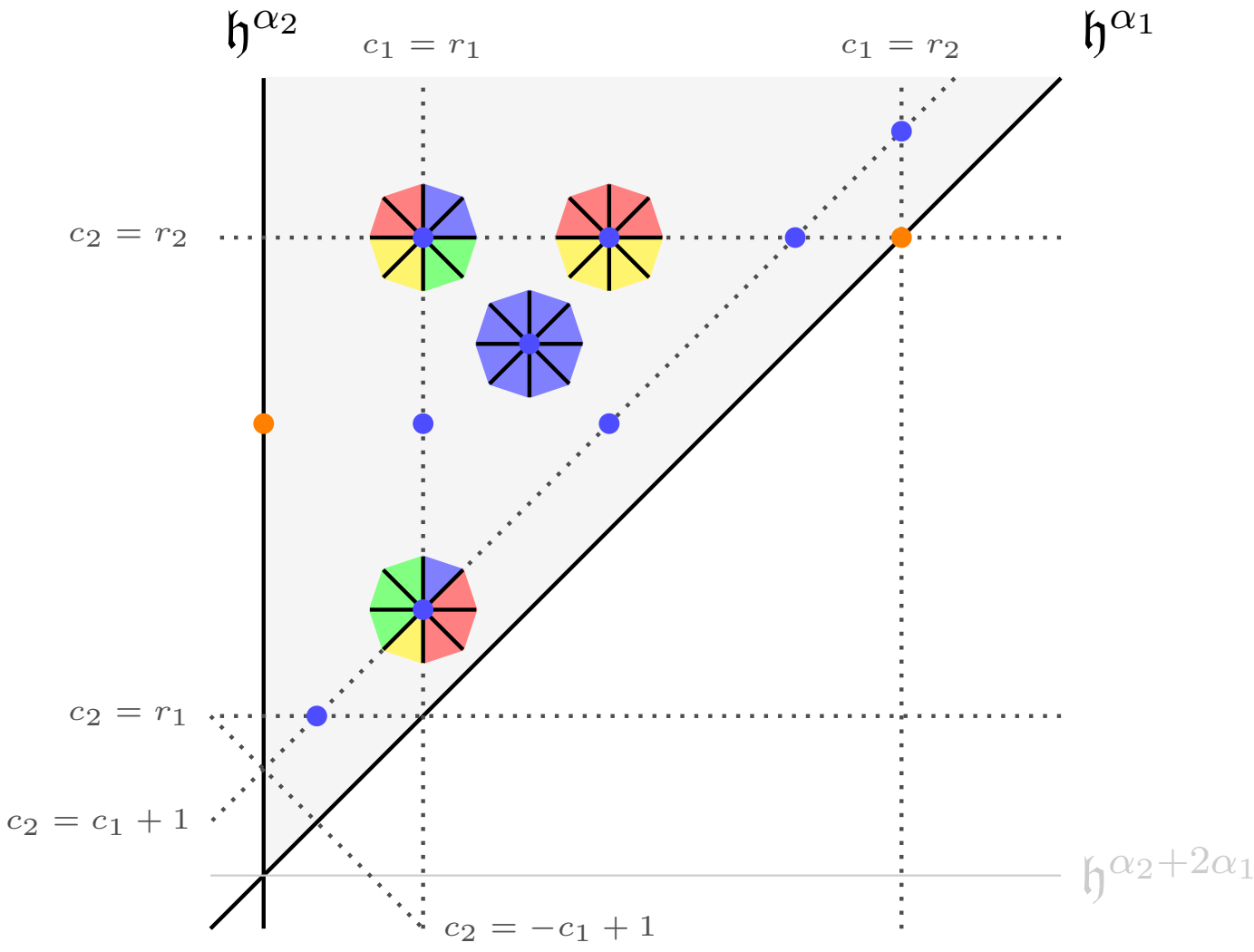

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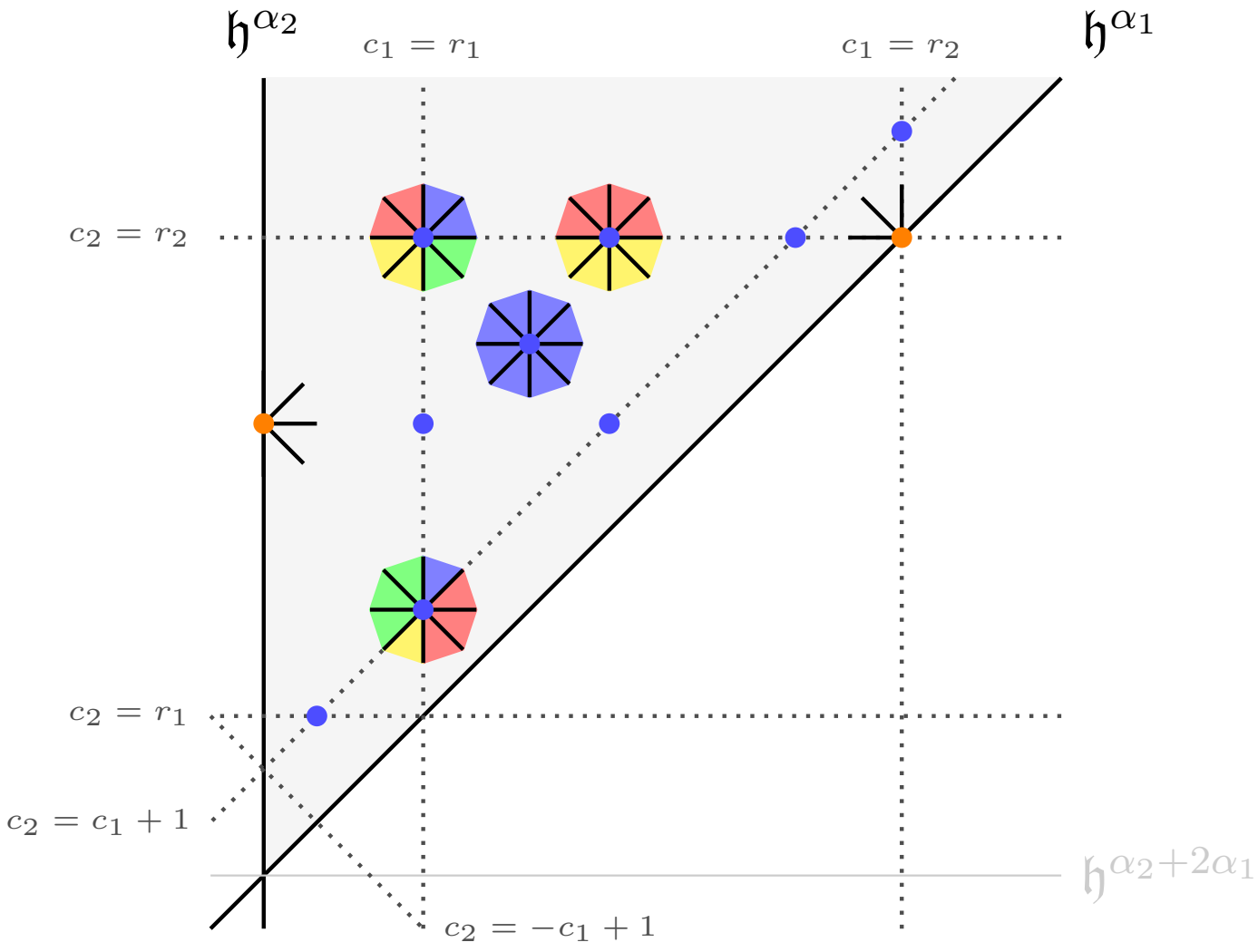

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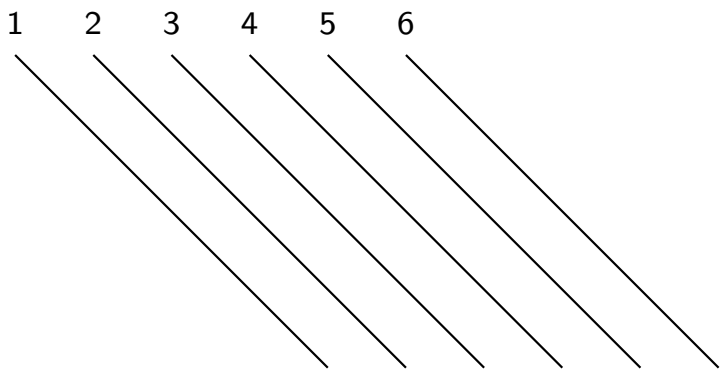

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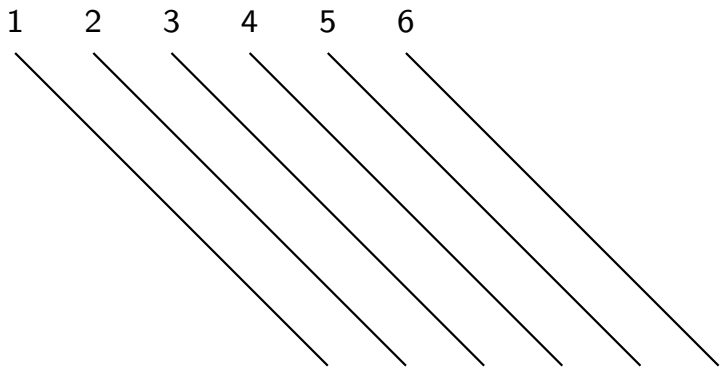

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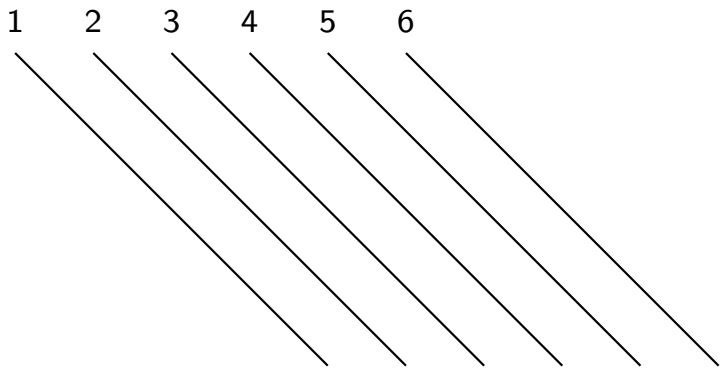

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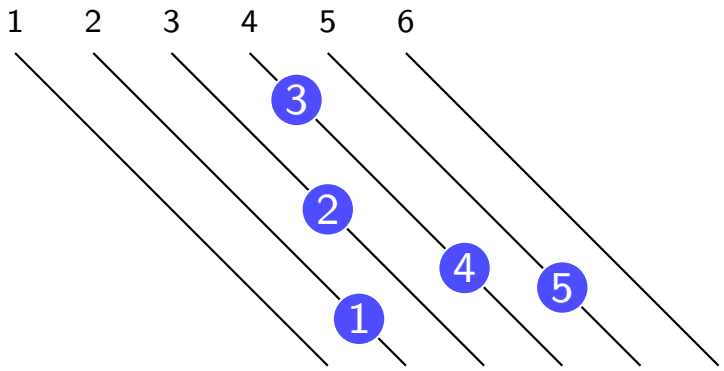

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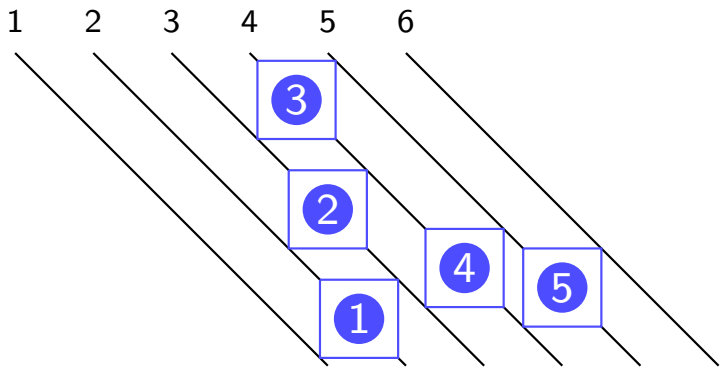

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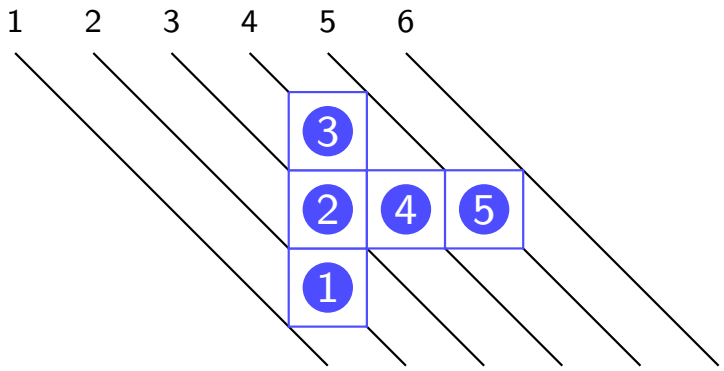

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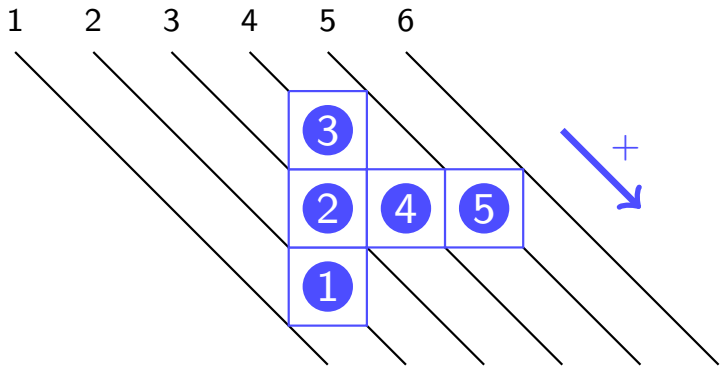

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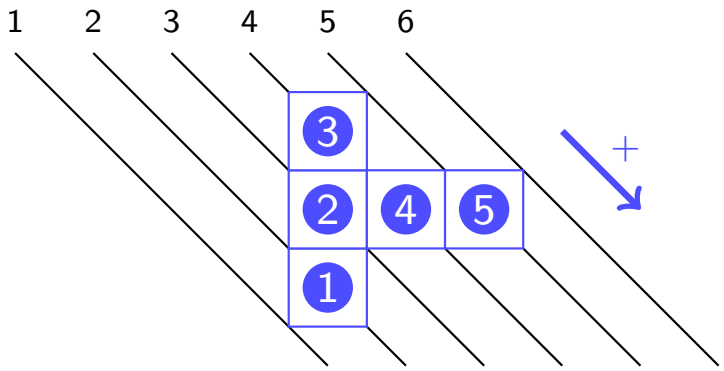

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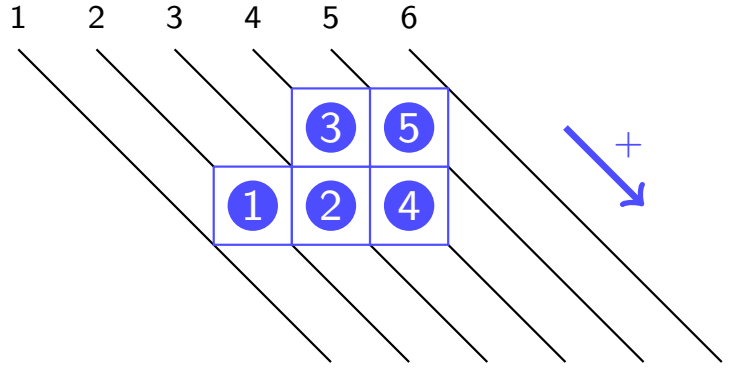

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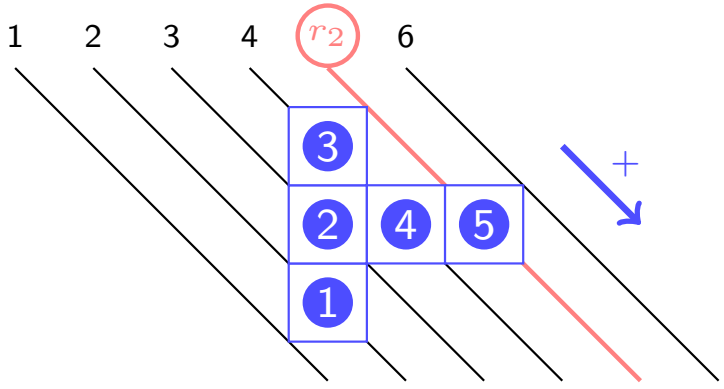

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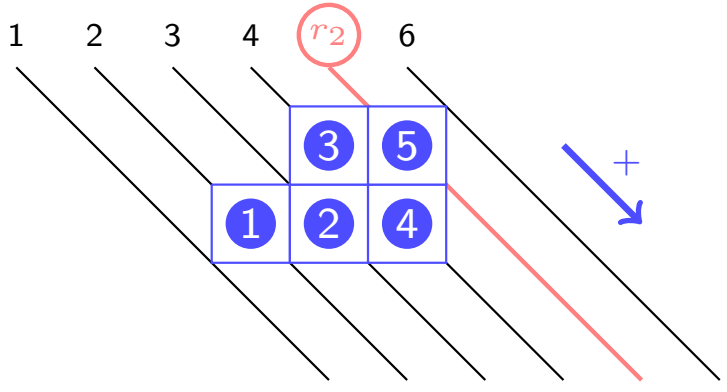

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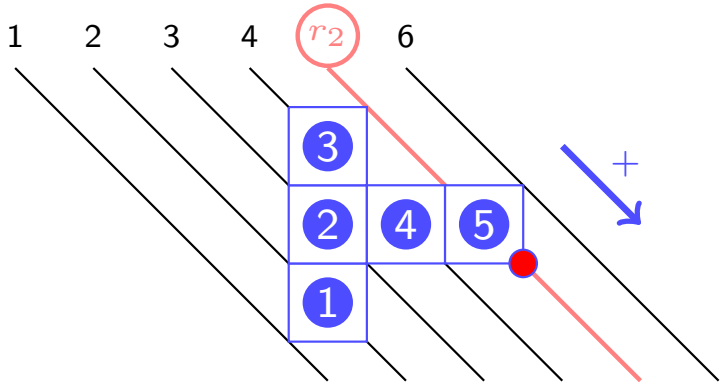

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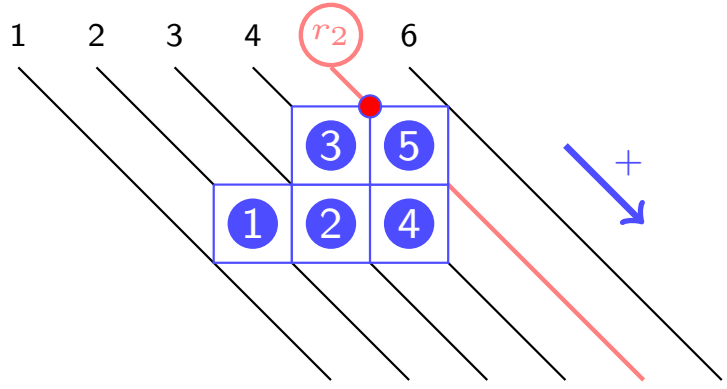

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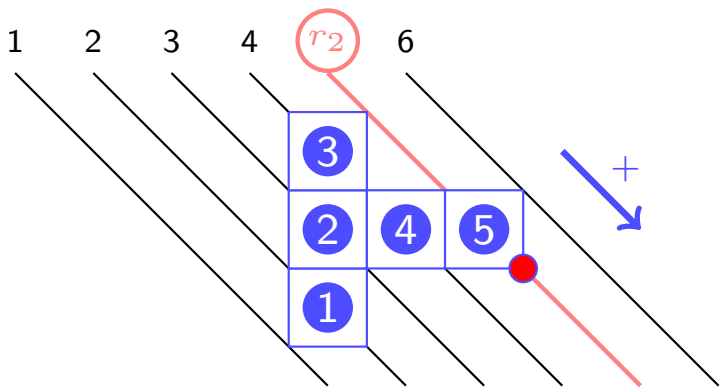

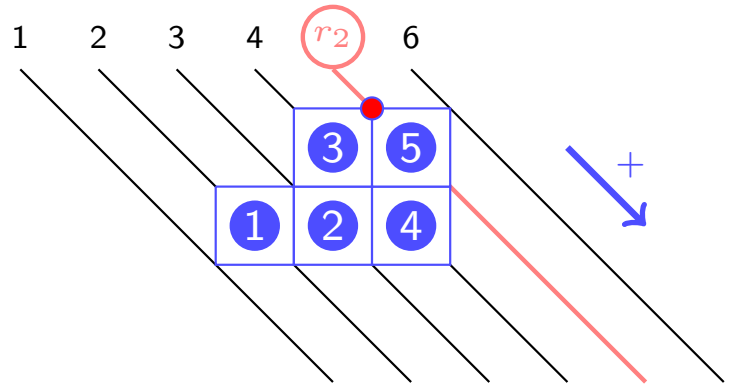

Basis indexed by standard fillings with $\{\pm 1, \ldots, \pm k\}$ with restrictions:

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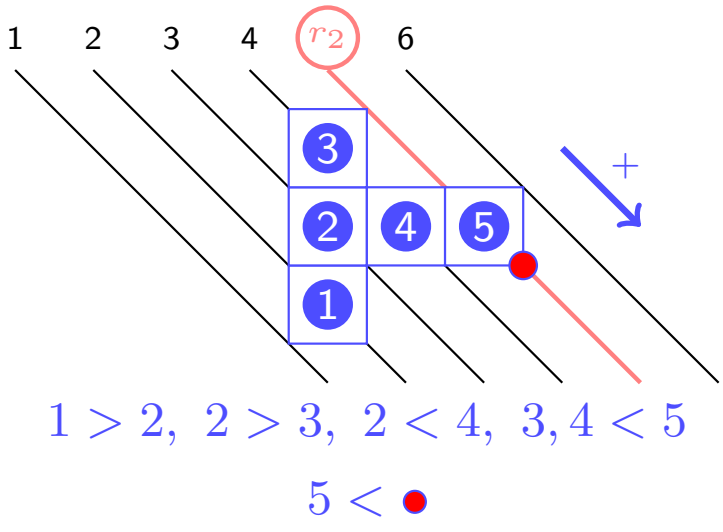

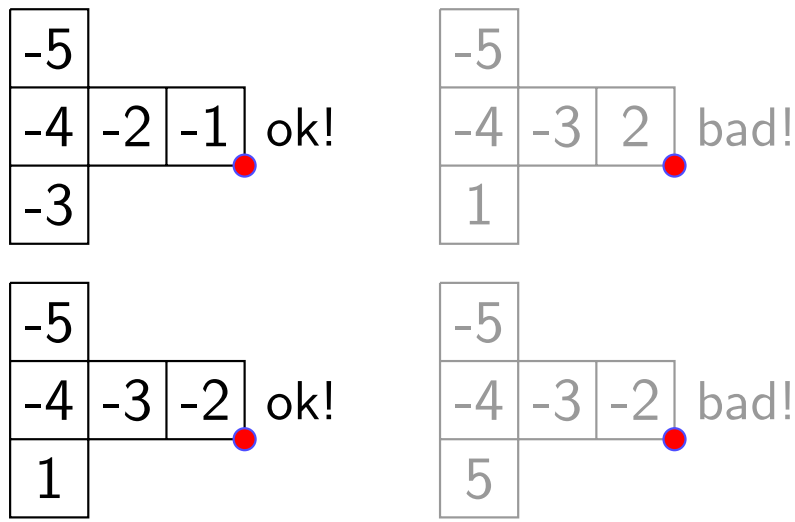

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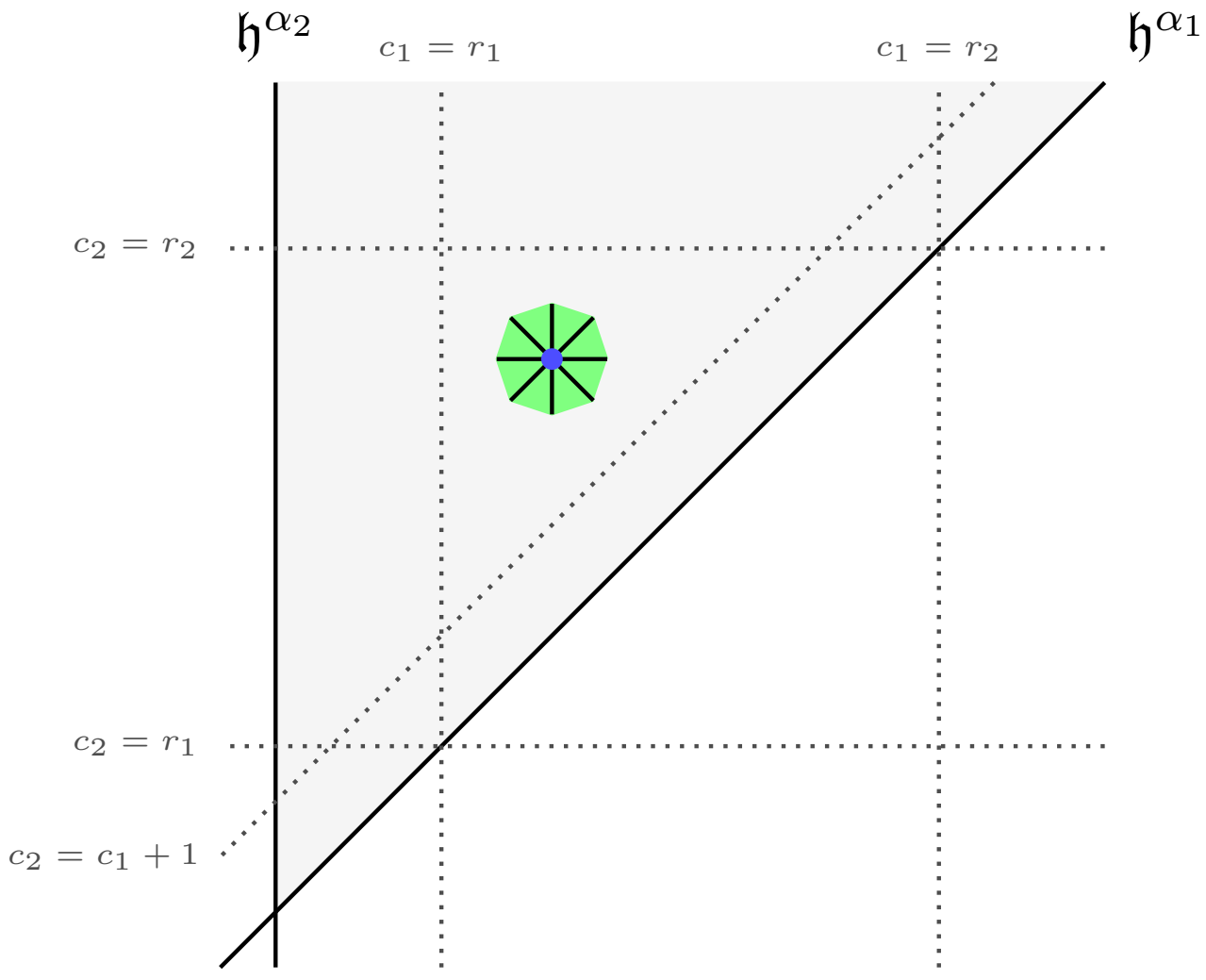

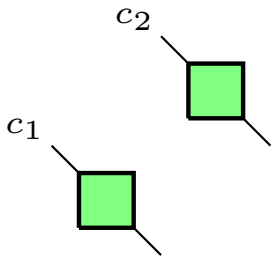

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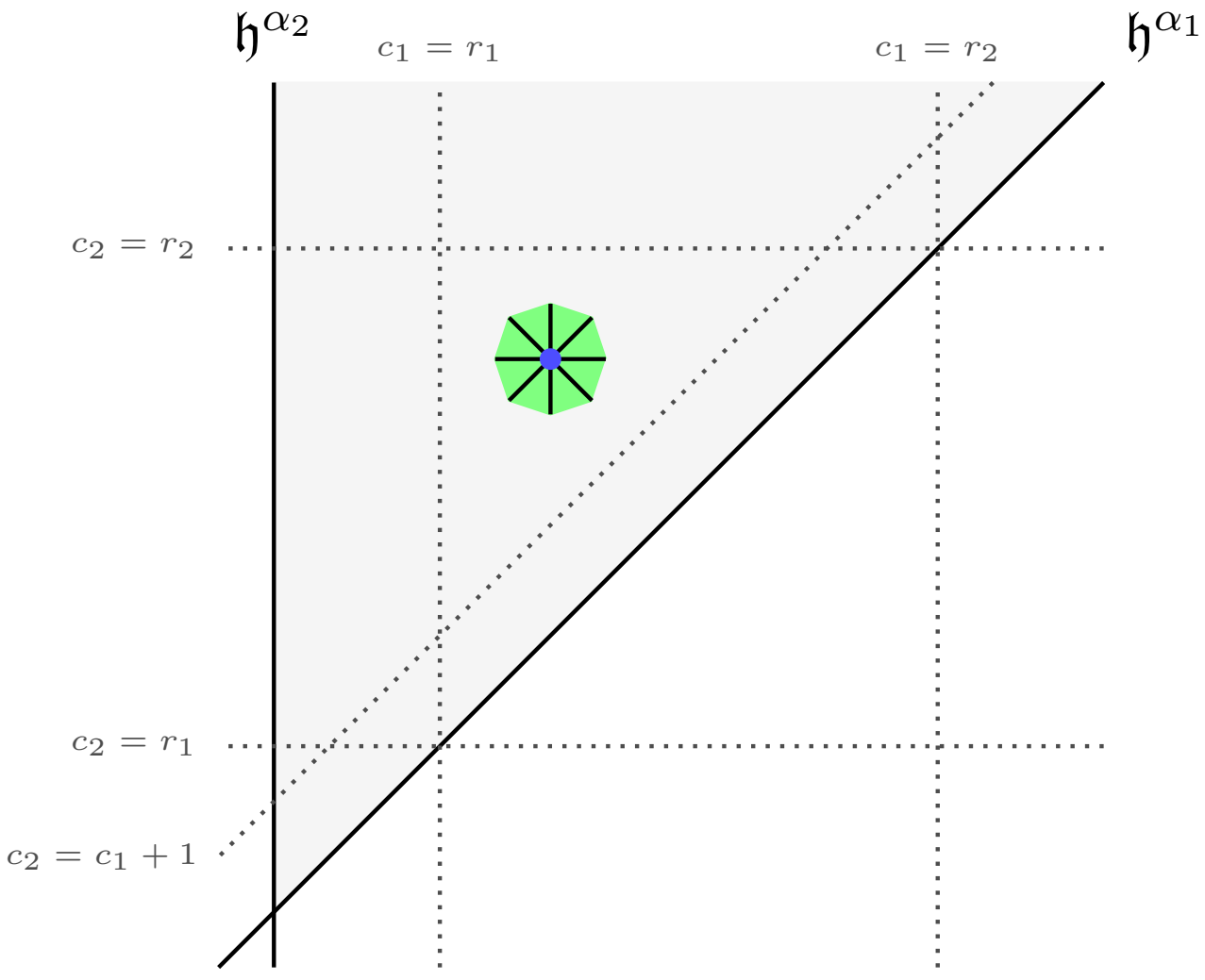

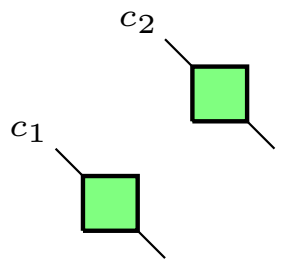

2 1	2 -1
1 2	-1 2
-2 -1	-2 1
-1 -2	1 -2

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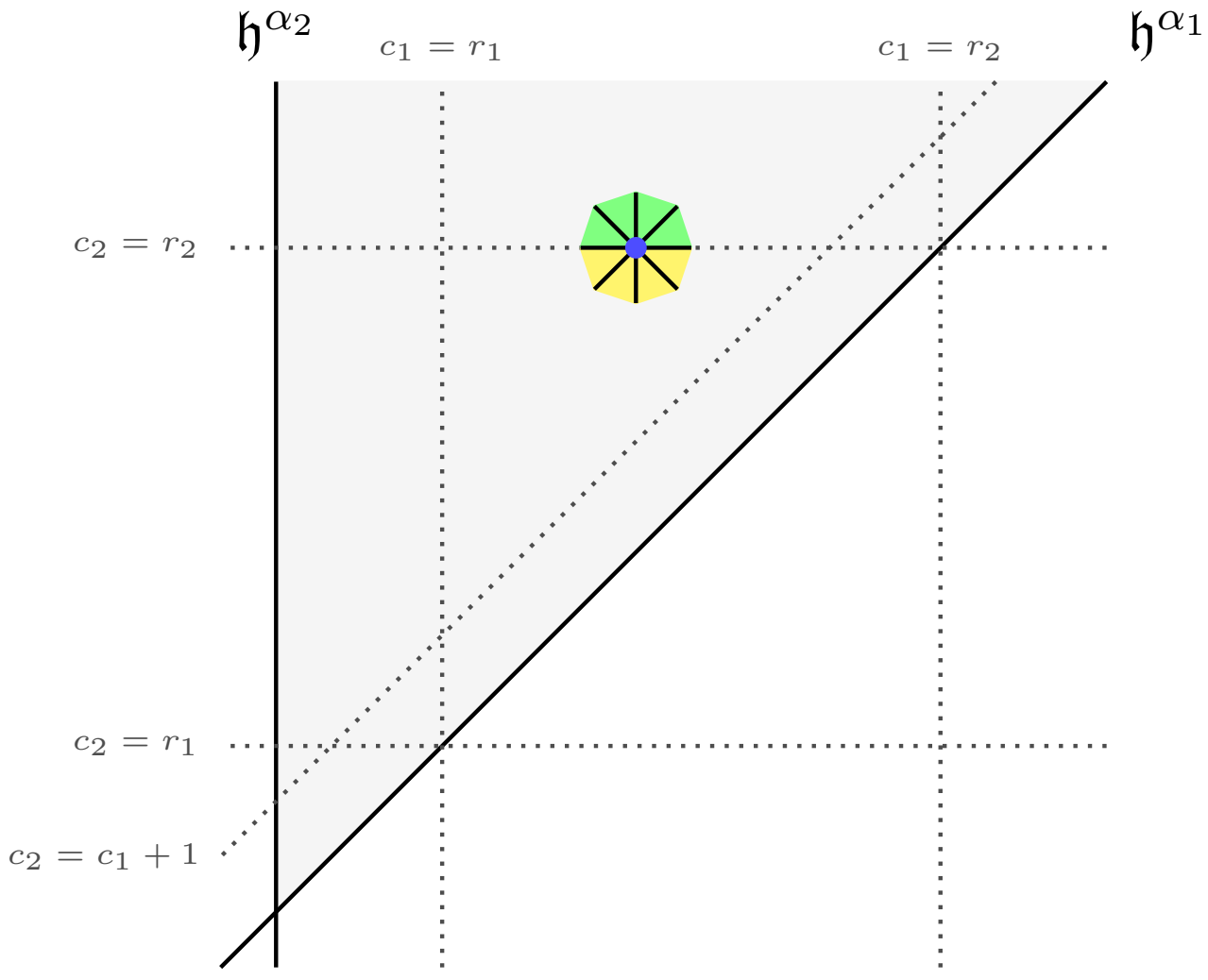

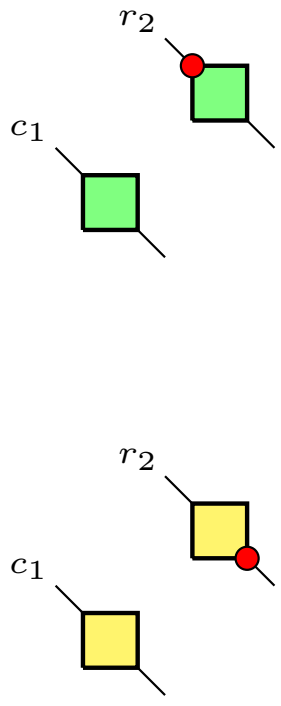

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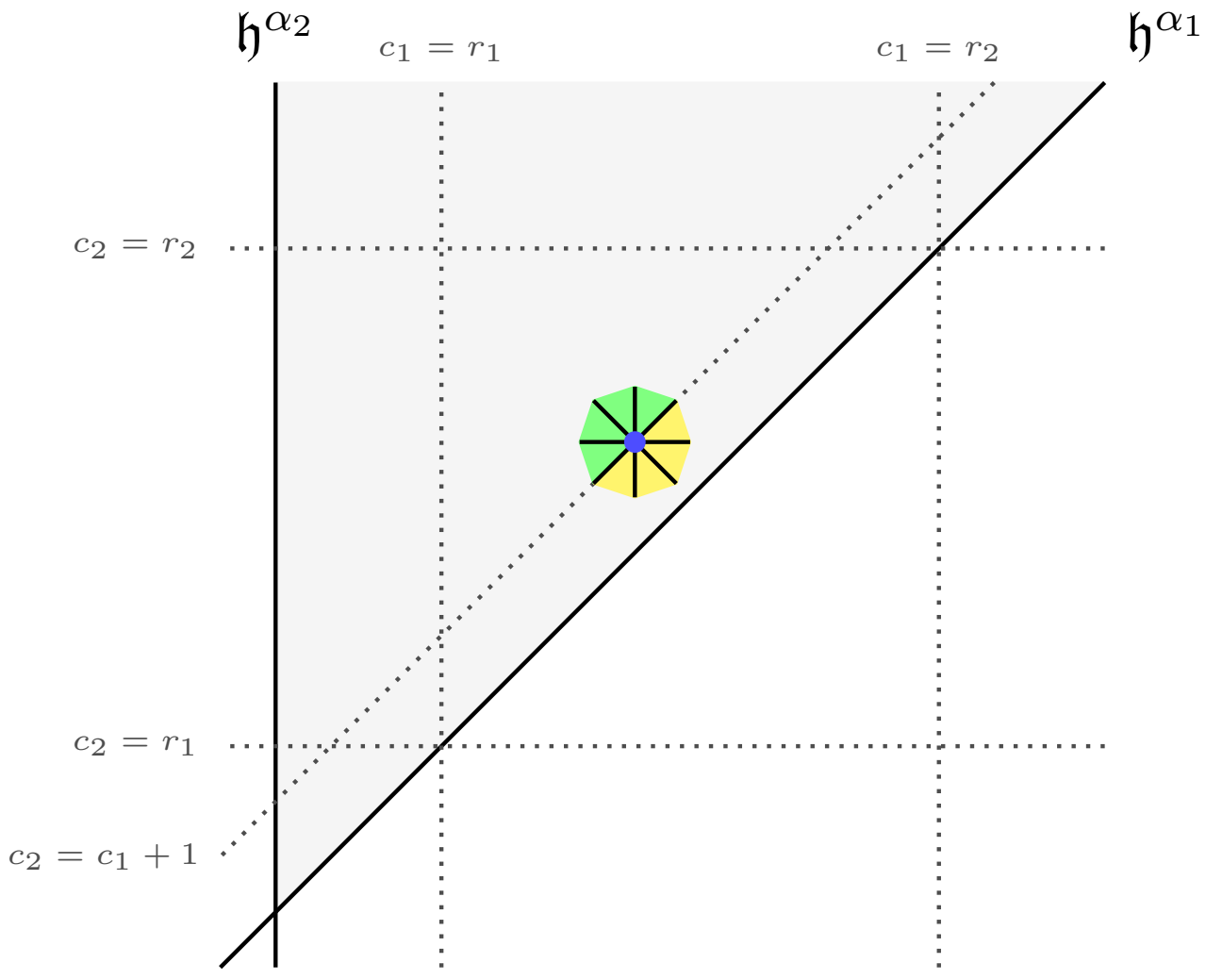

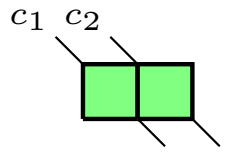

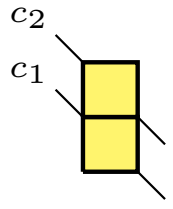

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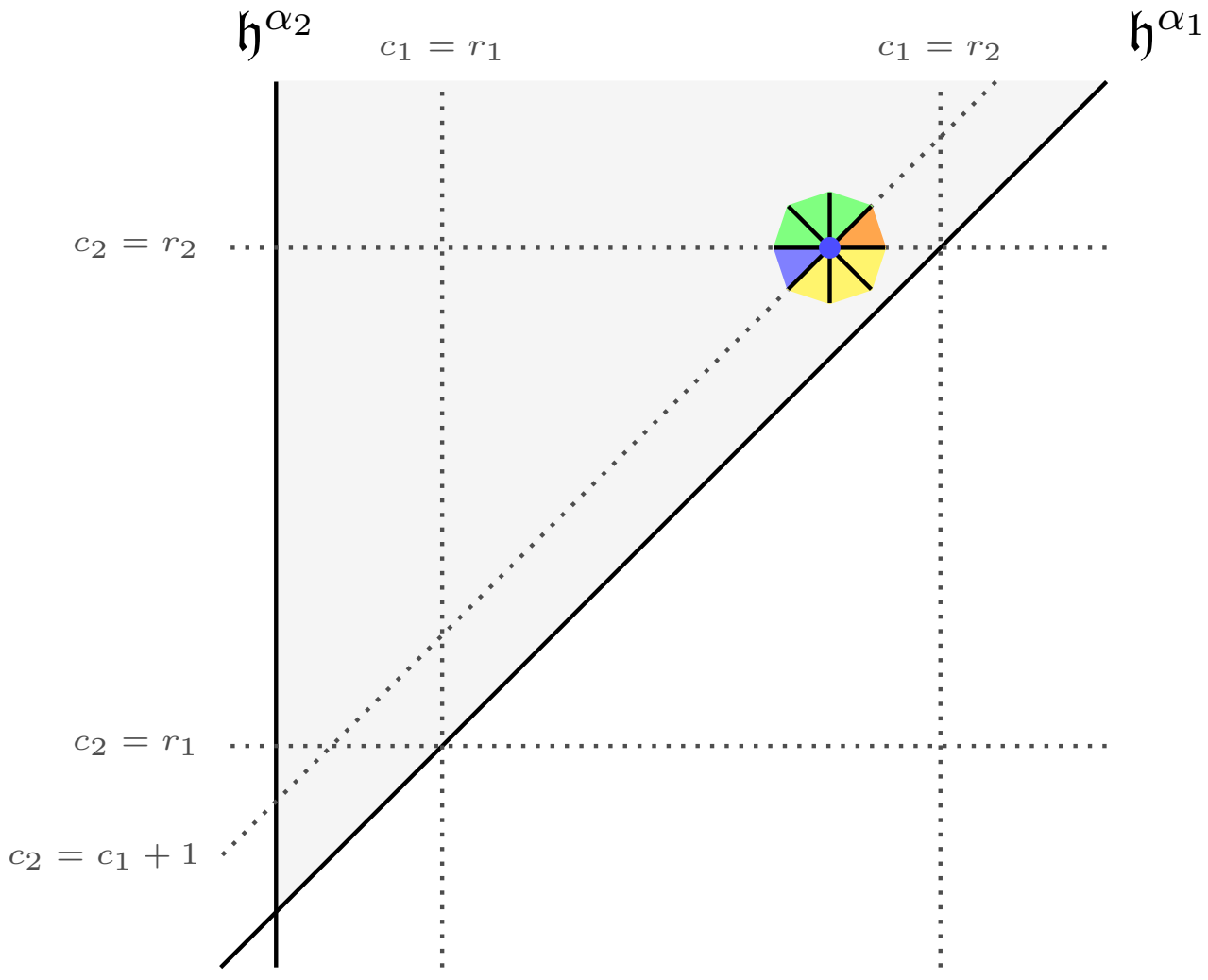

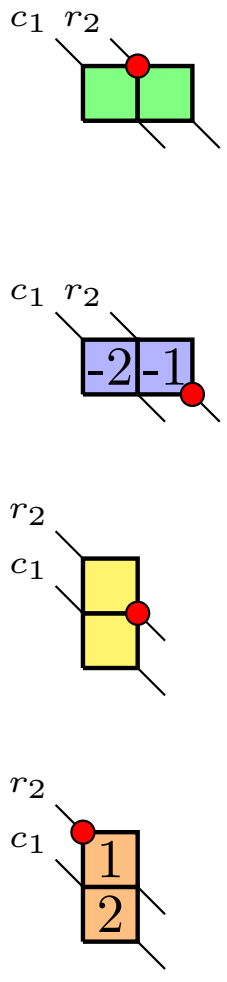

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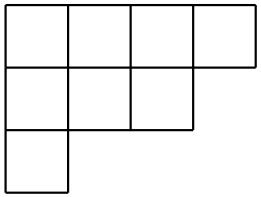
Representation arise in Schur-Weyl duality with certain $U_q\mathfrak{gl}_n$ reps.

Centralizer properties

Let $U = U_q\mathfrak{gl}_n$ be the quantum group for $\mathfrak{gl}_n(\mathbb{C})$. We're interested in certain finite dimensional simple U-modules $L(\lambda)$ indexed by partitions:

$\lambda =$ 

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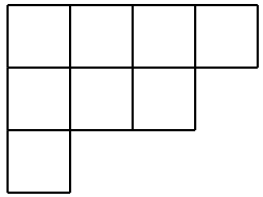
$\lambda =$ 

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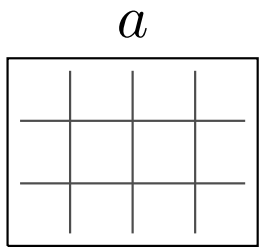
In particular, rectangular partitions:

$(a^c) =$ c 

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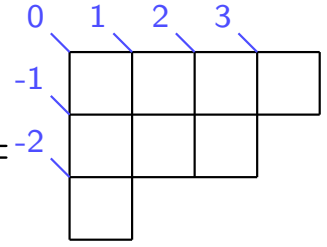
$\lambda =$ 

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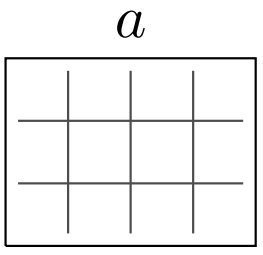
$(a^c) =$ 

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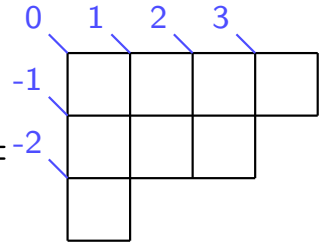
$\lambda =$ 

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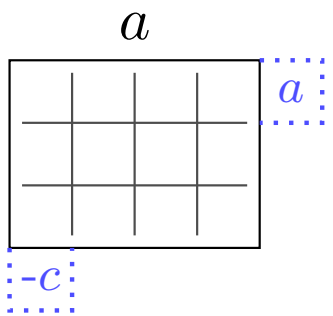
$(a^c) =$ 

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The eigenvalues of T_0 and T_k are controlled by the contents of addable boxes to (a^c) and (b^d).

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Theorem (D.-Ram)

1. *Let $U = U_q\mathfrak{g}$, and let A, B, and C be finite dim'l U-modules. The two-boundary braid group $\mathcal{B}_k$ acts on $B \otimes (C)^{\otimes k} \otimes A$ (via R-matrices) and this action commutes with that of U.*

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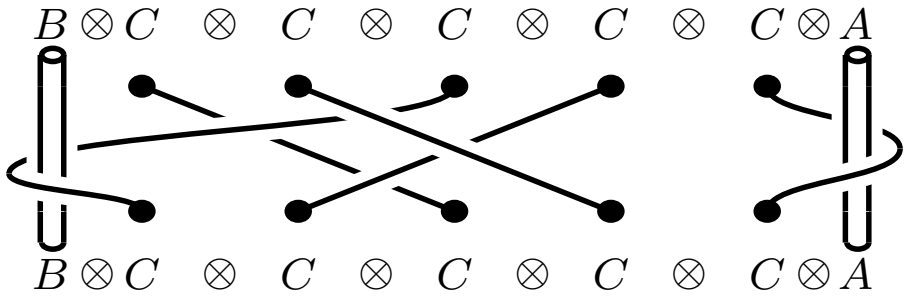

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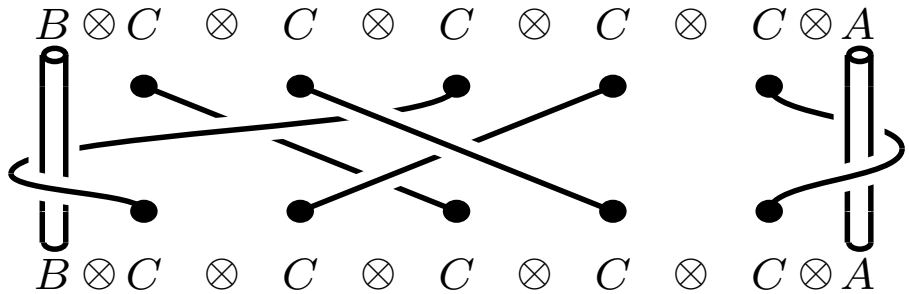
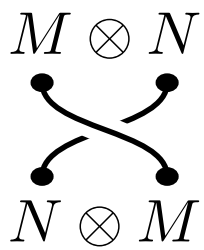


R-matrices: U has an associated invertible element $R = \sum_{\mathcal{R}} R_1 \otimes R_2$ of $U \otimes U$ that gives us a map

$$\check{R}_{MN} \colon M \otimes N \longrightarrow N \otimes M$$



This map acts on a component $L(\lambda)$ of $L(\mu) \otimes L(\square)$ by $q^{2c(\lambda/\mu)}$.

Centralizer properties

Theorem (D.-Ram)

2. *If* $\mathfrak{g} = \mathfrak{gl}_n$, $A = L((a^c))$, $B = L((b^d))$, *and* $C = L(\square)$, *then the action in 1. factors through the quotient by*

$$0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$

where $t = q^2$, $t_0 = t^{\frac{1}{2}(b+d)}$, *and* $t_k = t^{\frac{1}{2}(a+c)}$.

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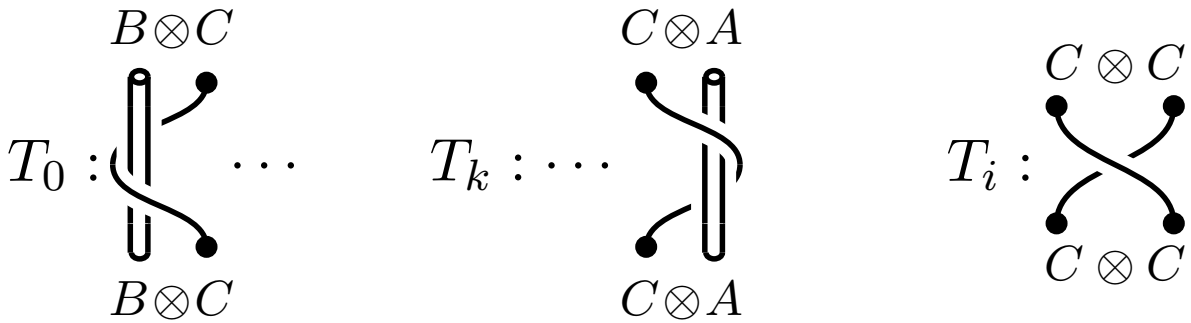
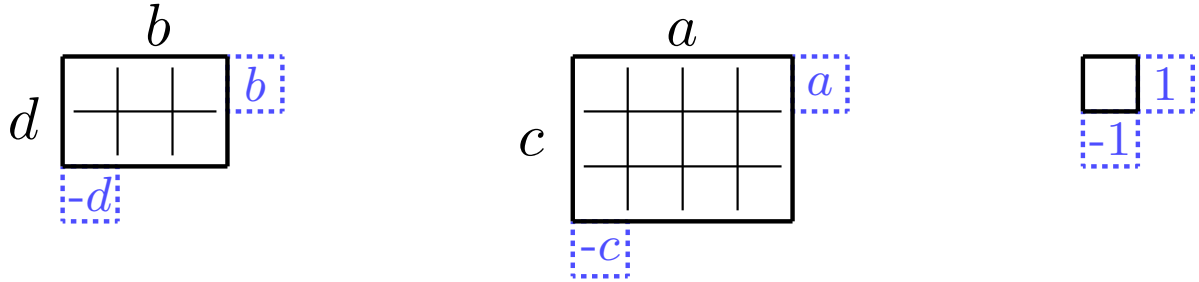
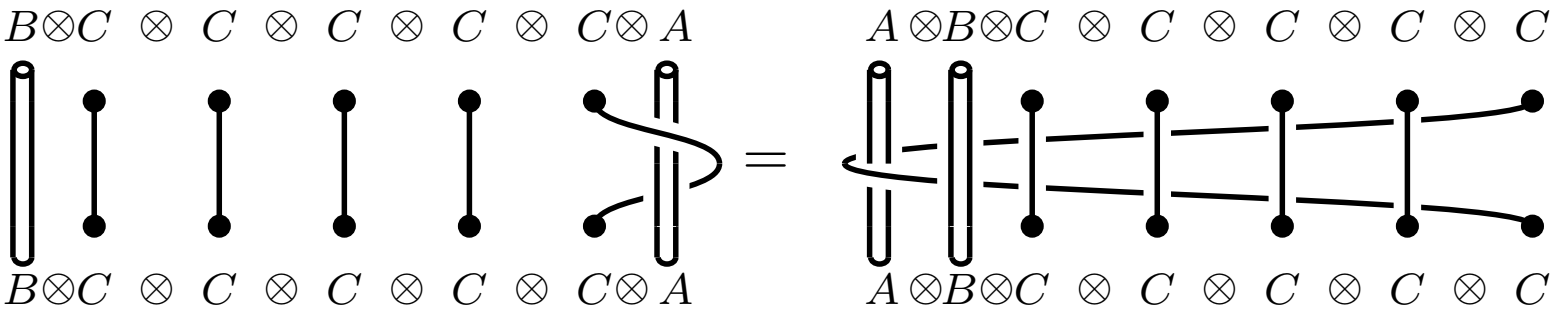




Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$

Move the right pole to the left:



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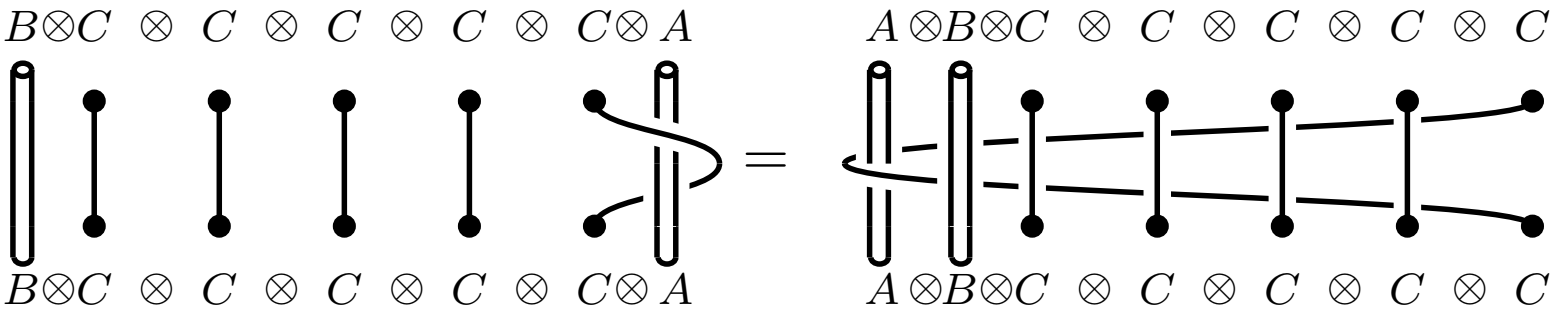
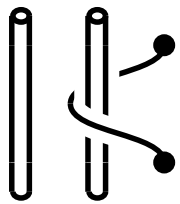
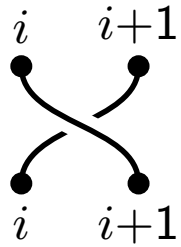
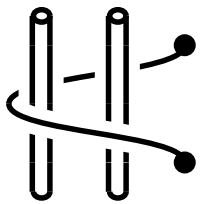
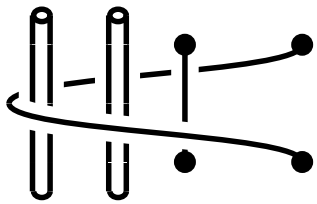


New favorite generators:

$T_0 =$ 

$T_i =$ 

and $Y_1 =$  .

Let $Y_2 = T_1 Y_1 T_1 =$  .

Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$

Products of rectangles:

$$L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

where Λ is the following set of partitions:

Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$

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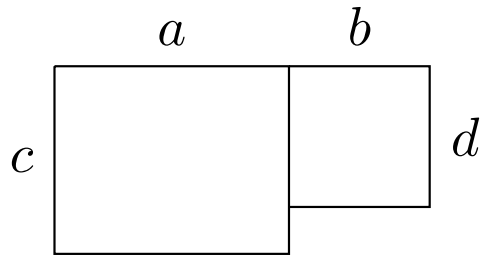

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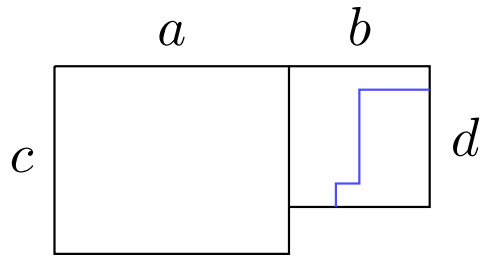

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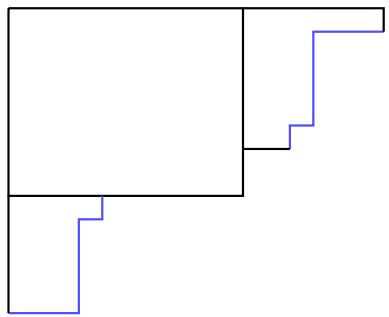

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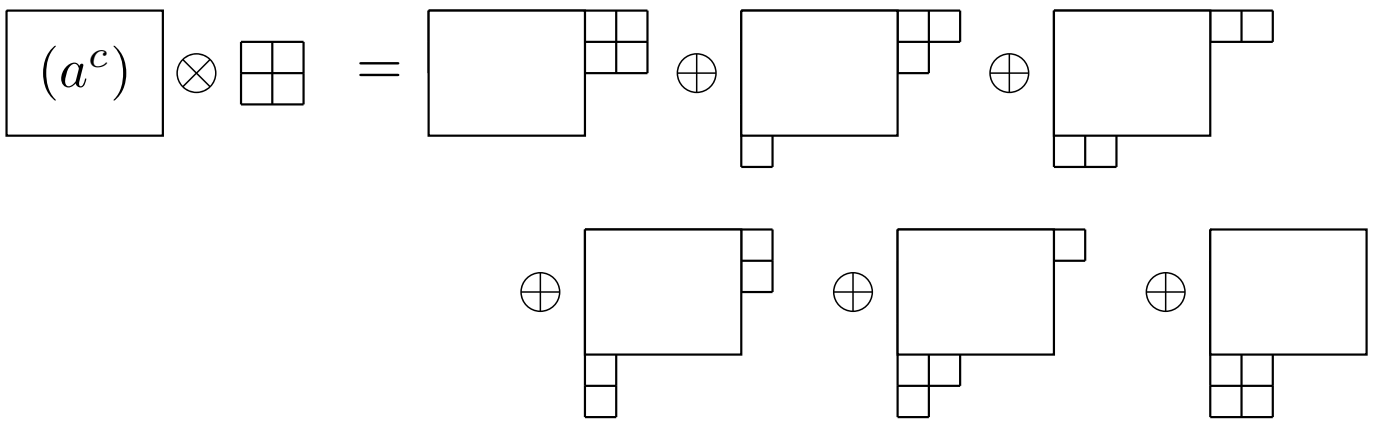
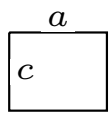
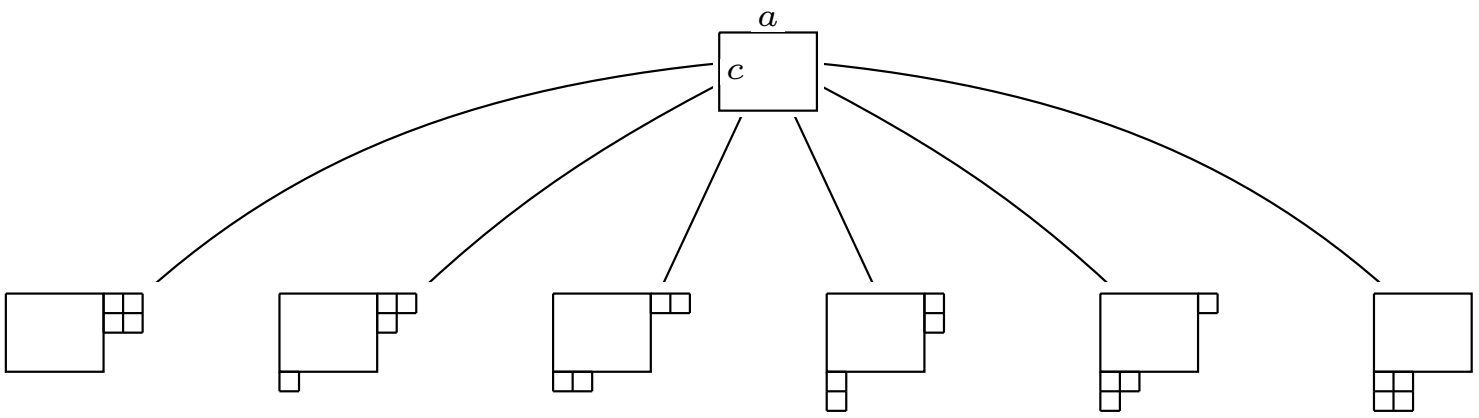


Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$



$k = 0$

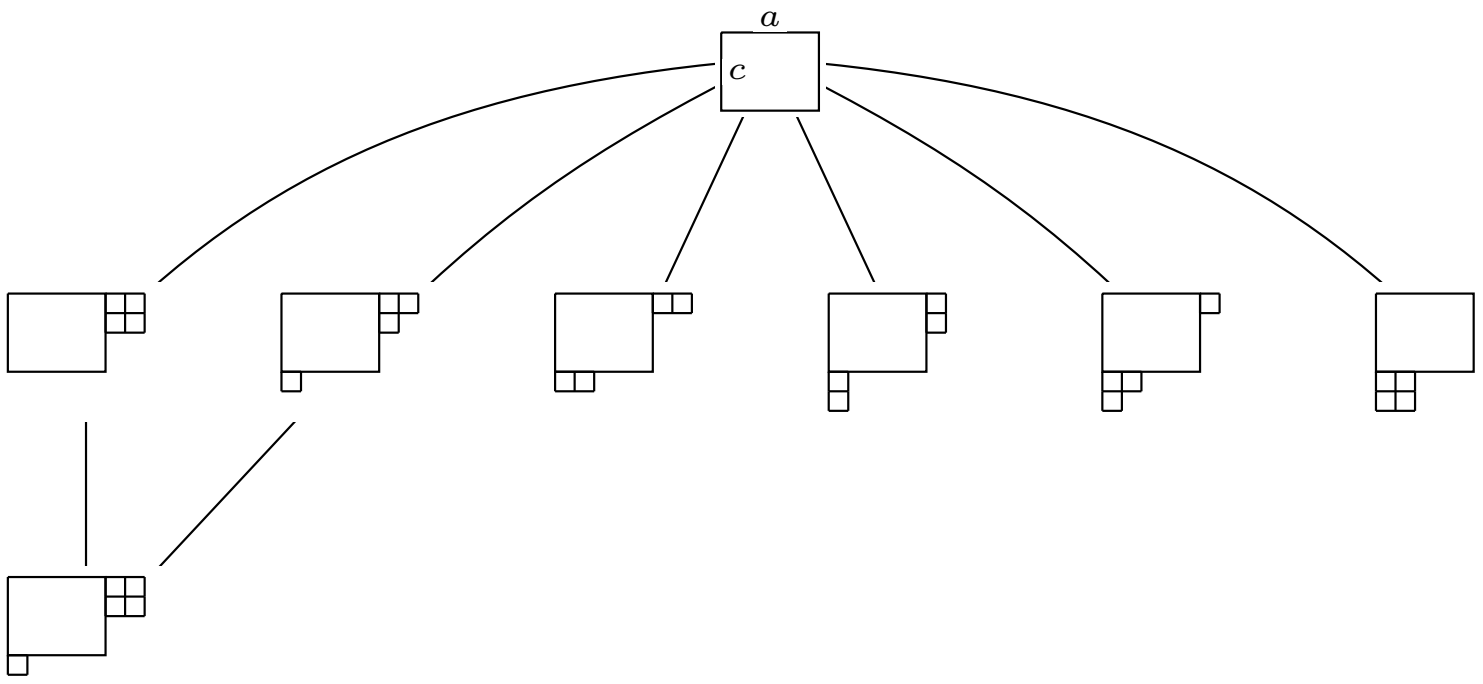
Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$



$k = 0$

$k = 1$

Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$



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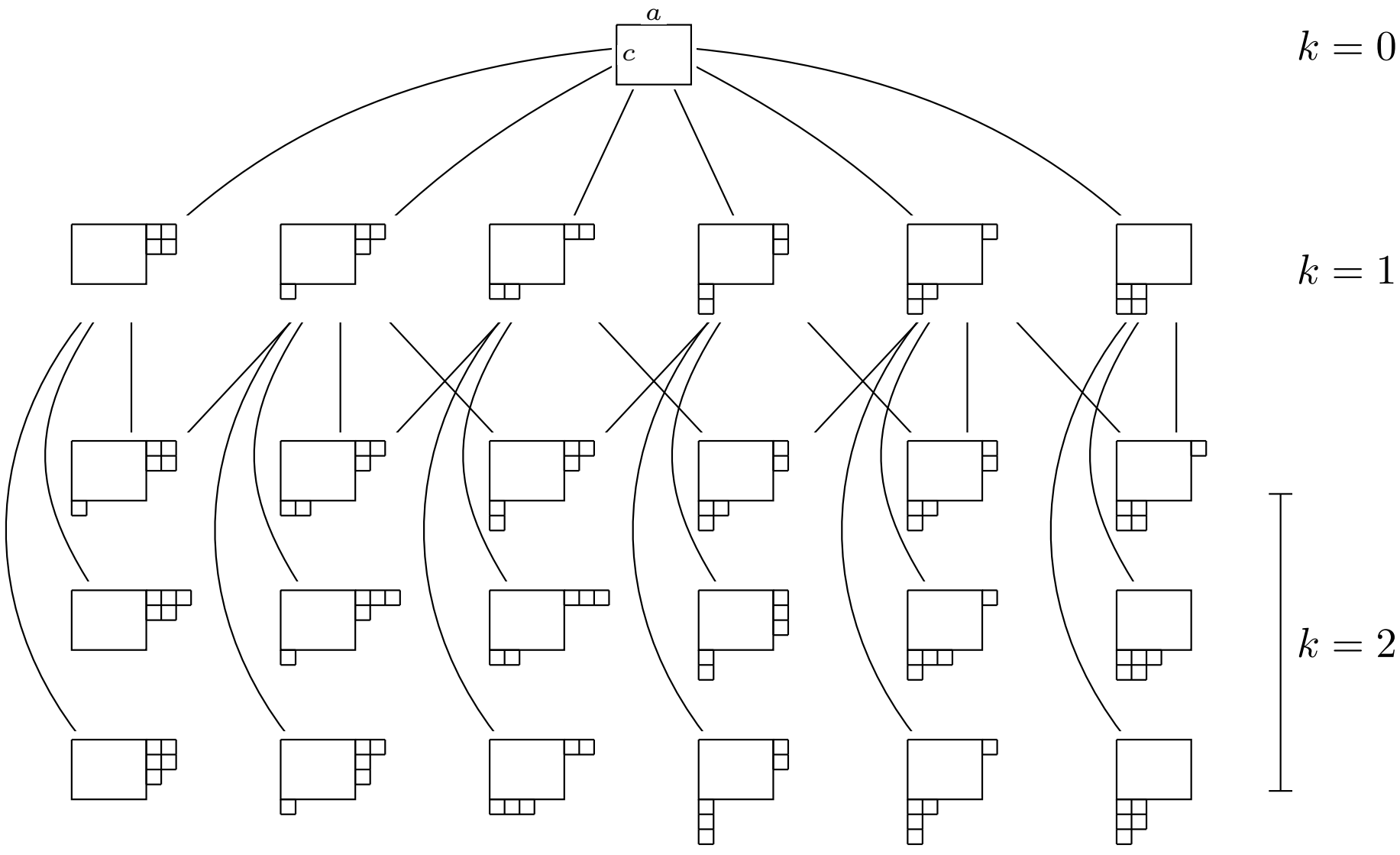
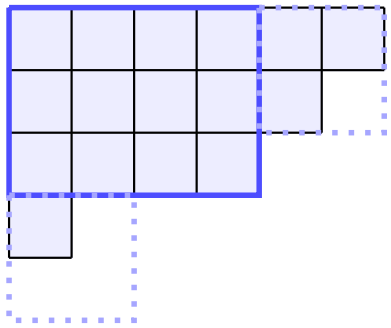
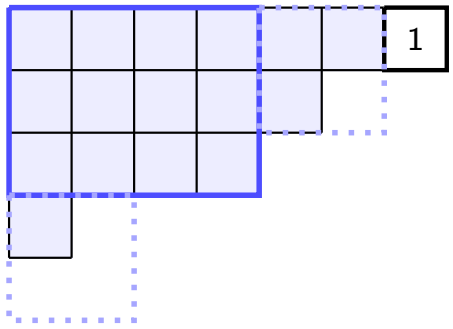


$$L\left(\begin{array}{|c|c|c|c|}\hline&&&\\\hline&&&\\\hline&&&\\\hline\end{array}\right)\otimes L\left(\begin{array}{|c|c|}\hline&\\\hline&\\\hline\end{array}\right)$$



$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2}\right) \otimes L\left(\ydiagram{1}\right)$$



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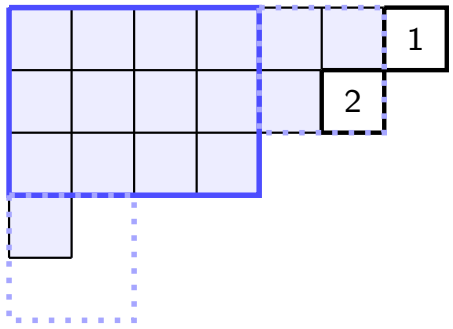
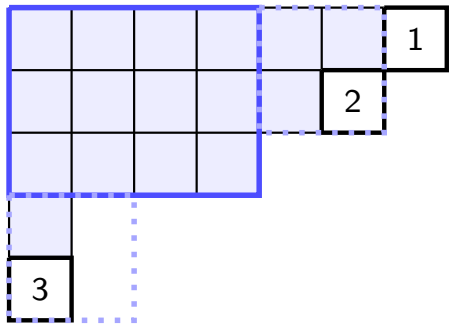
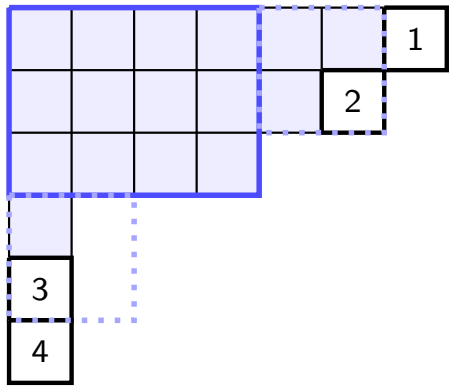


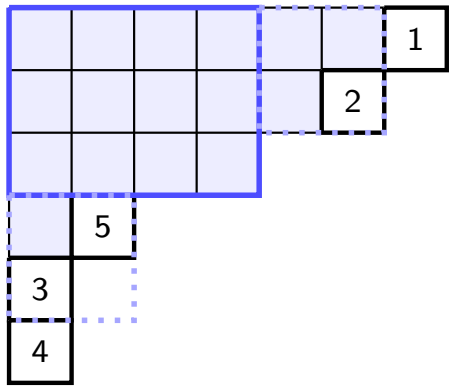
$$L\left(\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right)$$



$$L\left(\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right)$$

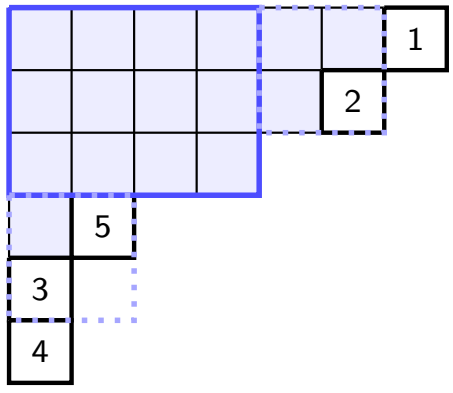


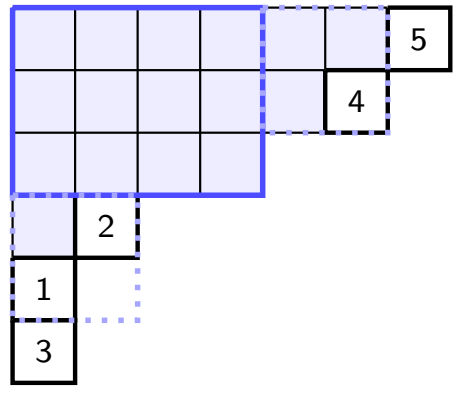
$$L\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|}\hline \ \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|}\hline \ \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|}\hline \ \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|}\hline \ \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|}\hline \ \\ \hline \end{array}\right)$$



$(*)$ H_k representations in tensor space are labeled by certain partitions λ.

$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2,2}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right)$$





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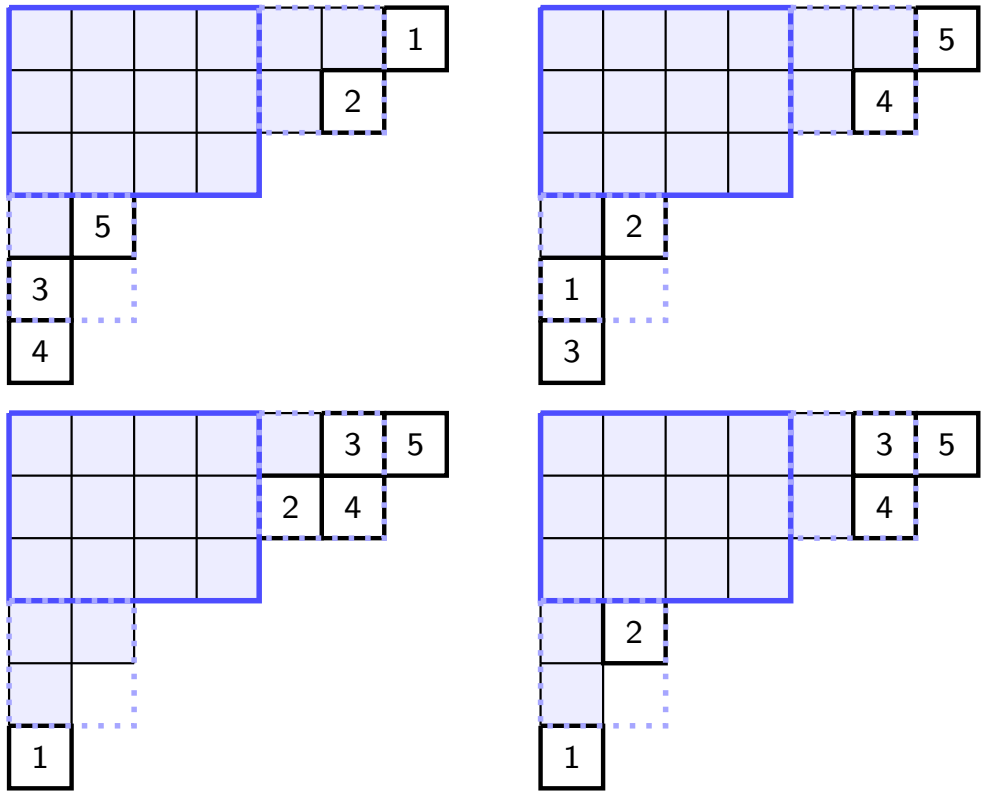

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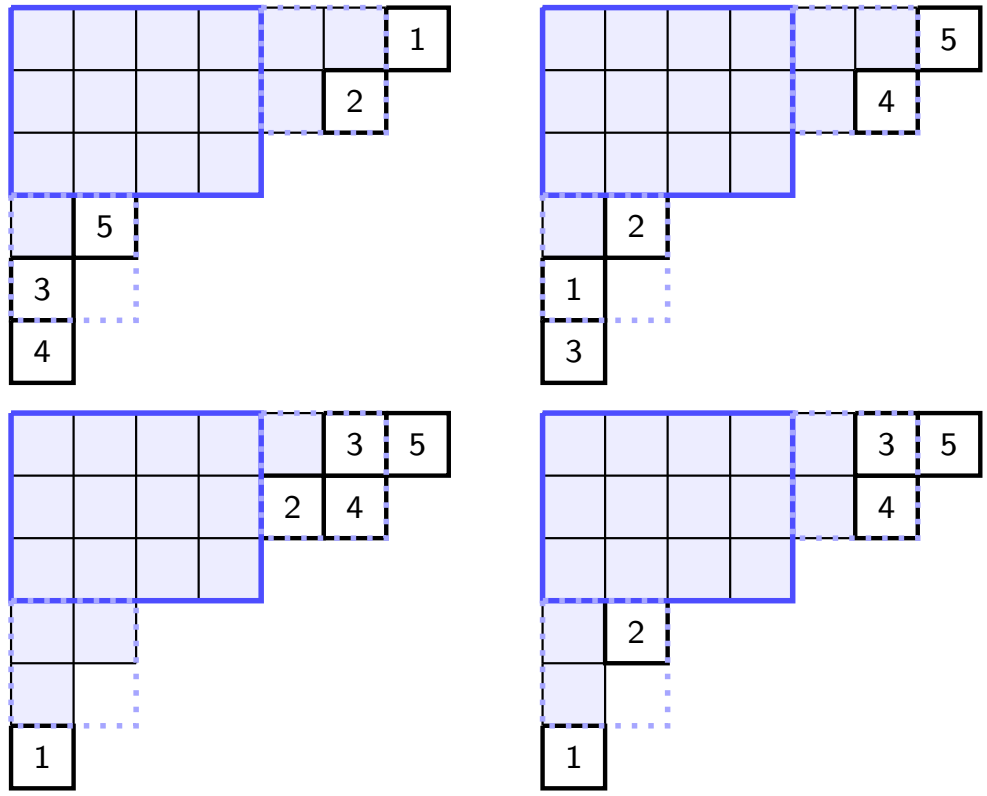
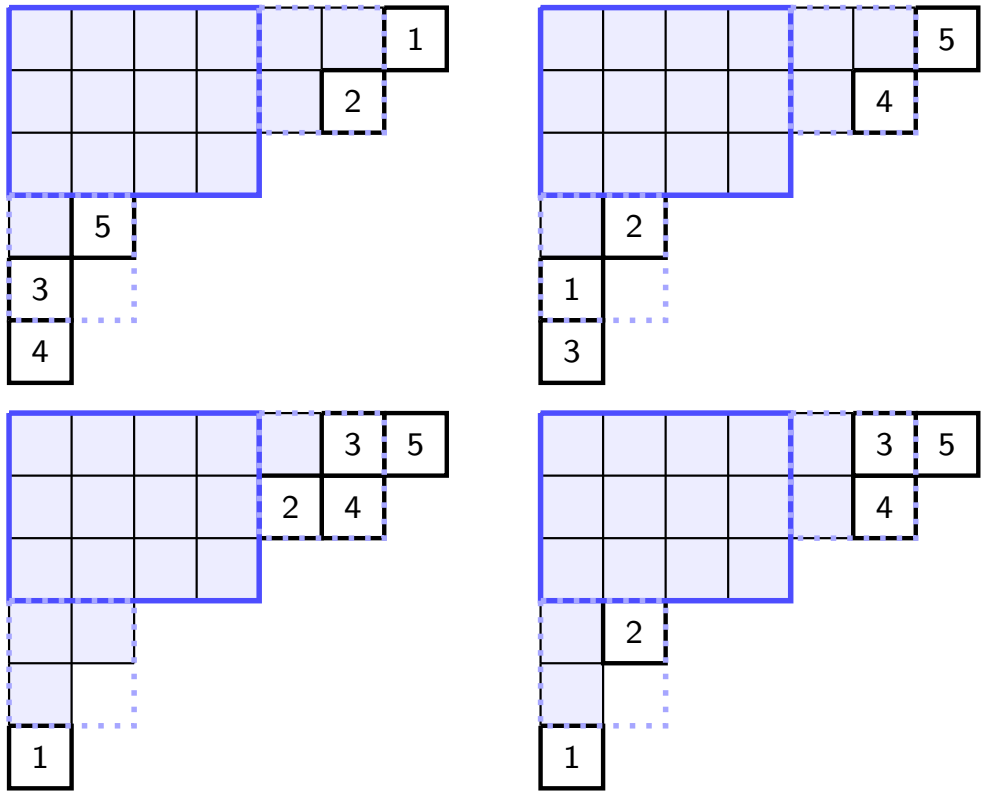


(∗) H_k representations in tensor space are labeled by certain partitions λ.

(∗) Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.

$$L\left(\ydiagram{4,4,4}\right) \otimes L\left(\ydiagram{2,2,2}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right) \otimes L\left(\ydiagram{1}\right)$$



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(∗) Calibrated

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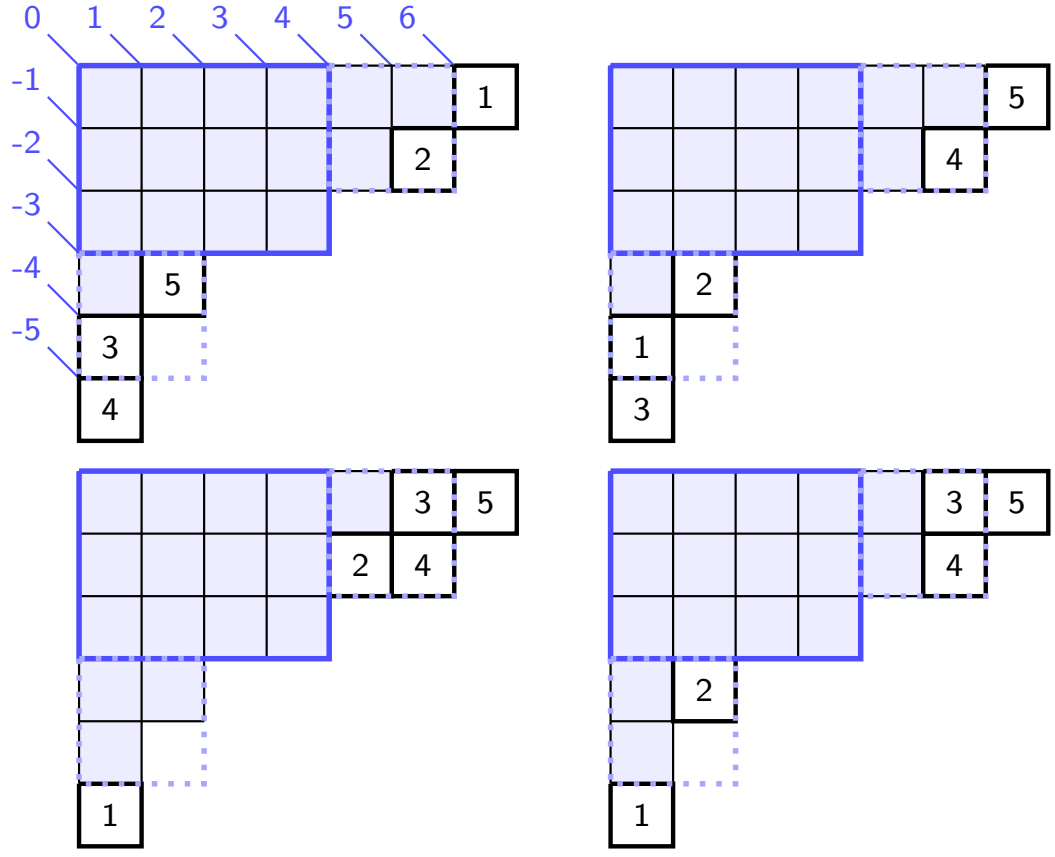

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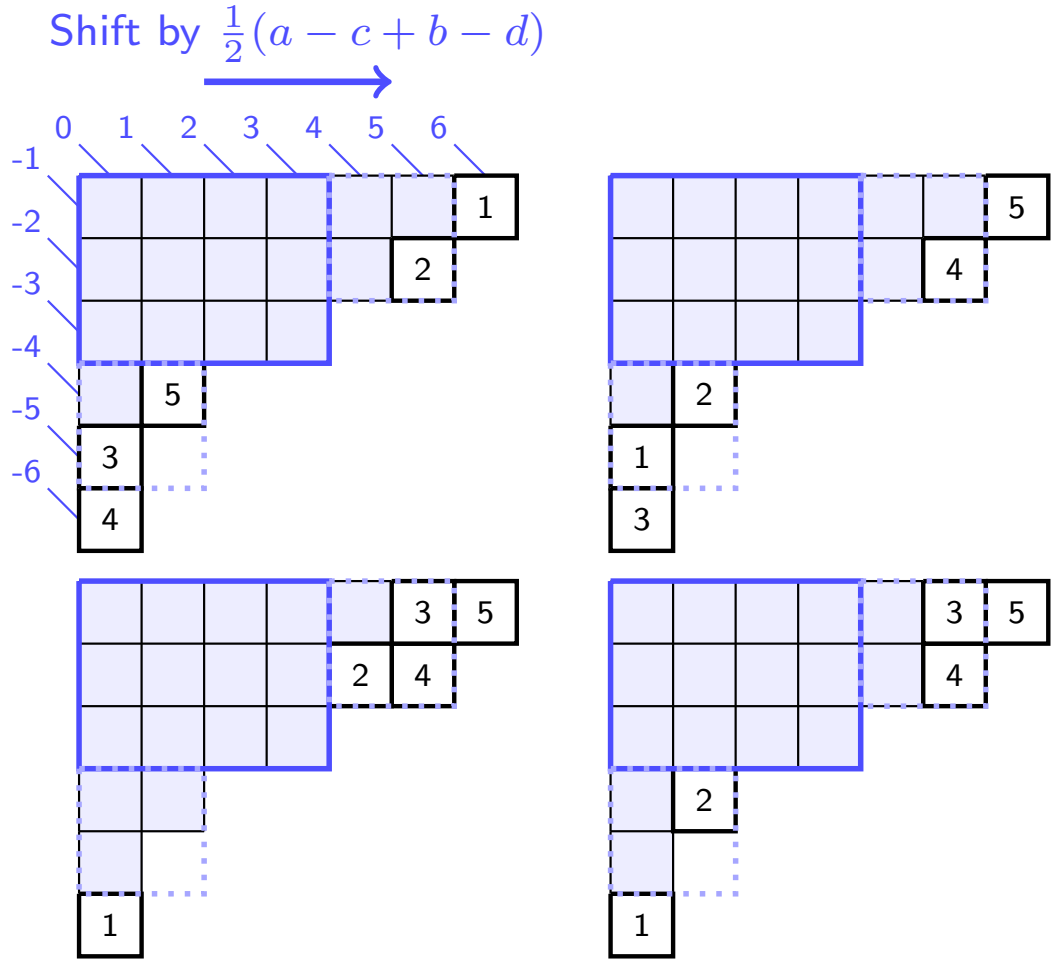

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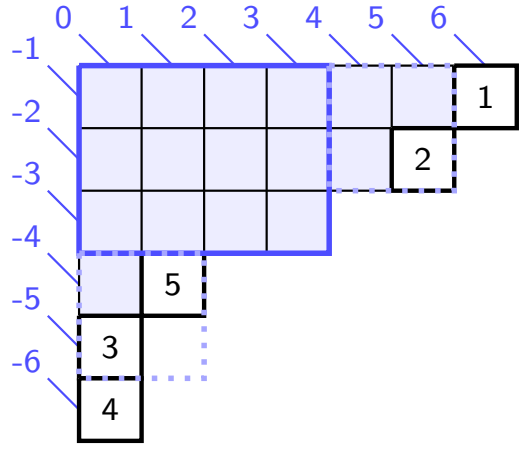
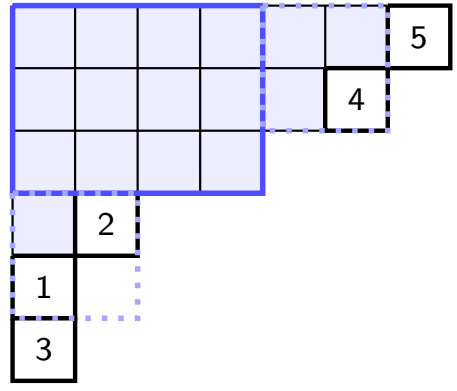
Shift by $\frac{1}{2}(a-c+b-d)$

$Y_1 \mapsto t^{5.5}$
$Y_2 \mapsto t^{3.5}$
$Y_3 \mapsto t^{-4.5}$
$Y_4 \mapsto t^{-5.5}$
$Y_5 \mapsto t^{-2.5}$





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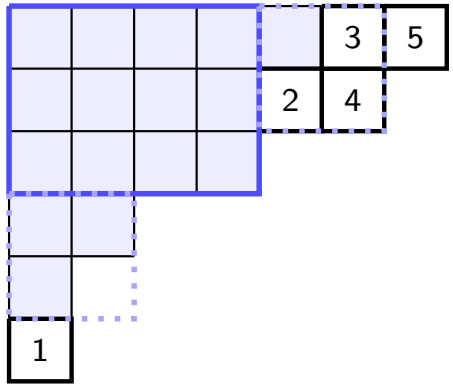

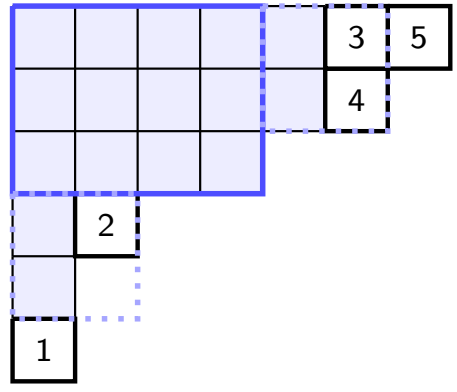

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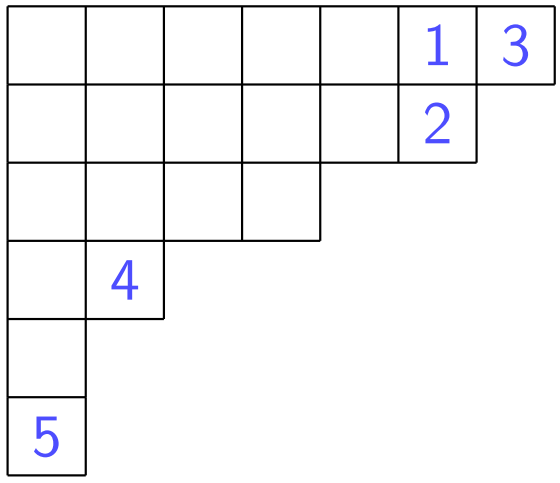
(∗) H_k representations in tensor space are labeled by certain partitions λ.

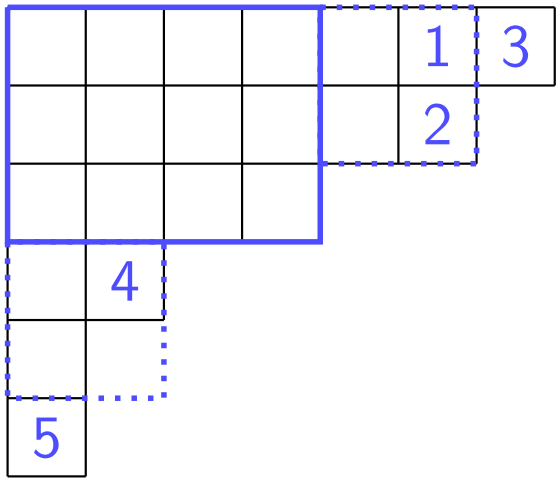
(∗) Basis labeled by tableaux from *some* partition μ in $(a^c) \otimes (b^d)$ to λ.

(∗) Calibrated: Y_i acts by t to the shifted content of box_i.

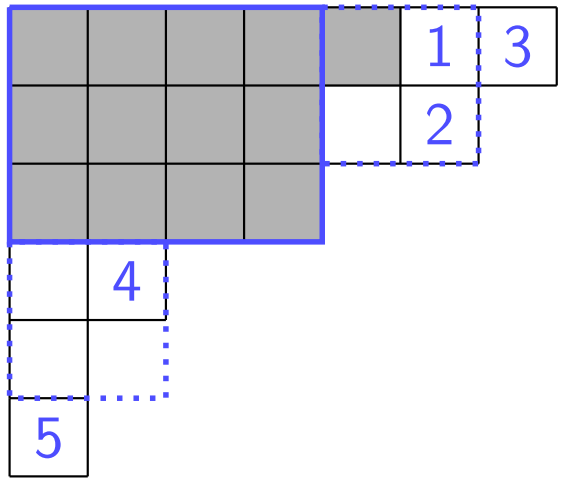
From {partitions in tensor space} to {box arrangements}



From {partitions in tensor space} to {box arrangements}

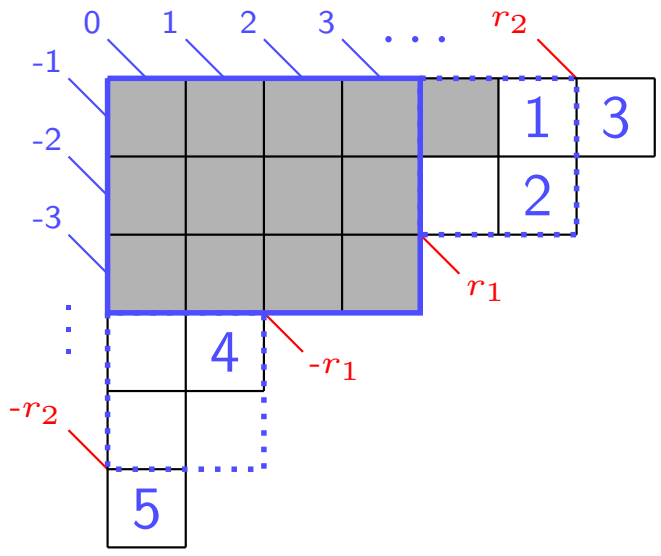


From {partitions in tensor space} to {box arrangements}



$\square$ = boxes that must appear in the partition at level 0.

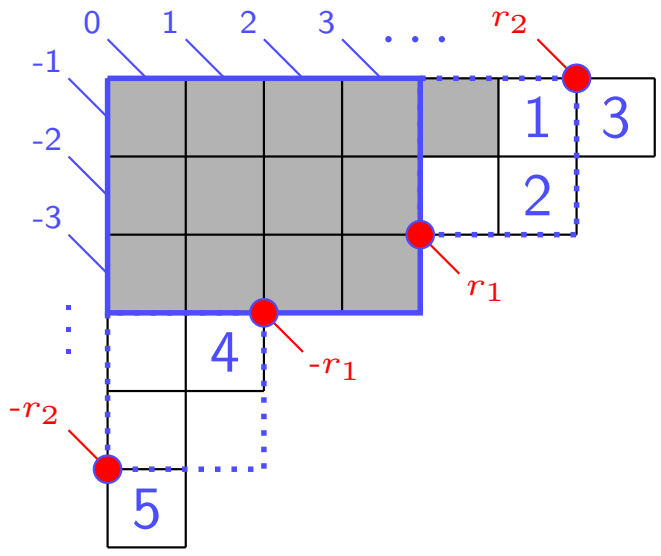
From {partitions in tensor space} to {box arrangements}



$\square$ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5},\ \gamma(Y_2) = t^{3.5},\ \gamma(Y_3) = t^{r_2},\ \gamma(Y_4) = t^{-2.5},\ \gamma(Y_5) = t^{-r_2}.$$

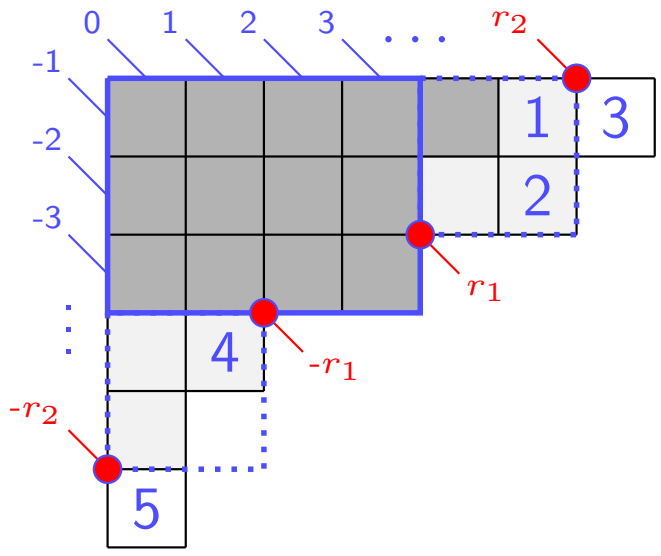
From {partitions in tensor space} to {box arrangements}



$\square$ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5},\ \gamma(Y_2) = t^{3.5},\ \gamma(Y_3) = t^{r_2},\ \gamma(Y_4) = t^{-2.5},\ \gamma(Y_5) = t^{-r_2}.$$

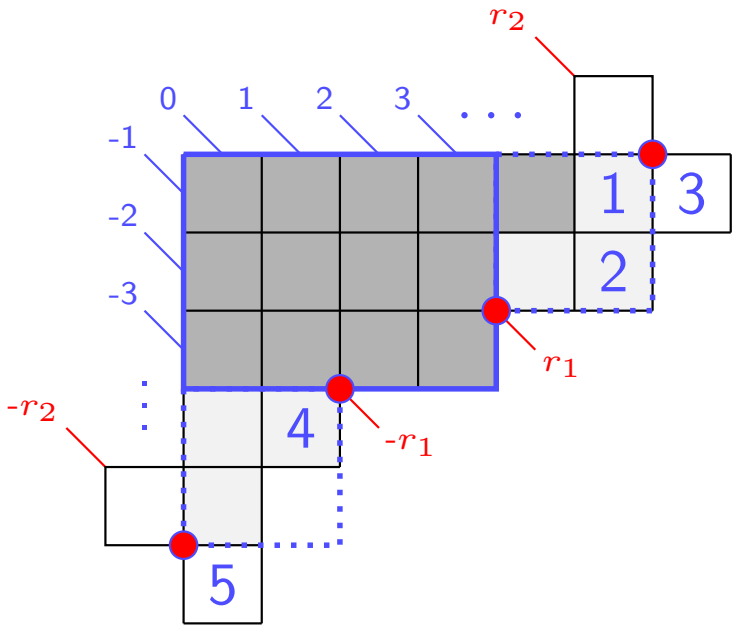
From {partitions in tensor space} to {box arrangements}



$\square$ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5},\ \gamma(Y_2) = t^{3.5},\ \gamma(Y_3) = t^{r_2},\ \gamma(Y_4) = t^{-2.5},\ \gamma(Y_5) = t^{-r_2}.$$

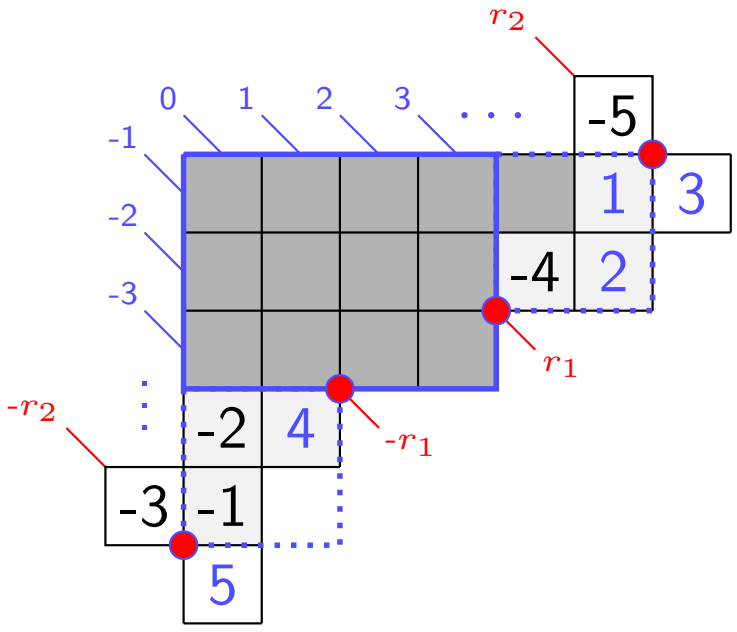
From {partitions in tensor space} to {box arrangements}



$\square$ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5},\ \gamma(Y_2) = t^{3.5},\ \gamma(Y_3) = t^{r_2},\ \gamma(Y_4) = t^{-2.5},\ \gamma(Y_5) = t^{-r_2}.$$

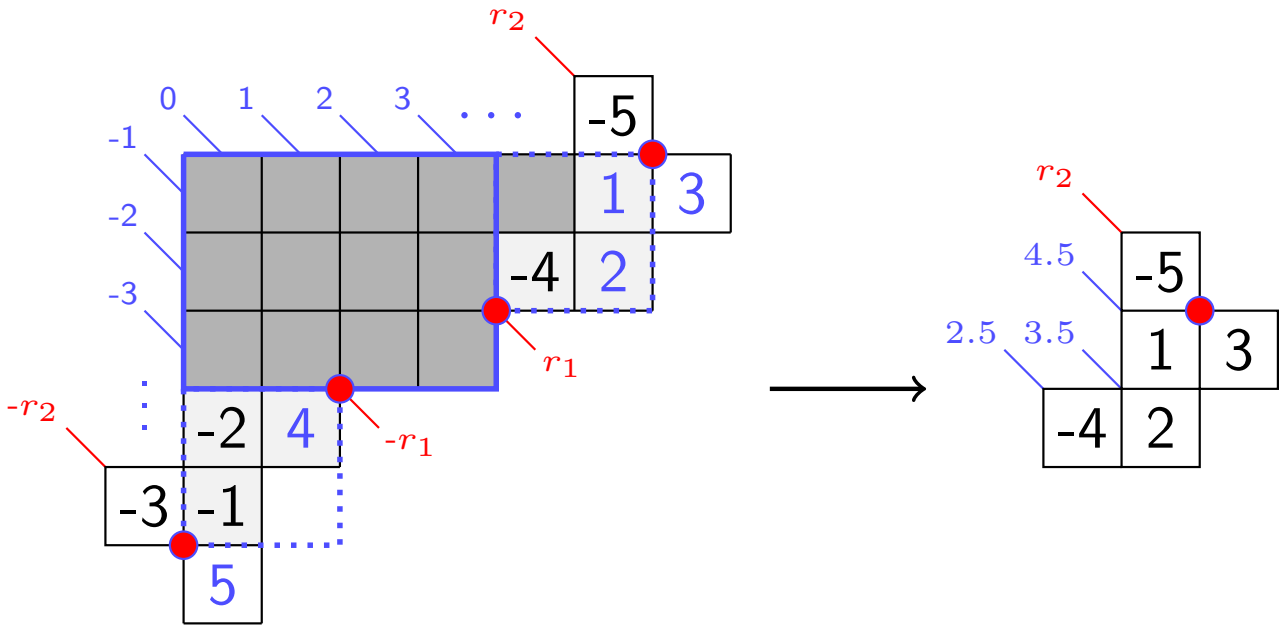
From {partitions in tensor space} to {box arrangements}



$\square$ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5},\ \gamma(Y_2) = t^{3.5},\ \gamma(Y_3) = t^{r_2},\ \gamma(Y_4) = t^{-2.5},\ \gamma(Y_5) = t^{-r_2}.$$

From {partitions in tensor space} to {box arrangements}



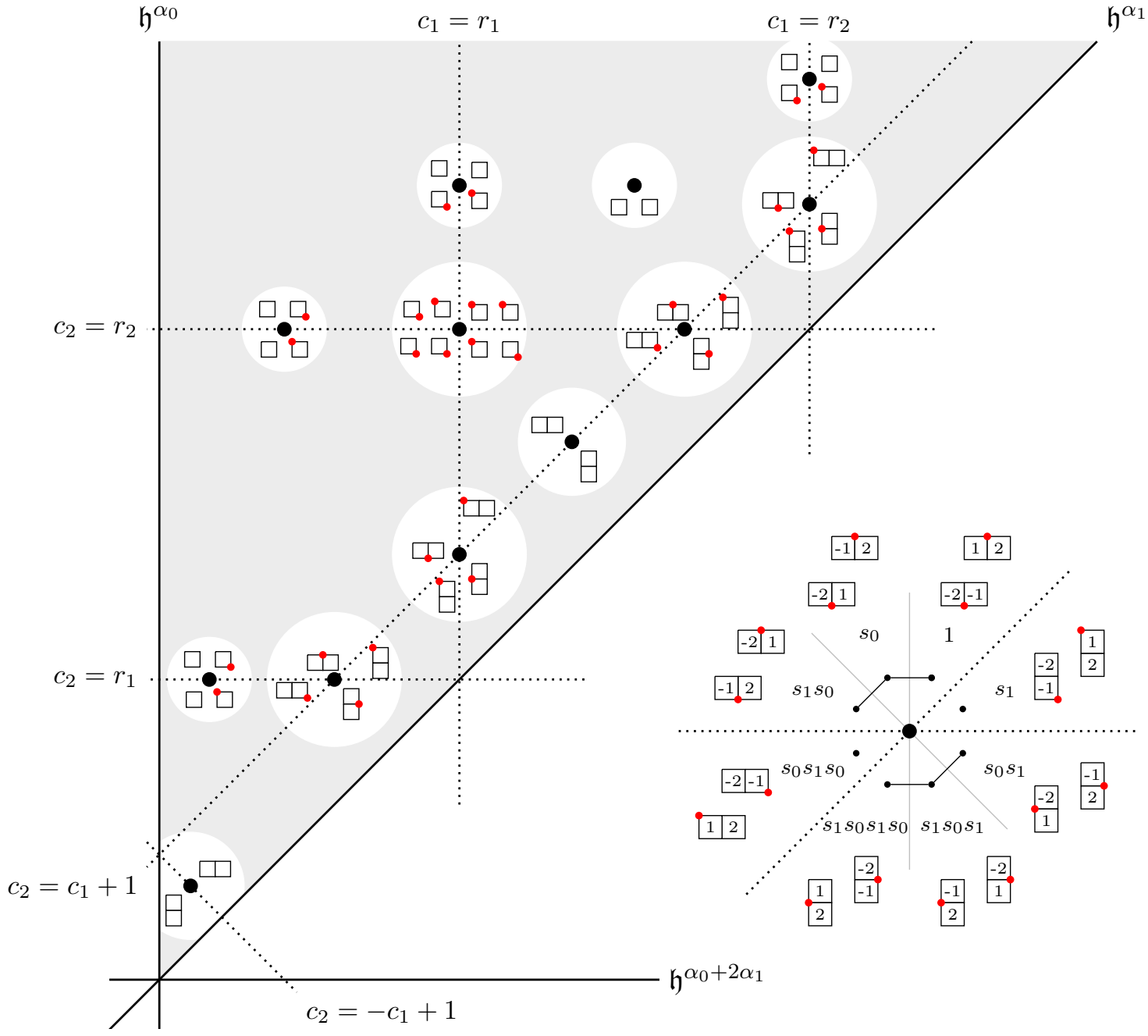
$\square$ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5},\ \gamma(Y_2) = t^{3.5},\ \gamma(Y_3) = t^{r_2},\ \gamma(Y_4) = t^{-2.5},\ \gamma(Y_5) = t^{-r_2}.$$

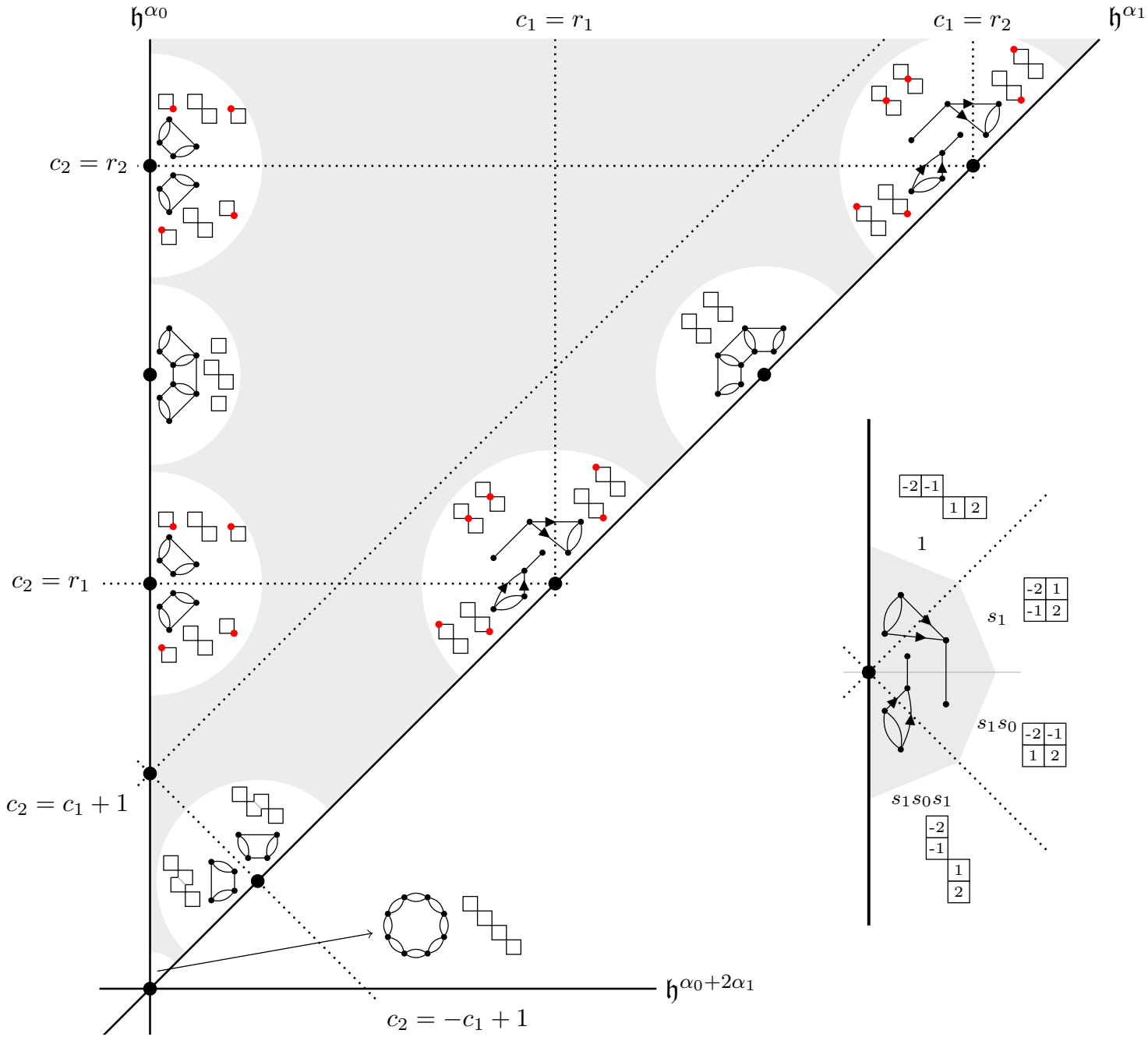
versus

$$\gamma(Y_1) = t^{4.5},\ \gamma(Y_2) = t^{3.5},\ \gamma(Y_3) = t^{r_2},\ \gamma(Y_4^{-1}) = t^{2.5},\ \gamma(Y_5^{-1}) = t^{r_2}.$$

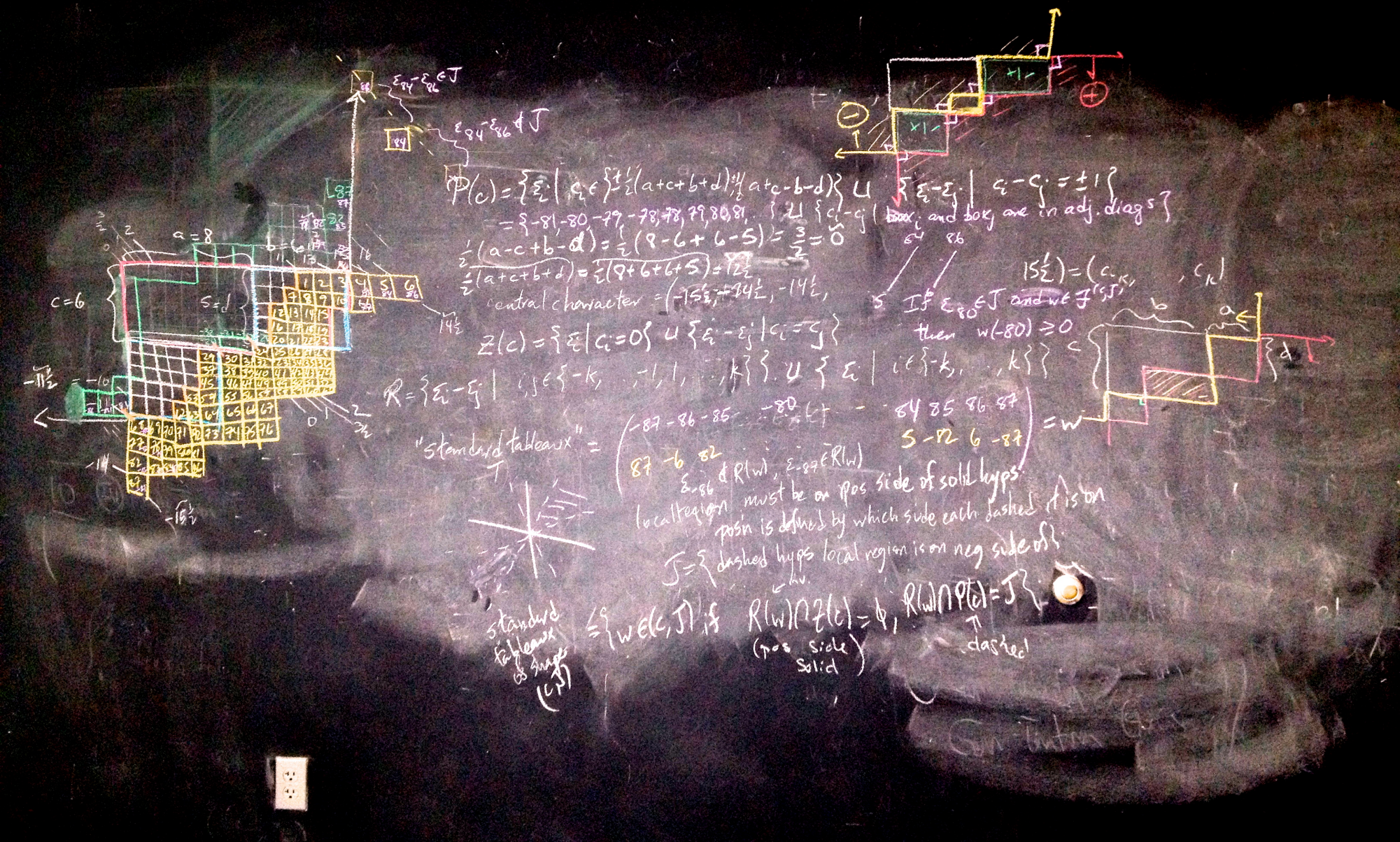
Result: All calibrated irred. representations correspond to some interior point. All such representations can be found in some tensor space.



Challenge: Can we use Verma modules to get the non-calibrated irreps?



Thanks!



central character
"standard tableaux"
localregion must be on pos side of solid hyps
posn is defined by which side each dashed it is on
J = {dashed hyps local region is on neg side of}
standard tableaux of shape (c,J)
(pos side solid)
dashed