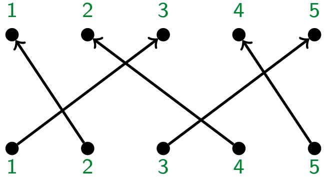
Tensor spaces, braid groups, and some remarkable quotients.

Zajj Daugherty

January 30, 2014

Motivating example: Schur-Weyl Duality

The **symmetric group** S_k (permutations) as diagrams:



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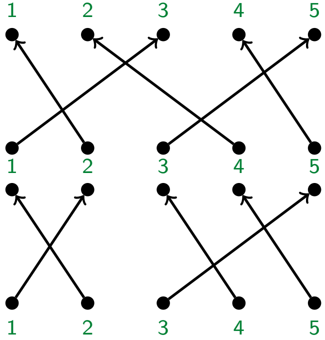

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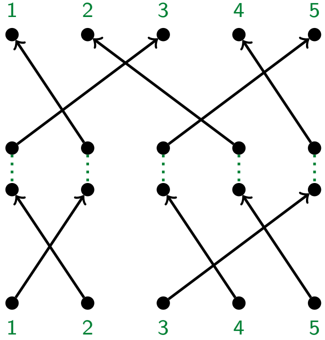

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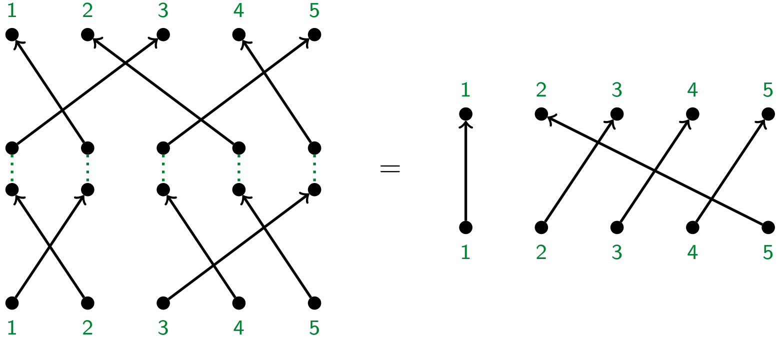

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$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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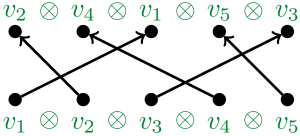

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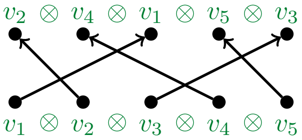
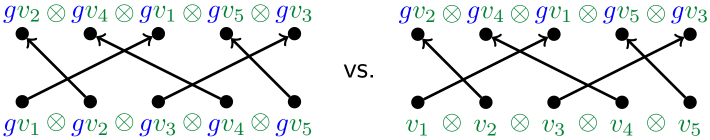


These actions commute!



Motivating example: Schur-Weyl Duality

Schur (1901): S_k and GL_n have commuting actions on $(\mathbb{C}^n)^{\otimes k}$.

Even better,

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\substack{\text{(all linear maps that}\\ \text{commute with } \mathrm{GL}_n\text{)}}} = \underbrace{\pi(\mathbb{C}S_k)}_{\substack{\text{(img of } S_k\\ \text{action)}}} \quad \text{and} \quad \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\substack{\text{(img of } \mathrm{GL}_n\\ \text{action)}}}.$$

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Why this is exciting:

The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

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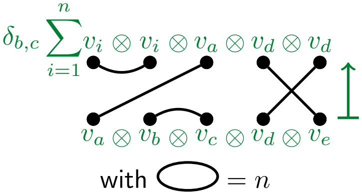
For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \left(G^{\yng(3)} \otimes S^{\yng(3)}\right) \oplus \left(G^{\yng(2,1)} \otimes S^{\yng(2,1)}\right) \oplus \left(G^{\yng(1,1,1)} \otimes S^{\yng(1,1,1)}\right)$$

More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize the **Brauer algebra**:



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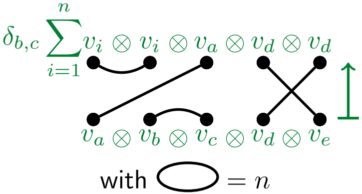

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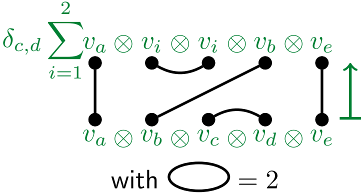

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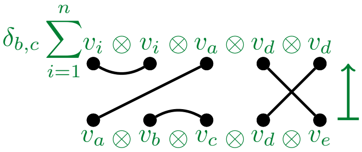

with $\bigcirc = n$

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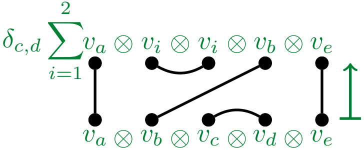

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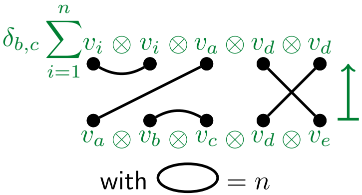

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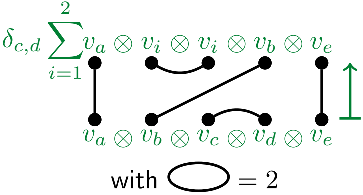

Big idea:
Diagrams encoding maps $V^{\otimes k} \to V^{\otimes k}$ that commute with the action of some classical algebra.

Quantum groups and braids

Let $\mathfrak{g}$ be a Lie algebra, and fix $q \in \mathbb{C}$.

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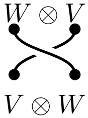
$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\check{\mathcal{R}}_{VW} \colon V \otimes W \longrightarrow W \otimes V$$



that (1) satisfies braid relations, and
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for any $\mathcal{U}$-module V.

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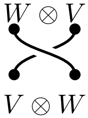

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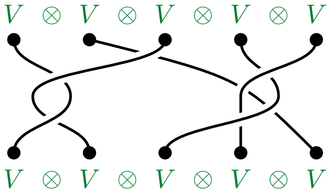
The braid group shares a commuting action with $\mathcal{U}$ on $V^{\otimes k}$:



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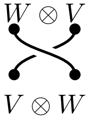

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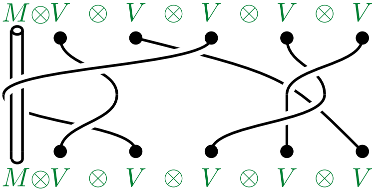
The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k}$:



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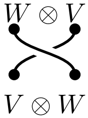

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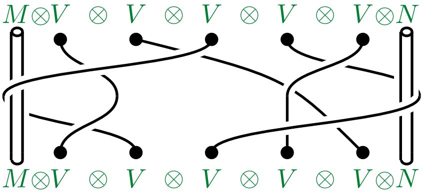
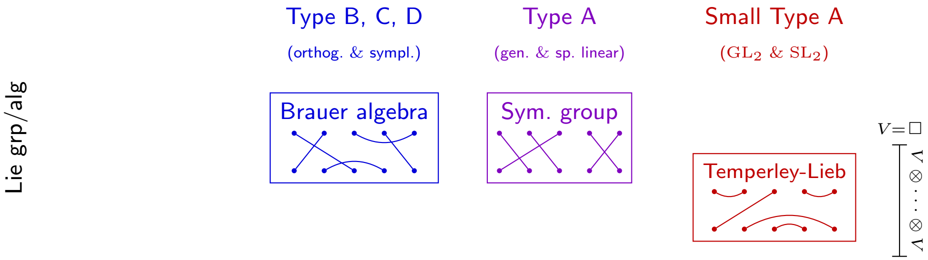
The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k} \otimes N$:




Lie grp/alg
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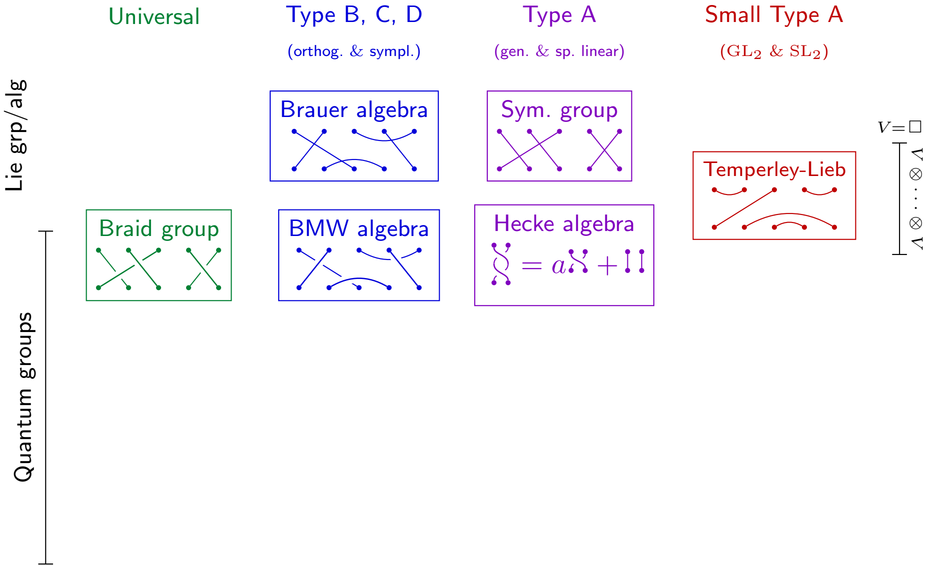
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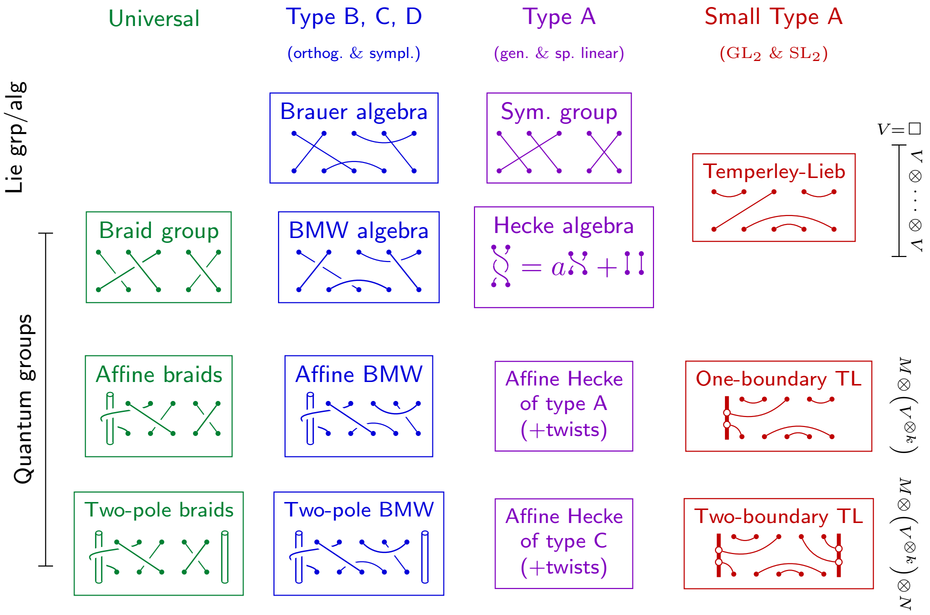

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Affine BMW
Affine Hecke of type A (+twists)
One-boundary TL
Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
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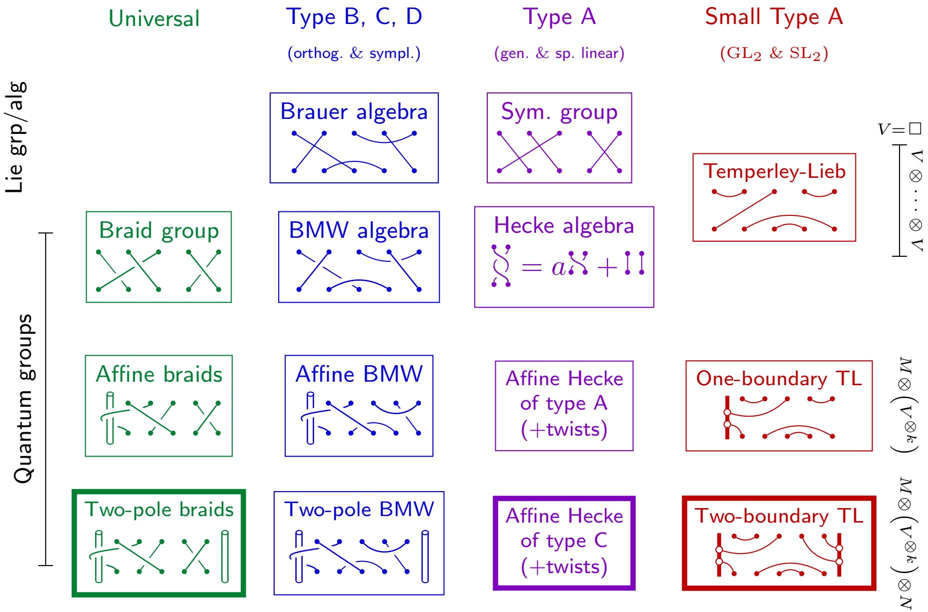
$V \otimes \cdots \otimes V$
$M \otimes (V^{\otimes k})$
$N \otimes (V^{\otimes k}) \otimes M$



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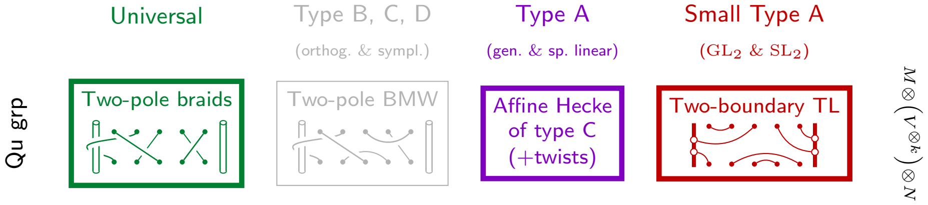

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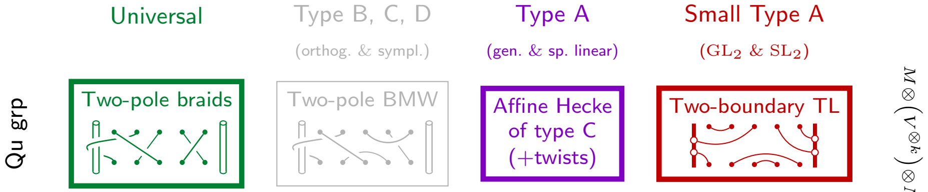

Two boundary algebras:

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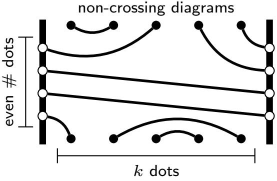

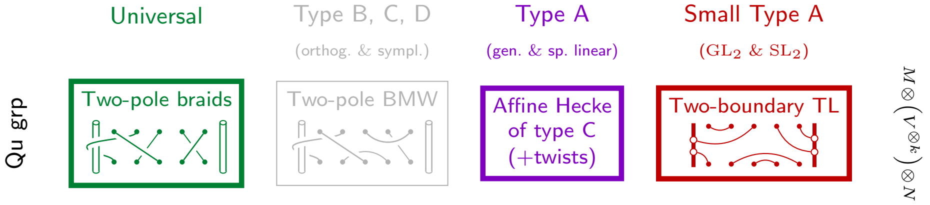

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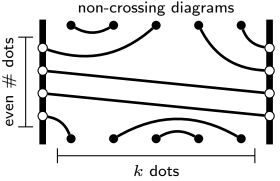

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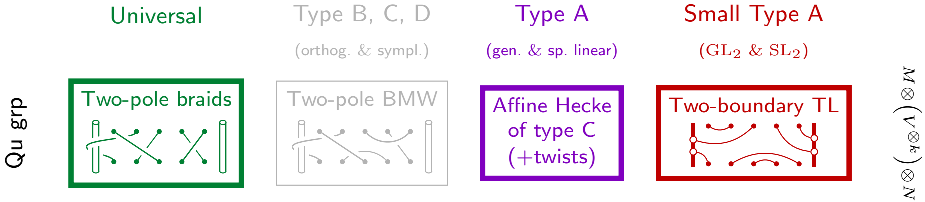

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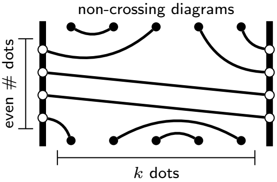

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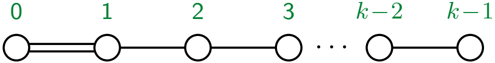
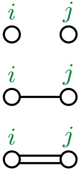
D. (2010): The centralizer of $\mathfrak{gl}_n$ acting on tensor space $M \otimes V^{\otimes k} \otimes N$ displays type C combinatorics for good choices of M, N, and V.

Type C Weyl group and affine Hecke algebra



$m_{i,j} =$

- 2 if
- 3 if
- 4 if



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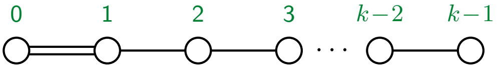

The **Weyl group of type C** is generated by $s_0, \ldots, s_{k-1}$ with relations $s_i^2 = 1$ and

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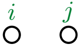
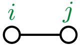
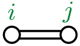
2 if 

3 if 

4 if 

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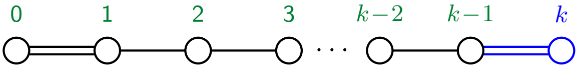

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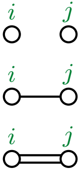
$$\underbrace{s_i s_j \ldots}_{m_{i,j} \text{ factors}} = \underbrace{s_j s_i \ldots}_{m_{i,j} \text{ factors}} \qquad \text{where} \qquad m_{i,j} = \begin{cases} 2 & \text{if} \\ 3 & \text{if} \\ 4 & \text{if} \end{cases}$$

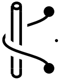
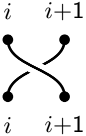


Fix constants a_0, a_k, and $a_1 = \cdots = a_{k-1}$. The **affine Hecke algebra of type C**, H_k, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$T_i^2 = (a_i - a_i^{-1})T_i + 1, \qquad \underbrace{T_i T_j \ldots}_{m_{i,j} \text{ factors}} = \underbrace{T_j T_i \ldots}_{m_{i,j} \text{ factors}}.$$

Why the two-boundary braid group is type C

The **two-boundary (two-pole) braid group** B_k is generated by

$T_k = \cdots$  $T_0 =$  $\cdots$ and $T_i =$ 

for $1 \leq i \leq k-1$.

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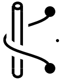
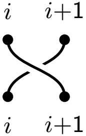
$T_k = \cdots$  $T_0 =$  $\cdots$ and $T_i =$ 

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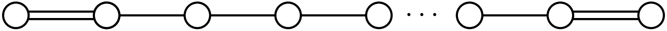

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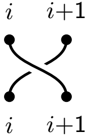
$T_k = \cdots$  $T_0 =$  $\cdots$ and $T_i =$ 

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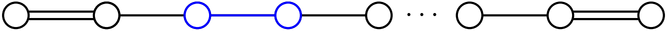
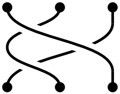
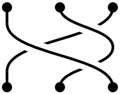


$T_iT_{i+1}T_i =$  $=$  $= T_{i+1}T_iT_{i+1}$

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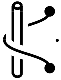
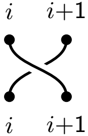
$T_k = \cdots$  $T_0 =$  $\cdots$ and $T_i =$ 

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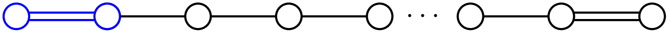
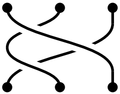
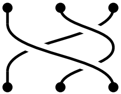
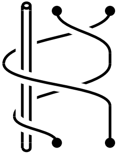
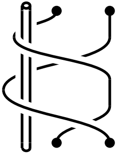


$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$

$T_0 T_1 T_0 T_1 =$  $=$  $= T_1 T_0 T_1 T_0$

(similar picture for $T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k$)

Theorem (D.-Ram, degenerate version in [Da])

1. *Let* $U = U_q\mathfrak{g}$ *for any complex reductive Lie algebras* $\mathfrak{g}$.
   Let M, N, *and* V *be finite-dimensional modules.*

   The two-boundary braid group B_k *acts on* $M \otimes (V)^{\otimes k} \otimes N$ *and this action commutes with the action of* U.

2. *If* $\mathfrak{g} = \mathfrak{gl}_n$, *then (for appropriate choices of* M, N, *and* V*),*
   the affine Hecke algebra of type C, H_k, *acts on* $M \otimes (V)^{\otimes k} \otimes N$ *and this action commutes with the action of* U.

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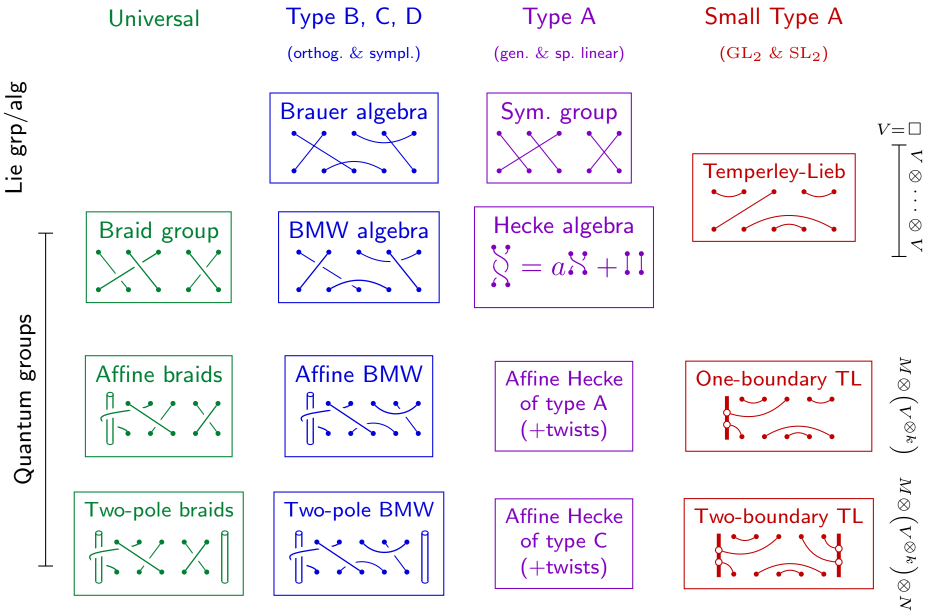
2. *If $\mathfrak{g} = \mathfrak{gl}_n$, then (for appropriate choices of M, N, and V), the affine Hecke algebra of type C, H_k, acts on $M \otimes (V)^{\otimes k} \otimes N$ and this action commutes with the action of U.*

Results:

(a) A combinatorial classification and construction of irreducible representations of H_k (type C with distinct parameters).

(b) A diagrammatic intuition behind otherwise unwieldy calculations in T_k and H_k.

(c) A classification of the representations of T_k in [dGN08] via central characters, including answers to open questions and conjectures regarding their irreducibility and isomorphism classes.



Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL_2 & SL_2)
Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
Temperley-Lieb
$V = \square$
$V \otimes \cdots \otimes V$
Braid group
BMW algebra
Hecke algebra
$= a$
Affine braids
Affine BMW
Affine Hecke of type A (+twists)
One-boundary TL
$M \otimes (V^{\otimes k})$
Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two-boundary TL
$M \otimes (V^{\otimes k}) \otimes N$