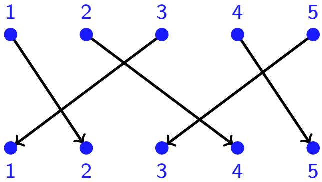
A taste of combinatorial representation theory (yum)

Zajj Daugherty
Dartmouth College & ICERM

March 14, 2013
(Happy π day!)

Permutations and the symmetric group

Permutation diagrams:



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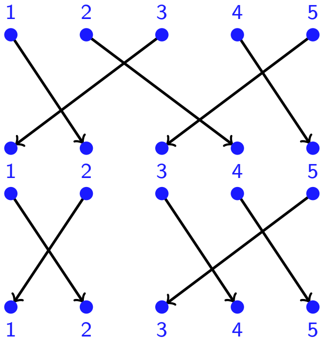

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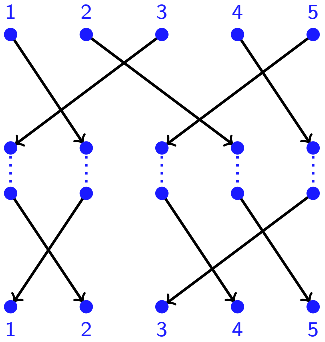

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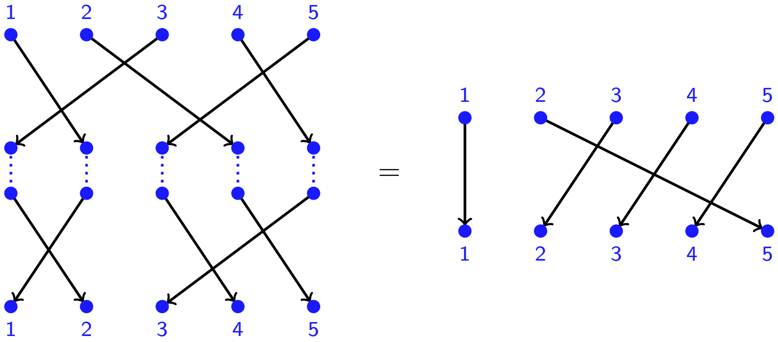

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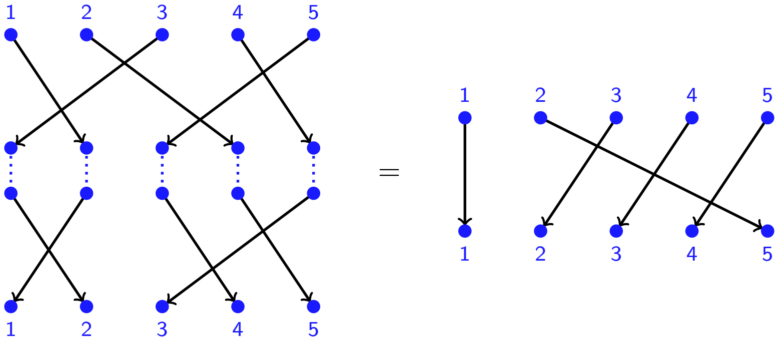


Permutations “multiply” by stacking and resolving.

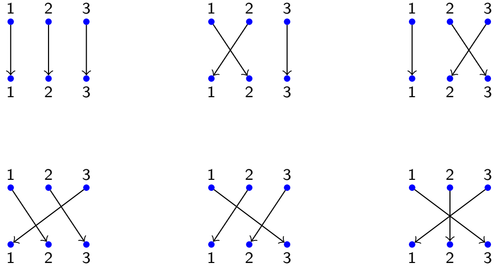
The symmetric group S_n is the group of permutations of $1, \dots, n$ with multiplication given by stacking and resolving diagrams.

Some examples:

S_1 :

S_2 :

S_3 :



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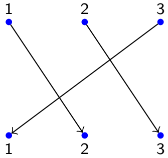

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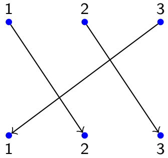

Pick a basis for $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

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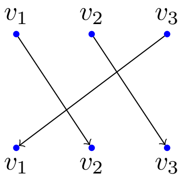

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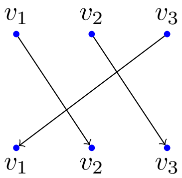

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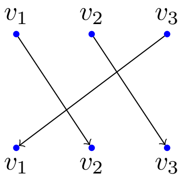

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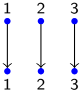
Aside: we actually have a representation of the group ring

$$\mathbb{R}S_n = \left\{ \sum_{\sigma \in S_n} r_\sigma \sigma \mid r_\sigma \in \mathbb{R} \right\},$$ with multiplication like polynomials



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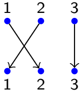

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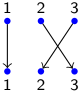

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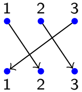

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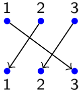

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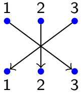

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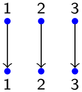

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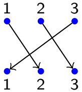

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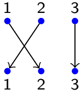

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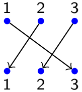

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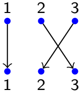

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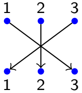

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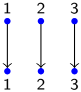

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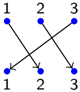

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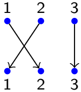

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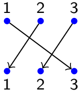

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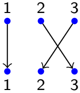

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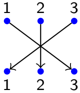

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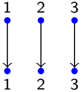

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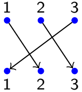

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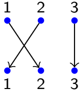

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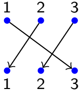

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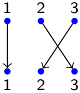

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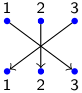

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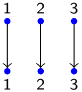

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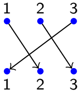

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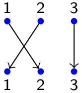

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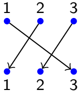

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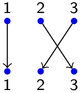

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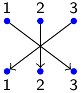

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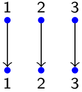

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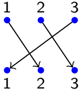

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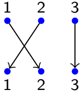

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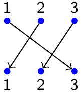

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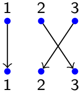

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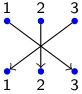

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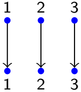

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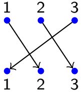

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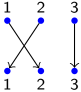

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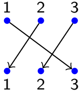

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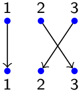

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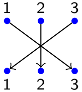

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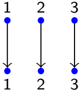

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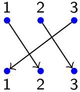

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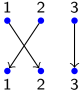

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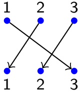

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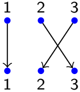

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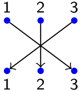

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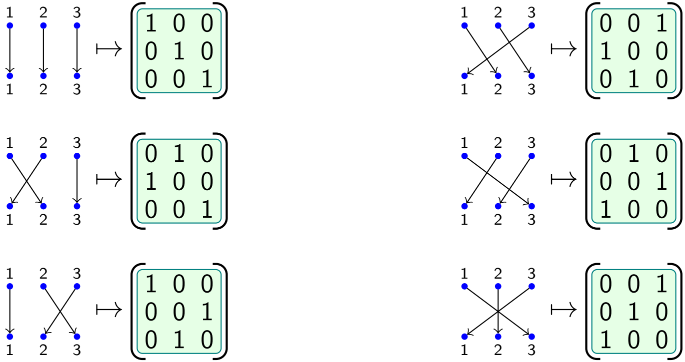

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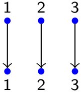

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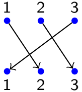

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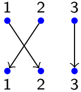

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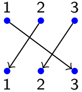

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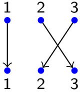

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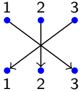

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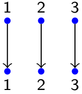

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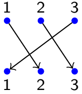

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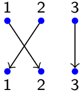

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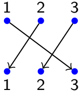

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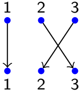

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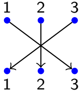

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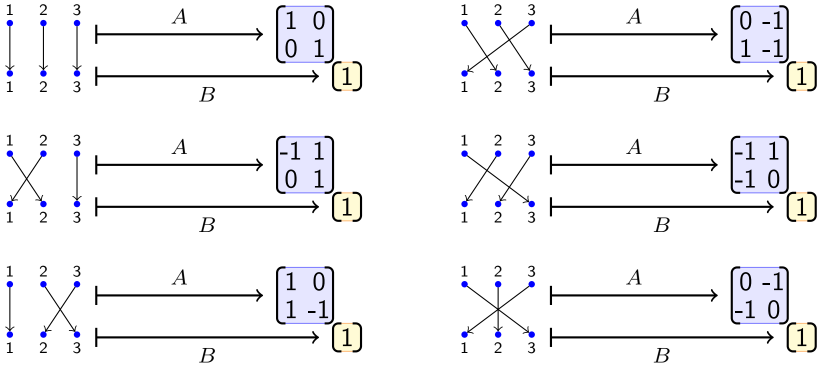

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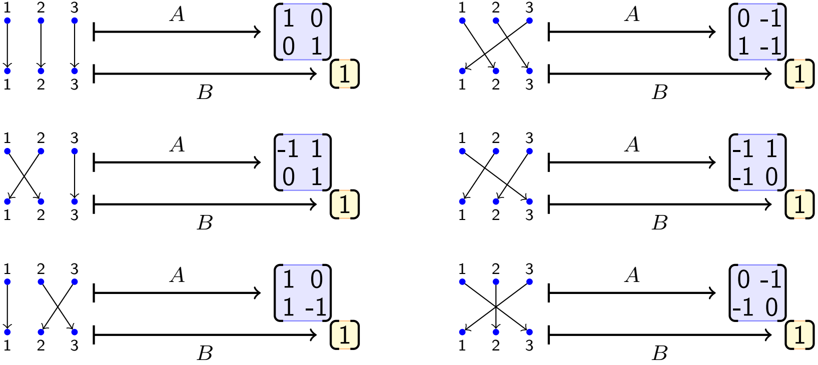

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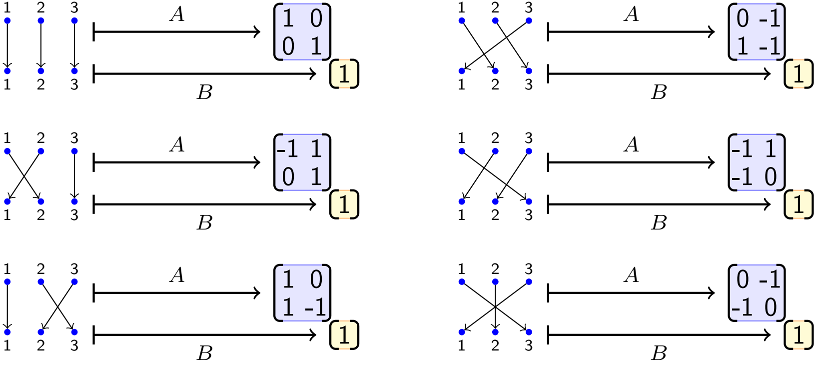
$$P \cong A \oplus B$$



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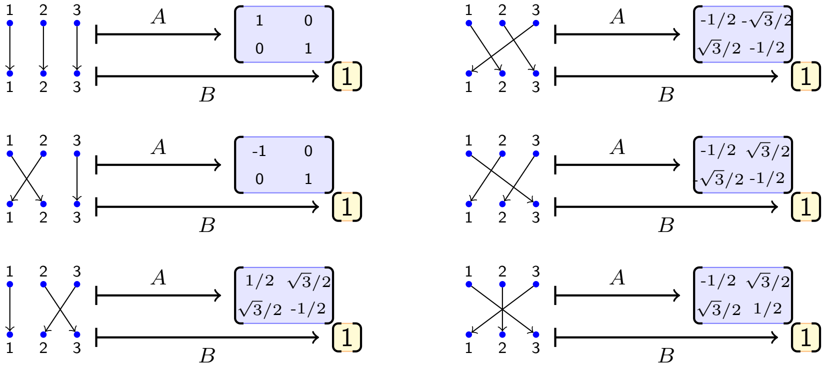

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How about some combinatorics?

Let n be a non-negative integer.
A partition λ of n is a non-ordered list of positive integers which sum to n.

Example: the partitions of 3 are (3), $(2, 1)$, and $(1, 1, 1)$.

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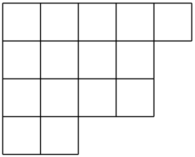
We draw partitions as n boxes piled up and to the left, where the parts are the number of boxes in a row:

$$\lambda = (5, 4, 4, 2) =$$



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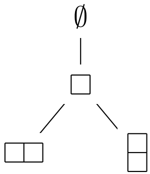

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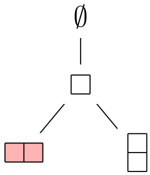

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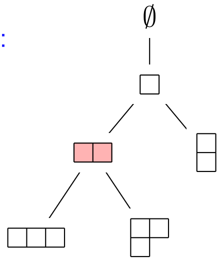

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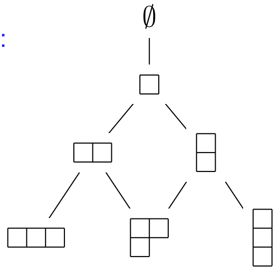

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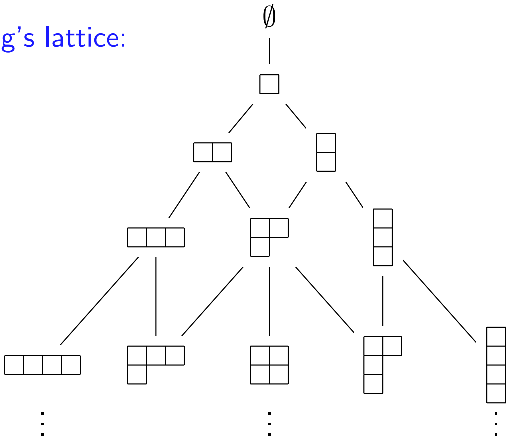

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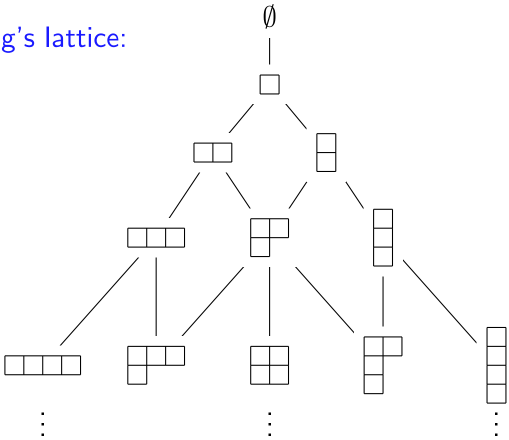

λ-Tableau: a path from $\emptyset$ down to a partition λ.

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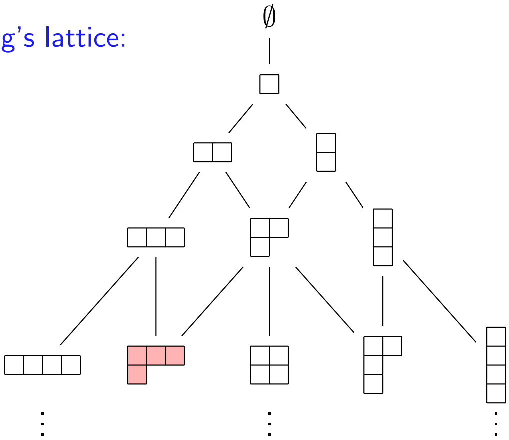

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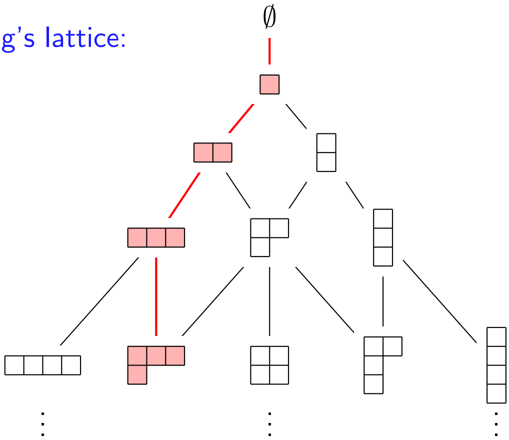

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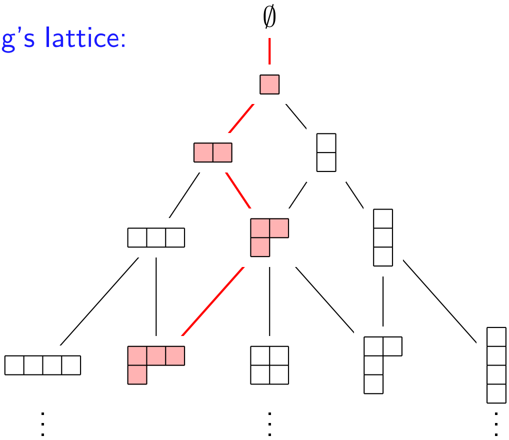

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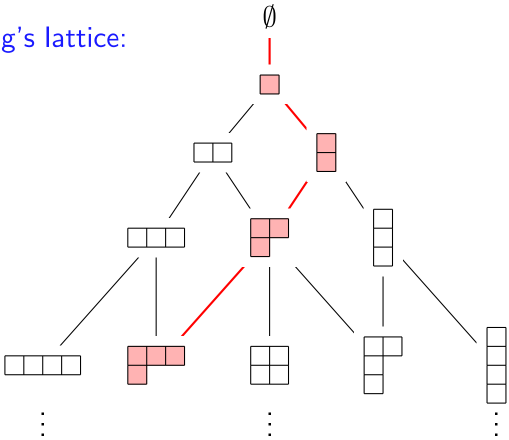

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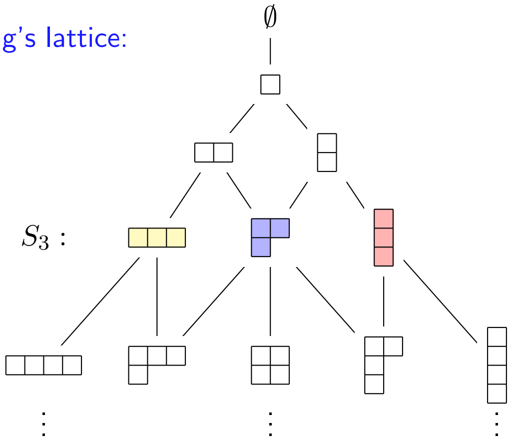

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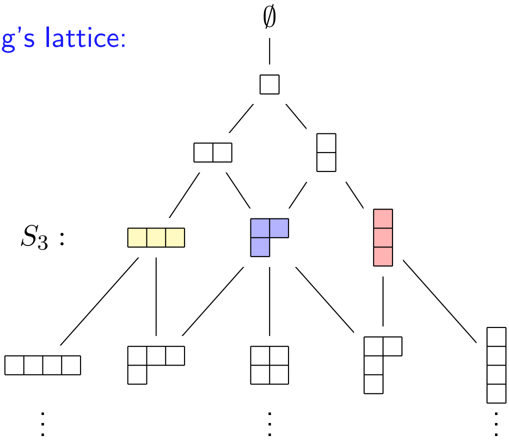

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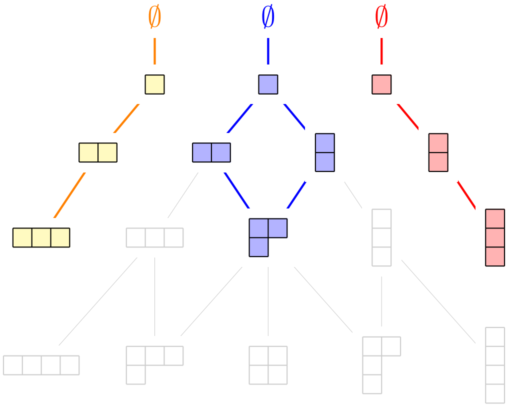

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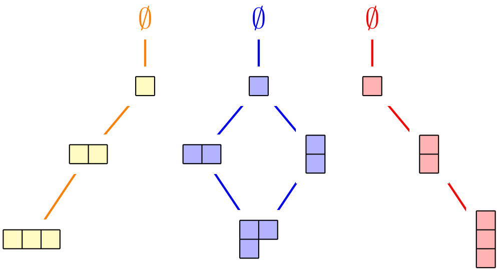

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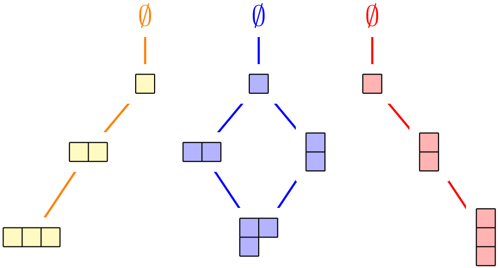

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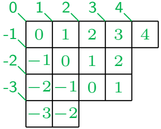

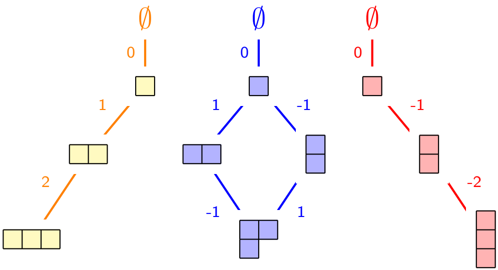

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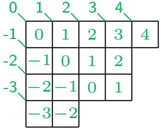

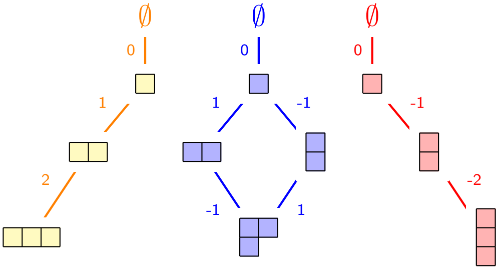

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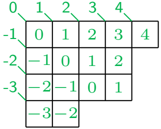

Again:

$*$ each partition is secretly a representation

$*$ each path is secretly a basis vector

Now: entries in matrices for s_1, s_2, . . . , are given by expressions in the contents of boxes added.

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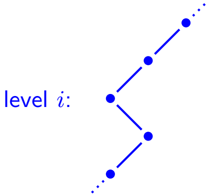

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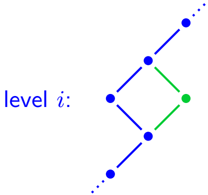
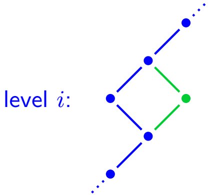


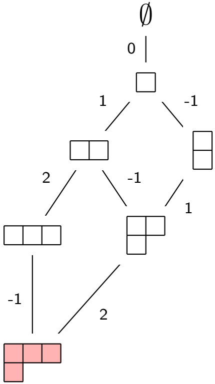
and u is almost the same, except at the ith step.

The rule for s_i:

Suppose v goes with the path

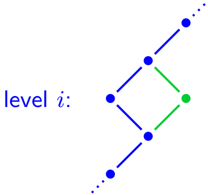


and u is almost the same, except at the ith step.

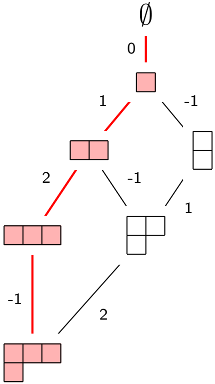


The rule for s_i:

Suppose v goes with the path

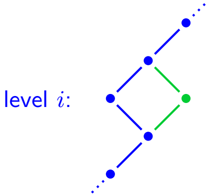


and u is almost the same, except at the ith step.

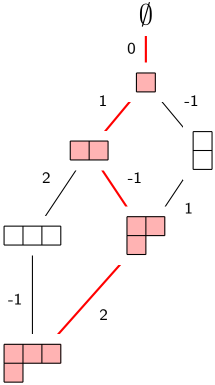


The rule for s_i:

Suppose v goes with the path

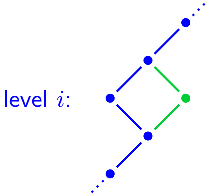


and u is almost the same, except at the ith step.

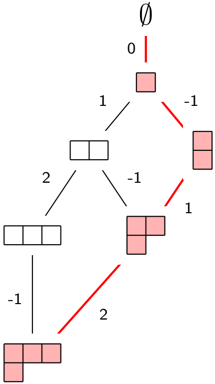


The rule for s_i:

Suppose v goes with the path

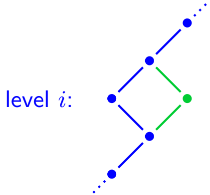


and u is almost the same, except at the ith step.



The rule for s_i:

Suppose v goes with the path

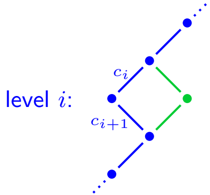


and u is almost the same, except at the ith step.

Let c_i be the content of the box added from $i-1$ to i.

The rule for s_i:

Suppose v goes with the path



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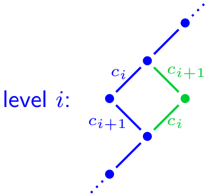

and u is almost the same, except at the ith step.

Let c_i be the content of the box added from $i-1$ to i.

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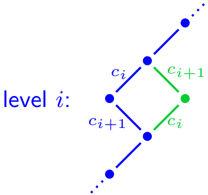
Suppose v goes with the path



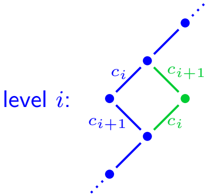
and u is almost the same, except at the ith step.

Let c_i be the content of the box added from $i-1$ to i.

Then the coefficient in $s_i \cdot v$
...on v is $1/(c_{i+1} - c_i)$
...on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$
...on any other path is 0.

The rule for s_i:

Suppose v goes with the path



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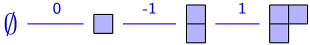
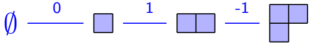
. . . on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$
. . . on any other path is 0.

Back to S_3:

v:

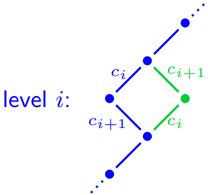


u:



The rule for s_i:

Suppose v goes with the path



and u is almost the same, except at the ith step.

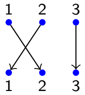
Let c_i be the content of the box added from $i-1$ to i.

Then the coefficient in $s_i \cdot v$

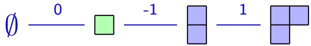
…on v is $1/(c_{i+1} - c_i)$

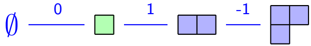
…on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$

…on any other path is 0.

$s_1 =$ 

Back to S_3:

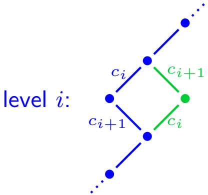
v: 

u: 

	v	u
v		
u		

The rule for s_i:

Suppose v goes with the path



and u is almost the same, except at the ith step.

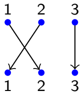
Let c_i be the content of the box added from $i-1$ to i.

Then the coefficient in $s_i \cdot v$

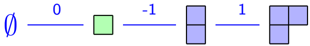
…on v is $1/(c_{i+1} - c_i)$

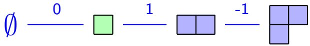
…on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$

…on any other path is 0.

$s_1 =$ 

Back to S_3:

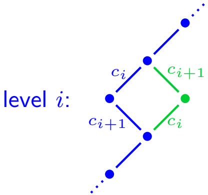
v: 

u: 

	v	u
v	$1/(-1-0)$	0
u	0	$1/(1-0)$

The rule for s_i:

Suppose v goes with the path



and u is almost the same, except at the ith step.

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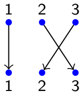
Then the coefficient in $s_i \cdot v$

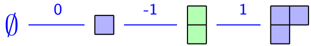
...on v is $1/(c_{i+1} - c_i)$

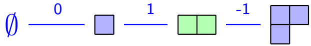
...on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$

...on any other path is 0.

$s_2 =$ 

Back to S_3:

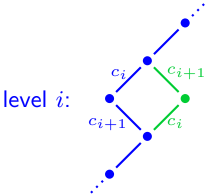
v: 

u: 

	v	u
v		
u		

The rule for s_i:

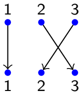
Suppose v goes with the path



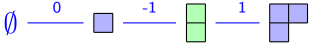
and u is almost the same, except at the ith step.

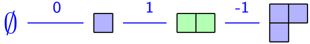
Let c_i be the content of the box added from $i-1$ to i.

Then the coefficient in $s_i \cdot v$
. . . on v is $1/(c_{i+1} - c_i)$
. . . on u is $\sqrt{1 - (1/(c_{i+1} - c_i))^2}$
. . . on any other path is 0.

$s_2 =$ 

Back to S_3:

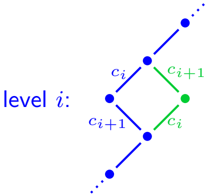
v: 

u: 

	v	u
v	$1/(1-(-1))$	
u		$1/(-1-1)$

The rule for s_i:

Suppose v goes with the path



and u is almost the same, except at the ith step.

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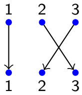
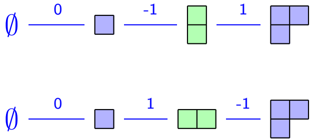
Then the coefficient in $s_i \cdot v$
…on v is $1/(c_{i+1}-c_i)$
…on u is $\sqrt{1-(1/(c_{i+1}-c_i))^2}$
…on any other path is 0.

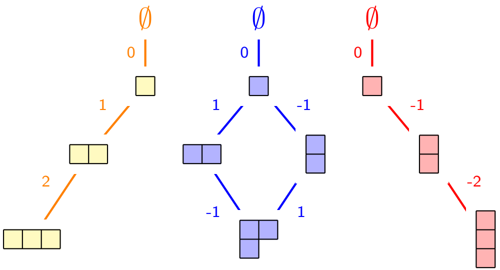
$s_2 =$ 

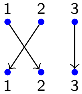
Back to S_3:

v: and u:

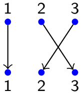


	v	u
v	$1/(1-(-1))$	$\sqrt{1-1/4}$
u	$\sqrt{1-1/4}$	$1/(-1-1)$





$$\begin{pmatrix}1\end{pmatrix} \qquad \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix} \qquad \begin{pmatrix}-1\end{pmatrix}$$

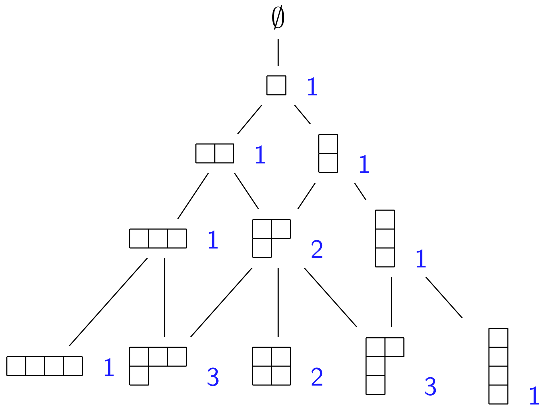


$$\begin{pmatrix}1\end{pmatrix} \qquad \begin{pmatrix}1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1\end{pmatrix} \qquad \begin{pmatrix}-1\end{pmatrix}$$

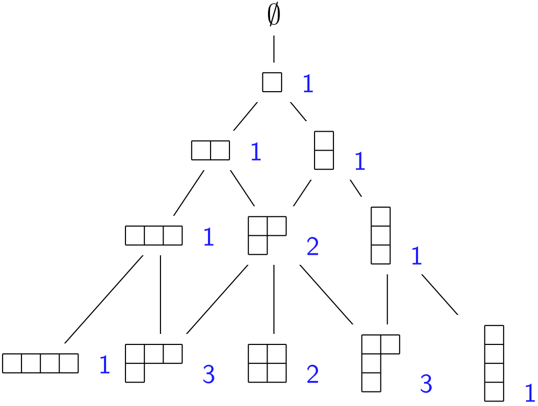
"trivial"

"alternating"

Counting tableaux and dimensions



Counting tableaux and dimensions



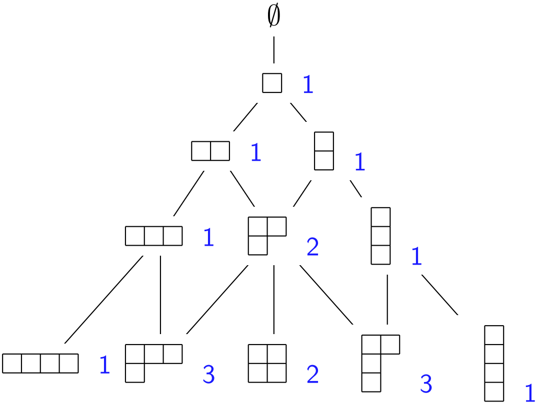
$$1 = 1$$

$$1 + 1 = 2$$

$$1 + 2 + 1 = 4$$

$$1 + 3 + 2 + 3 + 1 = 10$$

Counting tableaux and dimensions



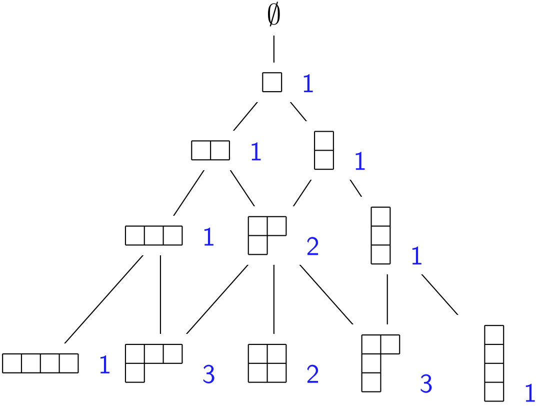
$$1^2 = 1$$

$$1^2 + 1^2 = 2$$

$$1^2 + 2^2 + 1^2 = 6$$

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 24$$

Counting tableaux and dimensions



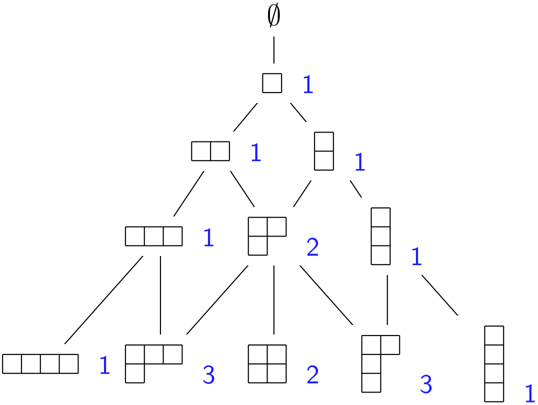
$$1^2 = 1!$$

$$1^2 + 1^2 = 2!$$

$$1^2 + 2^2 + 1^2 = 3!$$

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 4!$$

Counting tableaux and dimensions



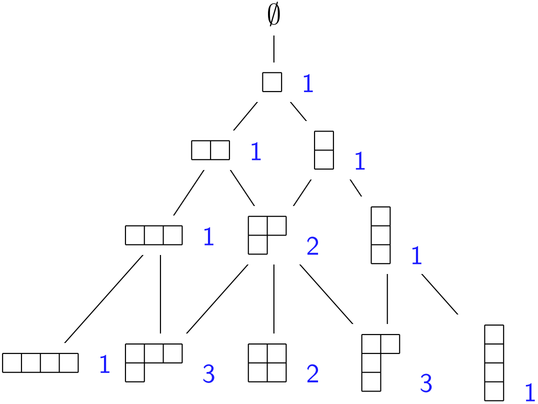
$$1^2 = |S_1|$$

$$1^2 + 1^2 = |S_2|$$

$$1^2 + 2^2 + 1^2 = |S_3|$$

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = |S_4|$$

Counting tableaux and dimensions



$1^2 = |S_1|$

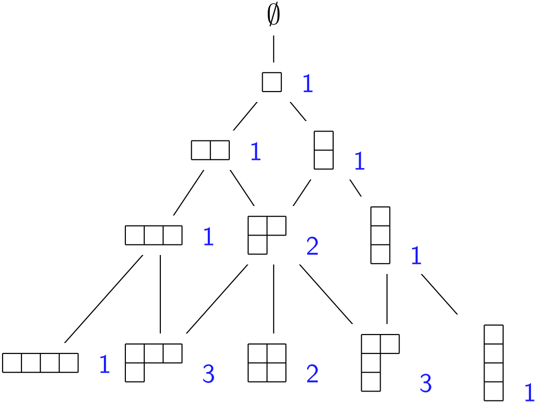
$1^2 + 1^2 = |S_2|$

$1^2 + 2^2 + 1^2 = |S_3|$

$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = |S_4|$

Wedderburn's theorem: "Nice" rings are isomorphic to the direct sum of matrix rings.

Counting tableaux and dimensions



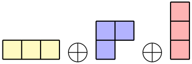
$1^2 = |S_1|$

$1^2 + 1^2 = |S_2|$

$1^2 + 2^2 + 1^2 = |S_3|$

$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = |S_4|$

Wedderburn's theorem: "Nice" rings are isomorphic to the direct sum of matrix rings.

$\mathbb{R}S_n \cong M_1(\mathbb{R}) \oplus M_2(\mathbb{R}) \oplus M_1(\mathbb{R}) \cong$ 

(The matrix ring on an m-dimensional v.s. is m^2-dimenstional)