The quasi-partition algebra

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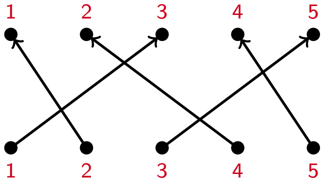
Joint with Rosa Orellana

Dartmouth College and ICERM

March 19, 2013

Diagram algebras

Everyone's favorite diagram algebra:
Group algebra of the symmetric group S_k



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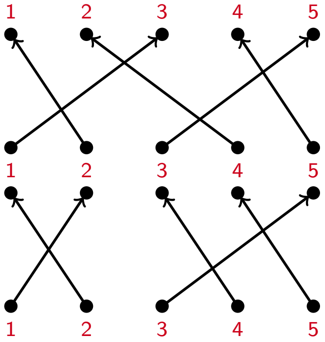

(with multiplication given by concatenation)

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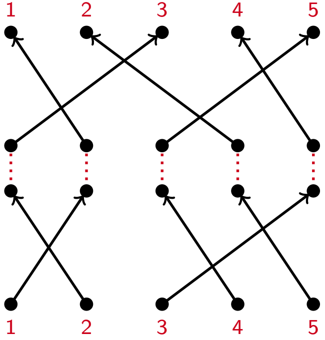

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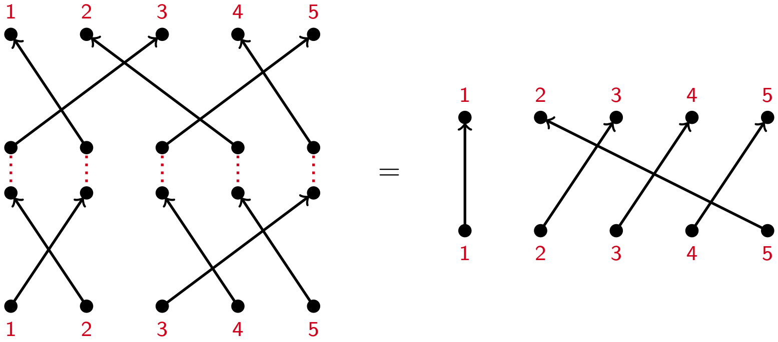

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Schur-Weyl duality and centralizer algebras

Classical example: (Schur 1901)

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$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

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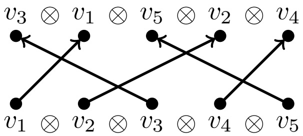

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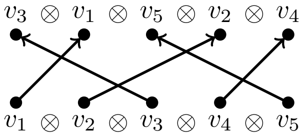
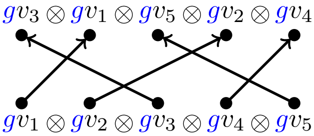


3. These actions commute!



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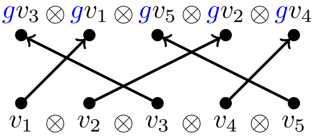
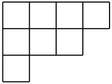


Modules (vector spaces, with the group or algebra acting as matrices)

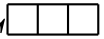
Simple S_k-modules are in bijection with partitions, $\lambda \vdash k$

$\lambda =$ 

4
+3
+1

(a collection of boxes piled up and to the left)

So, for example,

S S and S

are the simple S_3-modules (up to isomorphism).

Schur-Weyl duality and centralizer algebras

Classical example: S_k and GL_n have commuting actions on $(\mathbb{C}^n)^{\otimes k}$, and their images fully centralize each in $\mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)$.

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Why this is exciting:

Centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} L(\lambda) \otimes S^\lambda \qquad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

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For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \left(L(\ydiagram{3}) \otimes S^{\ydiagram{3}}\right) \oplus \left(L(\ydiagram{2,1}) \otimes S^{\ydiagram{2,1}}\right) \oplus \left(L(\ydiagram{1,1,1}) \otimes S^{\ydiagram{1,1,1}}\right)$$

Switching roles: the partition algebra

Let V be the permutation representation of S_n.

$n \times n$ *matrices with 1's and 0's* i.e. $\sigma \cdot v_i = v_{\sigma(i)}$

Now let S_n act diagonally on $V^{\otimes k}$:

$$\sigma \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)}$$

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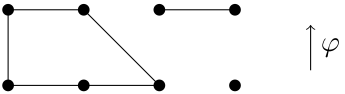

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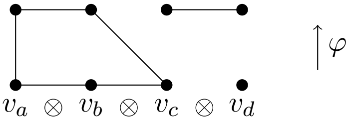

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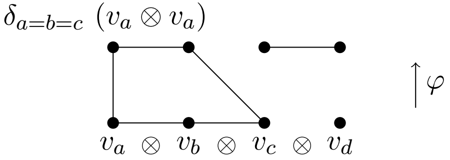

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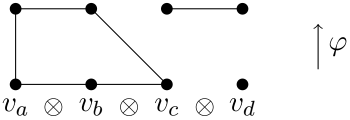
$$\delta_{a=b=c}\ (v_a \otimes v_a) \otimes \left(\sum_{i=1}^{n} v_i \otimes v_i \right)$$



Set partitions

Fix $k \in \mathbb{Z}_{>0}$, and let

$$[k] = \{1, \ldots, k\} \qquad \text{and} \qquad [k'] = \{1', \ldots, k'\}.$$

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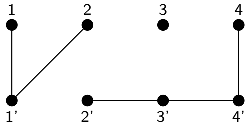
$$d = \{\{1, 2, 1'\}, \{3\}, \{2', 3', 4', 4\}\}$$

or as diagrams (considering connected components)



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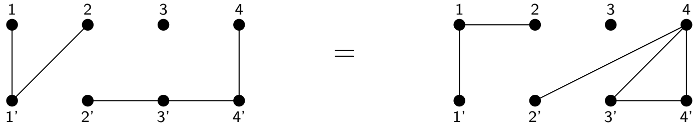
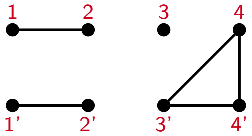


(Both encode the map $v_a \otimes v_b \otimes v_c \otimes v_d \mapsto \delta_{b=c=d}(v_a \otimes v_a) \otimes \sum_{i=1}^{n} v_i \otimes v_b$)

The partition algebra

Multiplying diagrams:



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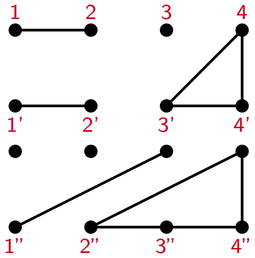

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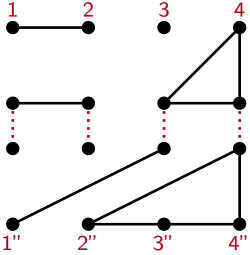

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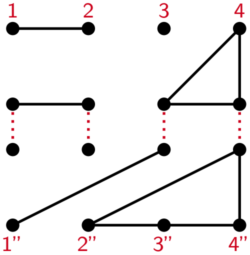

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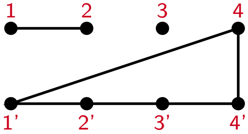

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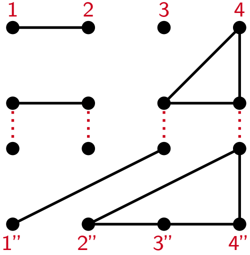

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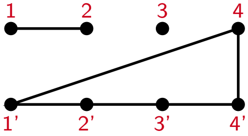

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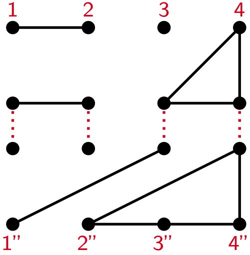

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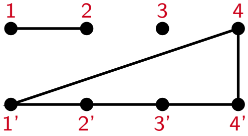

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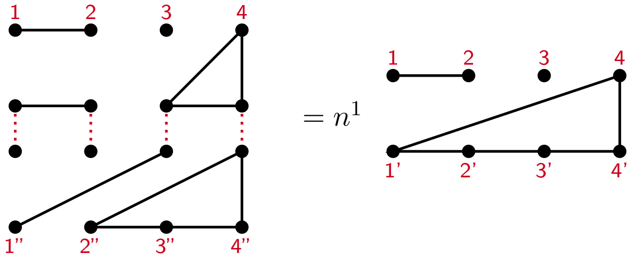

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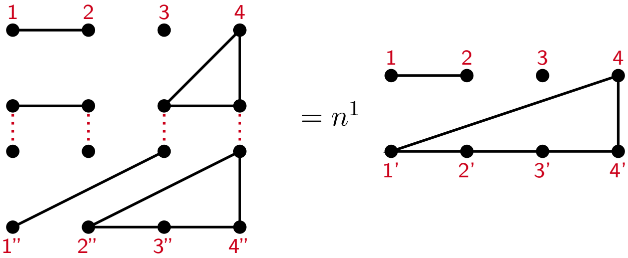

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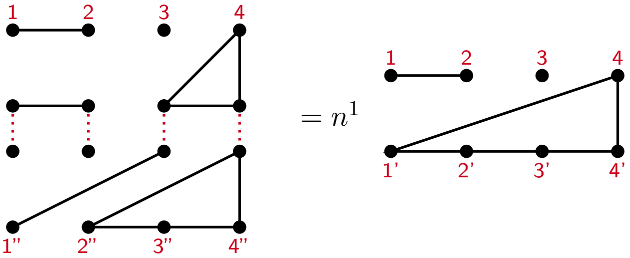

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(∗) Associative algebra with identity $1 = \{\{1, 1'\}, \ldots, \{k, k'\}\}$.
(∗) $\dim(P_k(n))$ = the Bell number $B(2k)$.
(∗) S_n and $P_k(n)$ centralize each other in $\mathrm{End}(V^{\otimes k})$.

Problem: V is not irreducible!

$$\begin{aligned} V &= \mathbb{C}\{v_1, \ldots, v_n\} \\ W &= \mathbb{C}\{w_2, \ldots, w_n\} && \text{where } w_i = v_i - v_1, \\ T &= \mathbb{C}v, && \text{where } v = v_1 + \cdots + v_n. \end{aligned}$$

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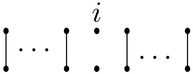
Let $p_i =$  Then $p_i : V^{\otimes k} \to V^{\otimes i-1} \otimes T \otimes V^{\otimes k-i}$.

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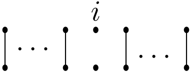
Let $p_i =$ 

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Any diagram d an isolated vertex satisfies $d = p_i d'$ or $d = d' p_i$.

Let d be a diagram on $2k$ vertices.
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2. Let $f : \{v_1, \ldots, v_{n-1}\} \to \{w_2, \ldots, w_n\} \qquad v_i \mapsto w_{i+1}$.
   Put $d \in P_k(n-1)$, and consider

$$[d] : W^{\otimes k} \xrightarrow{f^{-1}} V_{n-1}^{\otimes k} \xrightarrow{d} V_{n-1}^{\otimes k} \xrightarrow{f} W^{\otimes k}$$

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   Put $d \in P_k(n-1)$, and consider

$$[d] : W^{\otimes k} \xrightarrow{f^{-1}} V_{n-1}^{\otimes k} \xrightarrow{d} V_{n-1}^{\otimes k} \xrightarrow{f} W^{\otimes k}$$

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Let d be a diagram on $2k$ vertices.

Two maps in $\mathrm{End}(W^{\otimes k})$:

1. Put $d \in P_k(n)$, and project its image onto $W^{\otimes k}$:

$$\bar{d} : W^{\otimes k} \xrightarrow{d} V^{\otimes k} \xrightarrow{\pi_W^{\otimes k}} W^{\otimes k}.$$

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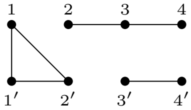
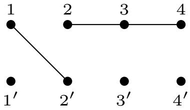
Goal: Express $\bar{d}$ in terms of $[d']$'s.

If X is a set of vertices, the isolation of d (at X) is d_X , the diagram constructed from d by isolating all vertices in X.

For example, if $X = \{1', 4'\}$ and

$d =$ 

then

$d_X =$ 

We can also place an order on diagrams, where $d' \leq d$ if d' is a refinement of d. In particular, $d_X \leq d$.

Define the quasi-partition algebra as $QP_k(n) = \mathrm{End}_{S_n}(W^{\otimes k})$.
Let $\mathcal{D} = \{$ diagrams d without isolated verticies $\}$.

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Theorem (D.-Orellana)

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$$\bar{d} = [d] + \sum_{X \subseteq [k] \cup [k']} c_X [d_X],$$

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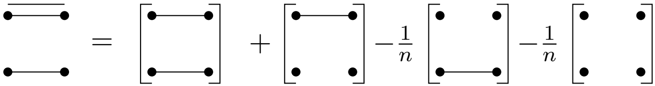

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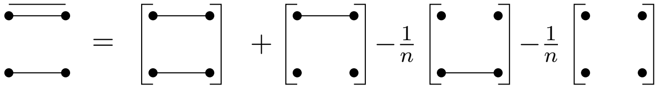

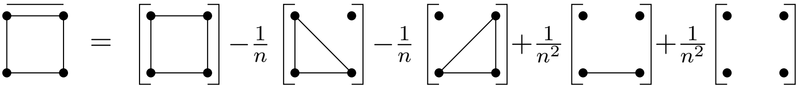

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Corollary

$QP_k(n)$ has basis $\{\bar{d} \mid d \in \mathcal{D}\}$, and thus has dimension

$$\sum_{j=1}^{2k} (-1)^{j-1} B(2k-j) + 1, \qquad \text{where } B(r) \text{ is the Bell number.}$$

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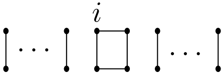
Corollary

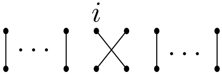
If $d_1, d_2 \in \mathcal{D}$,

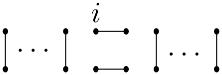
$$\bar{d}_1 \bar{d}_2 = \sum_{d \leq d_1 d_2} c_d \bar{d}.$$

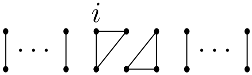
In particular, if $d_1 d_2 \notin \mathcal{D}$, then $\bar{d}_1 \bar{d}_2 = 0$.

So *functionally*, $QP_k(n)$ is a subalgebra of $P_k(n-1)$.
It's generated by projections of

$b_i =$ 

$s_i =$ 

$e_i =$ 

$t_i =$ 

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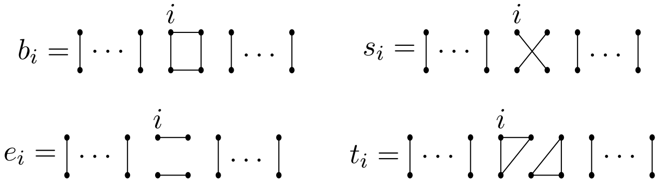

With relations that look like

in $P_k(n-1)$:	in $QP_k(n)$:
$s_i^2 = 1$	$\bar{s}_i^2 = 1$
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$	$\bar{s}_i \bar{s}_{i+1} \bar{s}_i = \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}$
$e_i^2 = (n-1)e_i$	$\bar{e}_i^2 = (n-1)\bar{e}_i$
$b_i^2 = b_i$	$\bar{b}_i^2 = \frac{n-2}{n}\bar{b}_i + \frac{1}{n^2}\bar{e}_i$

Representation theory

Tensoring rule for $W = S^{(n-1,1)}$

$$S^{\lambda} \otimes W = c(\lambda) S^{\lambda} \oplus \bigoplus_{\mu \in \Lambda} S^{\mu}$$

where Λ is the set of partitions gotten from λ by moving any corner box to another place, and $c(\lambda) = \#$ corner boxes -1.

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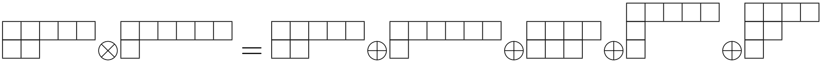

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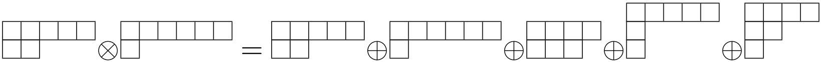
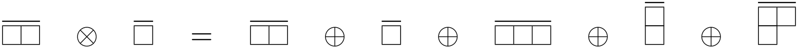


Assume $n >> 1$. We can forget the top row:



Bratteli diagram for $QP_k(n)$

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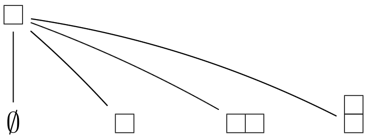

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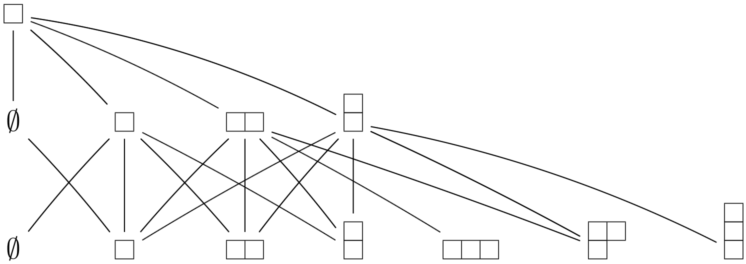

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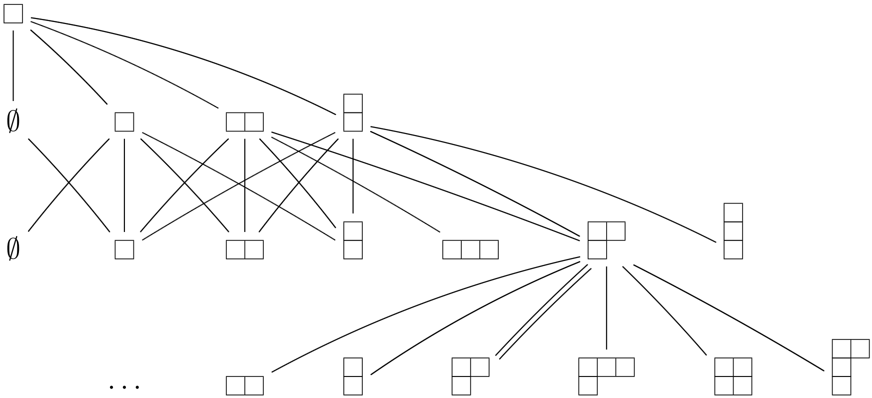

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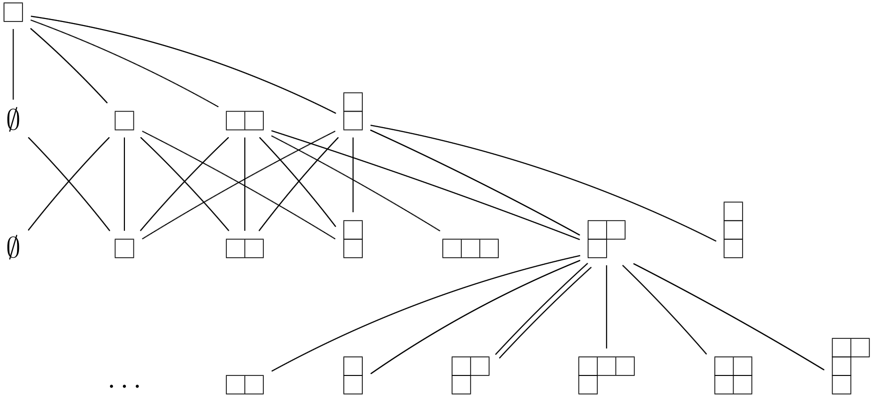

(Dimensions expressed explicitly in terms of numbers of standard tableaux and Bell numbers.)