The quasi-partition algebra

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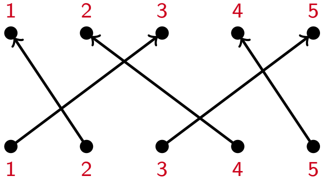
Combinatorial representation theory – a warm-up

Start with the symmetric group S_k: permutations of $1, \ldots, k$.
Depict using permutation diagrams:



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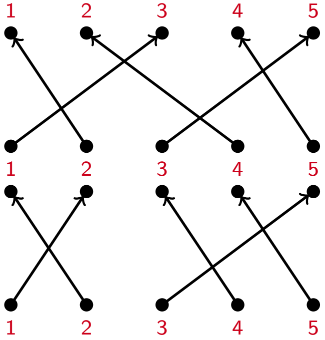

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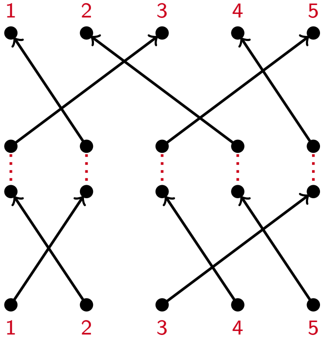

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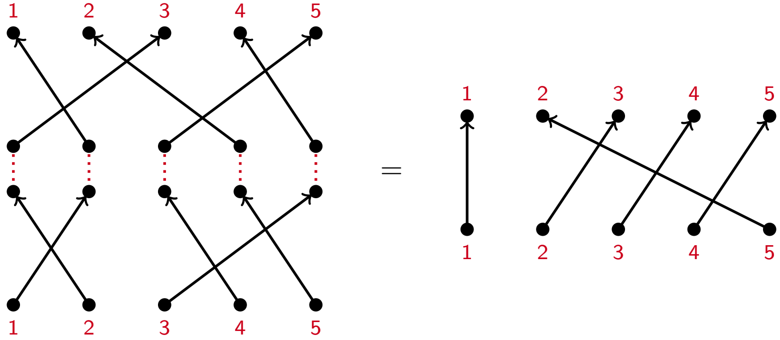

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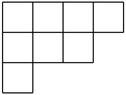
Simple S_k-modules are in bijection with partitions, $\lambda \vdash k$

$\lambda =$  $4 + 3 + 1$

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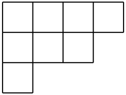
$\lambda =$  4 +3 +1

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So, for example,

S S and S

are the simple S_3-modules (up to isomorphism).

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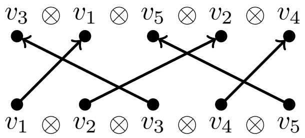
2. S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



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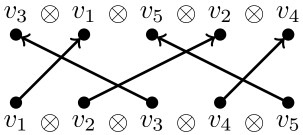
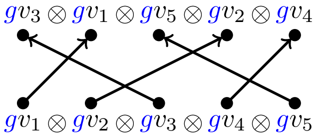


3. These actions commute!



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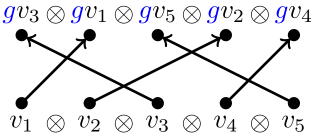

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Schur-Weyl duality: S_k and GL_n have commuting actions on $(\mathbb{C}^n)^{\otimes k}$, and their images fully centralize each in $\mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)$.

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Centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule},$$

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For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \cong \left(G^{\ydiagram{3}} \otimes S^{\ydiagram{3}}\right) \oplus \left(G^{\ydiagram{2,1}} \otimes S^{\ydiagram{2,1}}\right) \oplus \left(G^{\ydiagram{1,1,1}} \otimes S^{\ydiagram{1,1,1}}\right)$$

Switching roles: the partition algebra

Let V be the permutation representation of S_n.

$n \times n$ *matrices with 1's and 0's* i.e. $\sigma \cdot v_i = v_{\sigma(i)}$

Now let S_n act diagonally on $V^{\otimes k}$:

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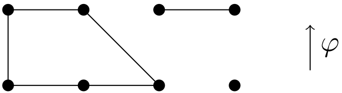

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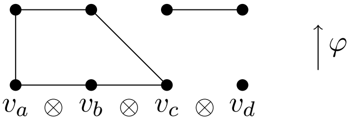

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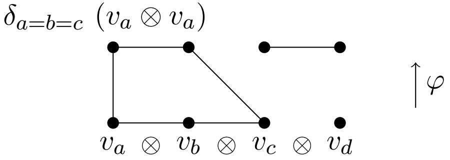

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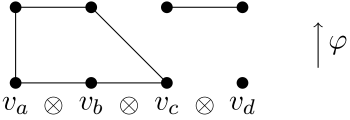
$$\delta_{a=b=c}\ (v_a \otimes v_a) \otimes \Big(\textstyle\sum_{i=1}^{n} v_i \otimes v_i \Big)$$



Set partitions

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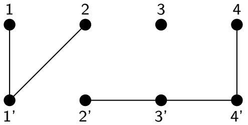
$$d = \{\{1, 2, 1'\}, \{3\}, \{2', 3', 4', 4\}\}$$

or as diagrams (considering connected components)



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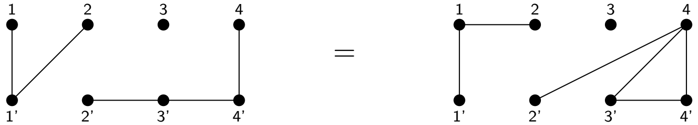
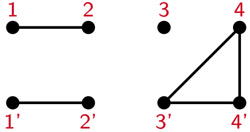


(Both encode the map $v_a \otimes v_b \otimes v_c \otimes v_d \mapsto \delta_{b=c=d}(v_a \otimes v_a) \otimes \sum_{i=1}^{n} v_i \otimes v_b$)

The partition algebra

Multiplying diagrams:



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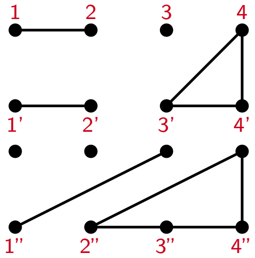

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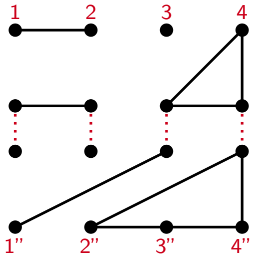

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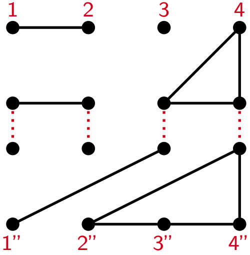
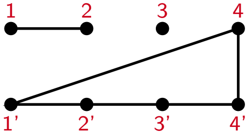


$= n^1$



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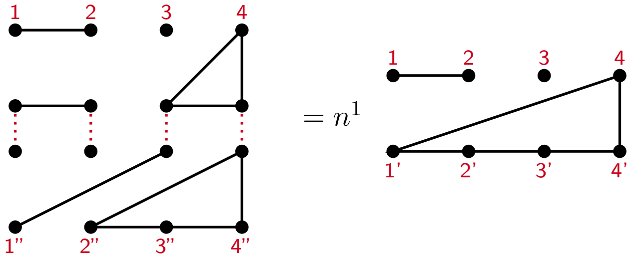

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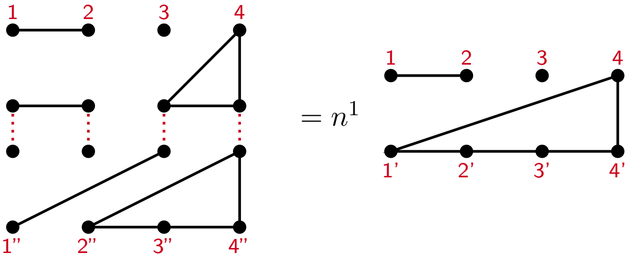

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Nice facts:

(∗) Associative algebra with identity $1 = \{\{1, 1'\}, \ldots, \{k, k'\}\}$.
(∗) $\dim(P_k(n)) =$ the Bell number $B(2k)$.
(∗) S_n and $P_k(n)$ centralize each other in $\mathrm{End}(V^{\otimes k})$.

Notice: V is not irreducible!

$$V = \mathbb{C}\{v_1, \dots, v_n\}$$
$$W = \mathbb{C}\{w_2, \dots, w_n\} \qquad \text{where } w_i = v_i - v_1,$$
$$T = \mathbb{C}v, \qquad \text{where } v = v_1 + \cdots + v_n.$$

Then $V = W \oplus T$ and so $V^{\otimes k} \cong W^{\otimes k} \oplus \left(\bigoplus_{i=1}^{k} \binom{k}{i} T^{\otimes i} \otimes W^{\otimes (k-i)} \right)$.

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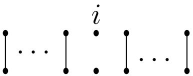
Let $p_i =$ 

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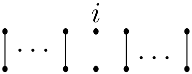
Let $p_i =$ 

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Any diagram d an isolated vertex satisfies $d = p_i d'$ or $d = d' p_i$.

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Two maps in $\mathrm{End}(W^{\otimes k})$**:**

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   Bad: They don't commute with S_n.

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2. Let $f : \{v_1, \ldots, v_{n-1}\} \to \{w_2, \ldots, w_n\} \qquad v_i \mapsto w_{i+1}.$
   Put $d \in P_k(n-1)$, and consider

$$[d] : W^{\otimes k} \xrightarrow{f^{-1}} V_{n-1}^{\otimes k} \xrightarrow{d} V_{n-1}^{\otimes k} \xrightarrow{f} W^{\otimes k}$$

   Good: We know what they are and how to multiply them.

   Bad: They don't commute with S_n.

Goal: Express $\bar{d}$ in terms of $[d']$'s.

Let’s calculate 

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Start with a basis element of $W \otimes W$:

$$
\begin{aligned}
w_a \otimes w_b &= (v_a - v_1) \otimes (v_b - v_1) && a, b \neq 1 \\
&= (v_a \otimes v_b) - (v_a \otimes v_1) - (v_1 \otimes v_b) + (v_1 \otimes v_1)
\end{aligned}
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$$\delta_{ab}\left(w_a - \tfrac{1}{n}w\right) \otimes \left(w_a - \tfrac{1}{n}w\right) + \left(\tfrac{1}{n}w\right) \otimes \left(\tfrac{1}{n}w\right) \qquad \text{with } w = \sum_{\ell=2}^{n} w_\ell$$

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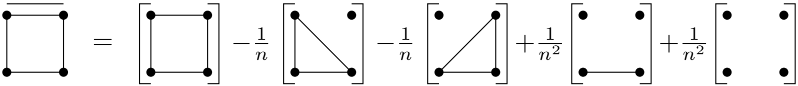
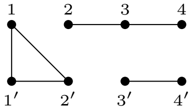
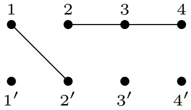


If X is a set of vertices, the isolation of d (at X) is d_X , the diagram constructed from d by isolating all vertices in X.

For example, if $X = \{1', 4'\}$ and

$d =$ 

then

$d_X =$ 

We can also place an order on diagrams, where $d' \leq d$ if d' is a refinement of d. In particular, $d_X \leq d$.

Define the quasi-partition algebra as $QP_k(n) = \mathrm{End}_{S_n}(W^{\otimes k})$.
Let $\mathcal{D} = \{$ diagrams d without isolated verticies $\}$.

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Theorem

If $d \in \mathcal{D}$ then

$$\bar{d} = [d] + \sum_{X \subseteq [k] \cup [k']} c_X [d_X],$$

where c_X is a (totally explicit) polynomials in n and $1/n$.

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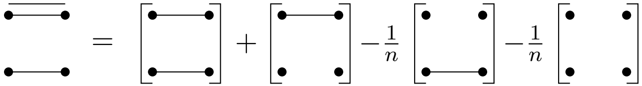

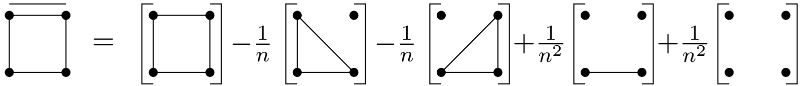

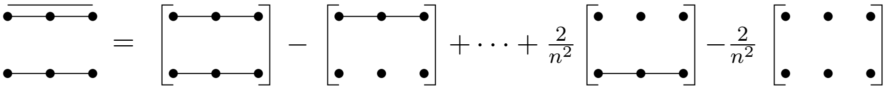

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Corollary

$QP_k(n)$ has basis $\{\bar{d} \mid d \in \mathcal{D}\}$, and thus has dimension

$$\sum_{j=1}^{2k} (-1)^{j-1} B(2k-j) + 1, \qquad \textit{where } B(r) \textit{ is the Bell number.}$$

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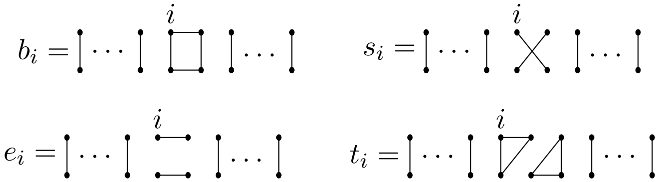
Corollary

If $d_1, d_2 \in \mathcal{D}$,

$$\bar{d}_1 \bar{d}_2 = \sum_{d \leq d_1 d_2} c_d \bar{d}.$$

In particular, if $d_1 d_2 \notin \mathcal{D}$, then $\bar{d}_1 \bar{d}_2 = 0$.

So $QP_k(n)$ is also a subalgebra of $P_k(n-1)$.
It's generated by projections of



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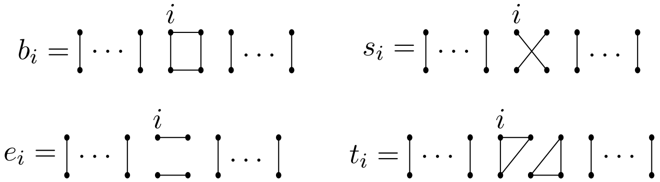

With relations that look like

in $P_k(n-1)$:	in $QP_k(n)$:
$s_i^2 = 1$	$\bar{s}_i^2 = 1$
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$	$\bar{s}_i \bar{s}_{i+1} \bar{s}_i = \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}$
$e_i^2 = (n-1) e_i$	$\bar{e}_i^2 = (n-1) \bar{e}_i$
$b_i^2 = b_i$	$\bar{b}_i^2 = \frac{n-2}{n} \bar{b}_i + \frac{1}{n^2} \bar{e}_i$

Representation theory

Recall that the centralizer relationship produces:

$$W^{\otimes k} \cong \bigoplus_{\lambda} QP^{\lambda} \otimes S^{\bar{\lambda}} \quad \text{as a } QP_k\text{-}S_n \text{ bimodule.}$$

$$\dim(QP^{\lambda}) = \text{multiplicity}(S^{\bar{\lambda}}) \quad \text{and} \quad \text{multiplicity}(QP^{\lambda}) = \dim(S^{\bar{\lambda}})$$

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Tensoring rule for $W = S^{(n-1,1)}$

$$S^{\lambda} \otimes W = c(\lambda) S^{\lambda} \oplus \bigoplus_{\mu \in \Lambda} S^{\mu}$$

where Λ is the set of partitions gotten from λ by moving any corner box to another place, and $c(\lambda) = \#$ corner boxes -1.

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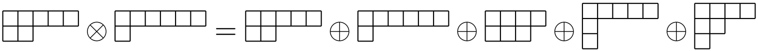
Example:



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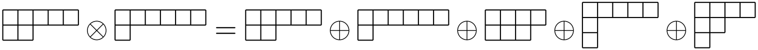

Assume $n >> 1$. We can forget the top row:

Representation theory: Bratteli diagram for $QP_k(n)$

1: $\square$

($*$) Modules for $QP_k(n)$ are indexed by partitions at the kth level of the Bratteli diagram.

($*$) Each module QP^λ has basis given by paths down to λ.

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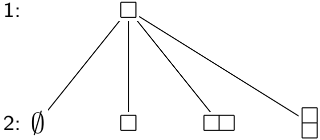

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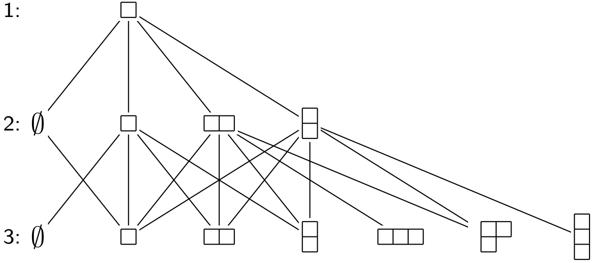

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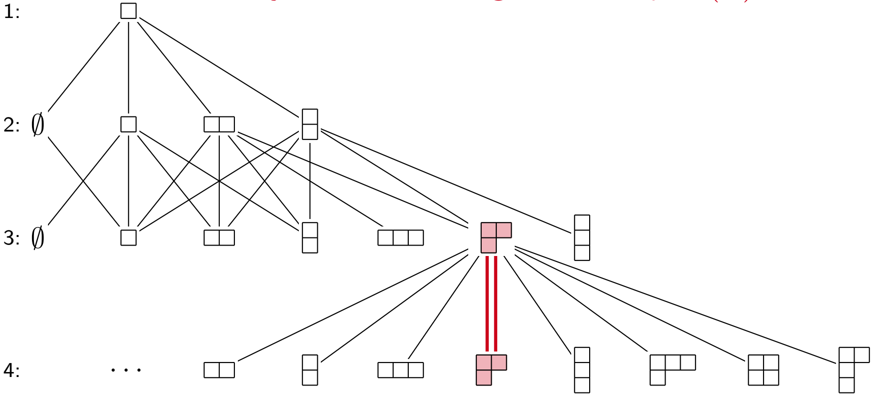

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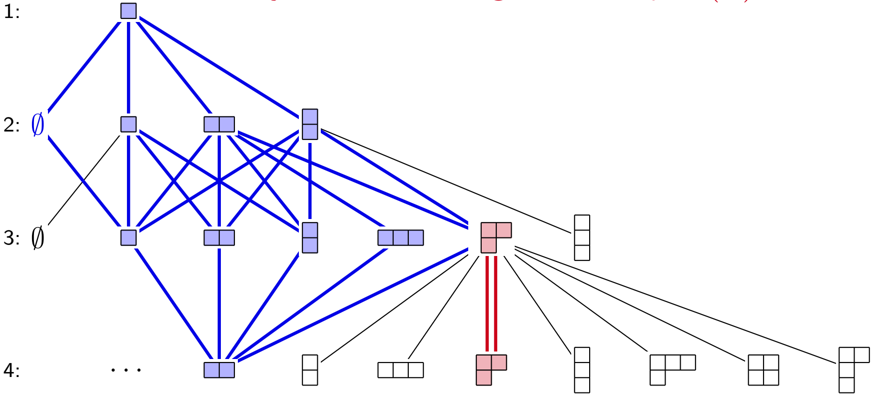

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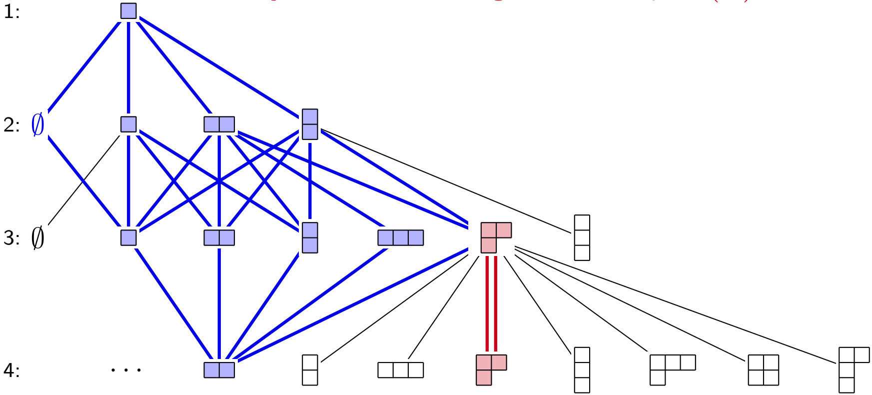

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Next: presentation, Jucys-Murphy elements and seminormal reps, central and primitive idempotents, Jones basic construction, induction/restriction rules. . .