Centralizers of the infinite symmetric group

Zajj Daugherty

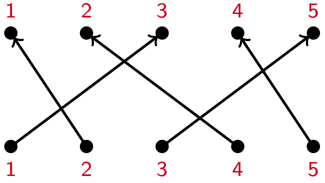
Joint with Peter Herbrich

Dartmouth College

November 26, 2013

Schur-Weyl duality – a warm-up

Start with the symmetric group S_k: permutations of $1, \ldots, k$.
Depict using permutation diagrams:



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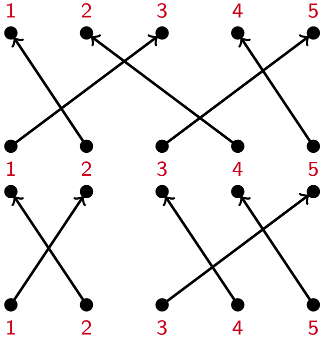

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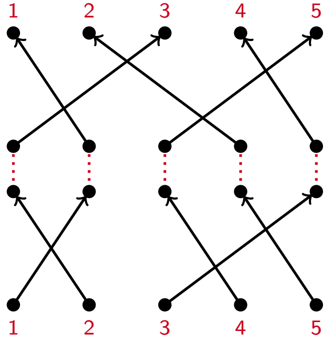

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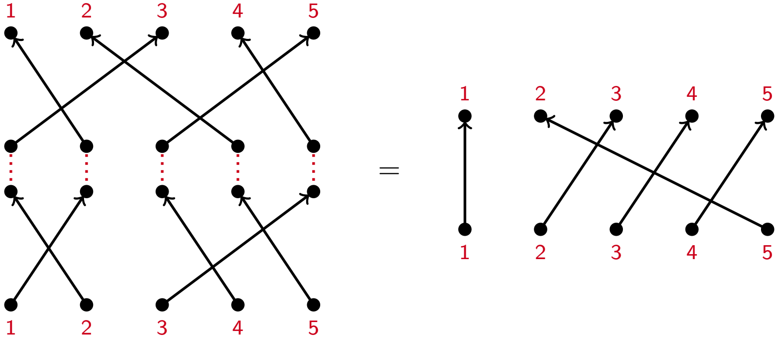

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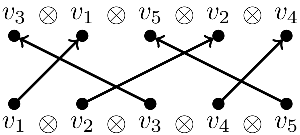

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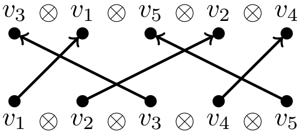
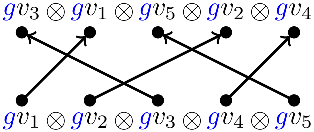


3. These actions commute!



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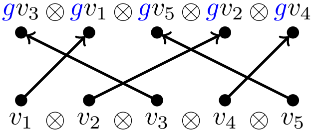

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Why this is exciting: Huge transfer of information!

Centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \qquad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

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For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \cong \left(G^{\yng(3)} \otimes S^{\yng(3)}\right) \oplus \left(G^{\yng(2,1)} \otimes S^{\yng(2,1)}\right) \oplus \left(G^{\yng(1,1,1)} \otimes S^{\yng(1,1,1)}\right)$$

Switching roles: the partition algebra

Let V be the permutation representation of S_n:

$n \times n$ *matrices with 1's and 0's* i.e. $\sigma \cdot v_i = v_{\sigma(i)}$

Now let S_n act diagonally on $V^{\otimes k}$:

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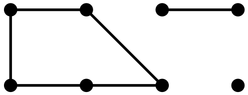

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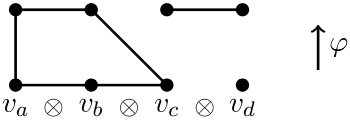

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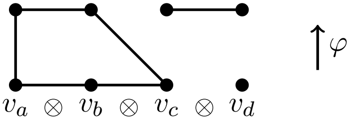

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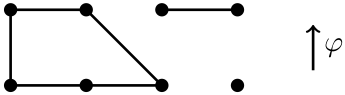
$$\delta_{a=b=c} \; (v_a \otimes v_a) \otimes \Big(\sum_{i=1}^{n} v_i \otimes v_i \Big)$$



$$v_a \otimes v_b \otimes v_c \otimes v_d$$

Set partitions

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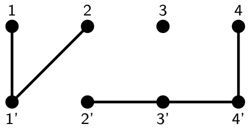
$$d = \{\{1, 2, 1'\}, \{3\}, \{2', 3', 4', 4\}\}$$

or as diagrams (considering connected components)



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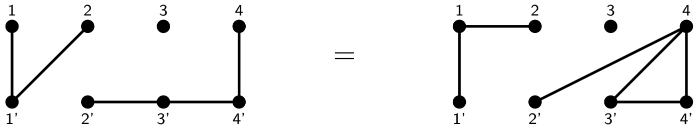
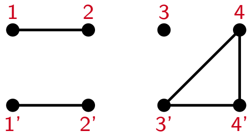


(Both encode the map $v_a \otimes v_b \otimes v_c \otimes v_d \mapsto \delta_{b=c=d}(v_a \otimes v_a) \otimes \sum_{i=1}^{n} v_i \otimes v_b$)

The partition algebra

Multiplying diagrams:



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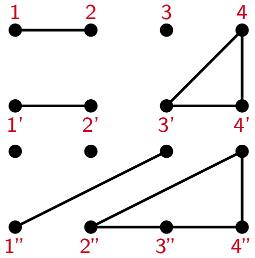

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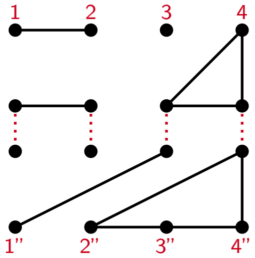

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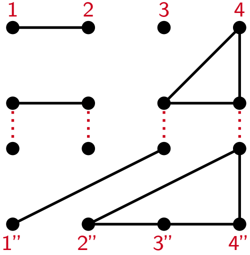
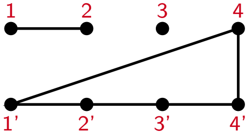


$= n^1$



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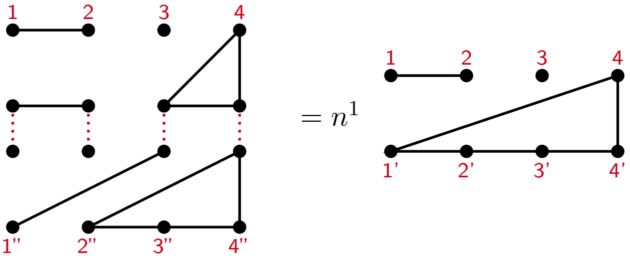

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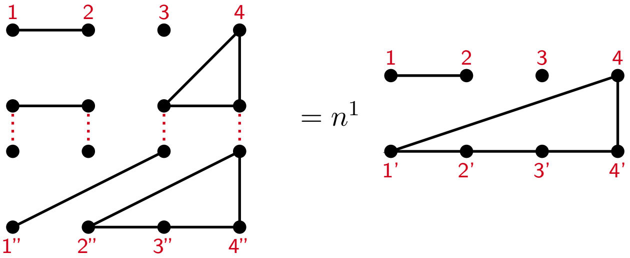

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Nice facts:

($*$) Associative algebra with identity $1 = \{\{1, 1'\}, \ldots, \{k, k'\}\}$.

($*$) $\dim(P_k(n)) =$ the Bell number $B(2k)$.

($*$) S_n and $P_k(n)$ centralize each other in $\mathrm{End}(V^{\otimes k})$.

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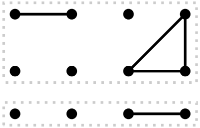

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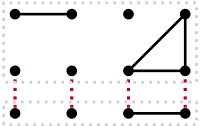

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Issue: finitely many versus countably many variables!

Moving to the infinite symmetric group

Natural inclusion: $S_n \subset S_{n+1}$ as permutations fixing $n+1$.
Consider the limit

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Want:

(1) A vector space V containing a countable linearly independent subset $\{v_i\}_{i\in\mathbb{N}}$;

(2) an appropriate notion of $V^{\otimes k}$; and

(3) an algebra of endomorphisms on $V^{\otimes k}$

   i. whose elements are determined by their image on v_i's, and

   ii. which contains S_∞ via the above action.

Three examples explored:

1. Countable dimensional vector space $V = \mathbb{C}^{(\mathbb{N})} = \mathbb{C}\{v_1, v_2, ...\}$.

2. Banach space of p-power summable sequences

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Good/Bad?

2 and 3 yield non-unitary and non-semisimple representations!

1. Countable dimensional vector space $V = \mathbb{C}^{(\mathbb{N})}$

If the $\varphi \in \mathrm{End}(V^{\otimes k})$ commutes with the action of S_∞, it acts like a linear combination of partition algebra diagrams.

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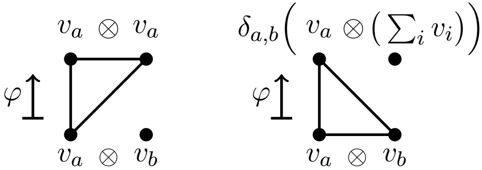

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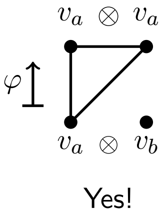

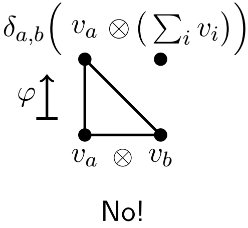

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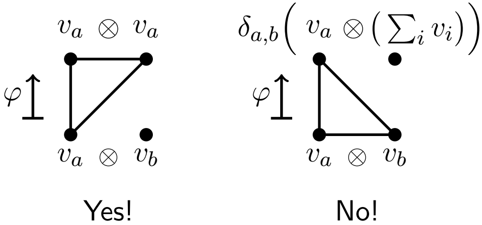

Result: The *top-propagating partition algebra*, generated by diagrams with no block isolated to the top.
(Sam-Snowden: the *upward partition category* glues all k together)

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Place a metric μ on $\mathbb{C}^{\mathbb{N}}$ so that

$$\Big|\Big| \sum_i v_i \Big|\Big|_p = ||(1, 1, 1, \dots)||_p = \sum_i \mu_i^p < \infty.$$

(Enough to get all expected invariants in the closure of $V^{\otimes k}$ for each k.)

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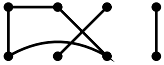
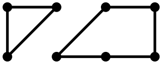
Result: The algebra of *uniform block permutations*, generated by diagrams whose blocks have the same size on top as on bottom.

Yes: 

No: 

(Same algebra as in Aguiar-Orellana!)

3. Banach space of ℓ^∞-bounded sequences

Sequences $(a_1, a_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ whose entries are bounded.

Issue: Even ℓ^∞-bounded endomorphisms are not determined by their images on $\{v_i\}_{i \in \mathbb{N}}$.

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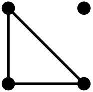
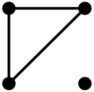
Result: The *bottom-propagating partition algebra*, generated by diagrams with no block isolated to the bottom. (Isomorphic to case 1)

Yes: 

No: 

Putting it back into context

Remark 1: Orellana et al. (in progress) show that if a diagram Hopf algebra (as in [MR95] or [AO08]) is built from partition diagrams, those diagrams can have no blocks isolated to the top or bottom rows.

Case 1: no application to symmetric functions.
Case 2: tied to symmetric functions in [AO08].
Question: Is there a fix for case 3?

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Remark 2: For all three cases, even for $k = 1$, the centralizer algebra is spanned by , so is isomorphic to $\mathbb{C}$.
However, in cases 2 and 3, we expected more since V has an invariant subspace. This discrepancy comes from the fact that the action of S_∞ is not semisimple.

Question: Can we use this framework to study certain non-unitary representations of S_∞?