# The quasi-partition algebra


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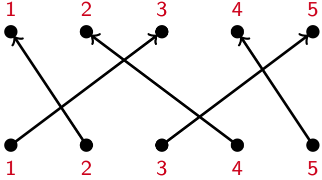
# Diagram algebras

Everyone's favorite diagram algebra:
Group algebra of the symmetric group $S_k$



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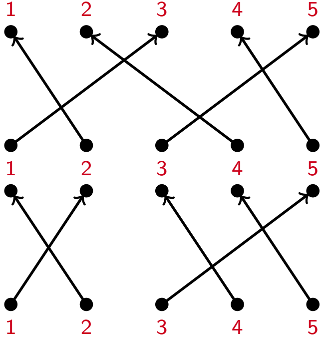


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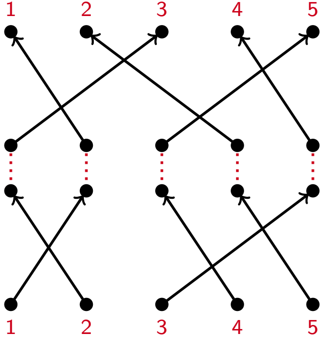


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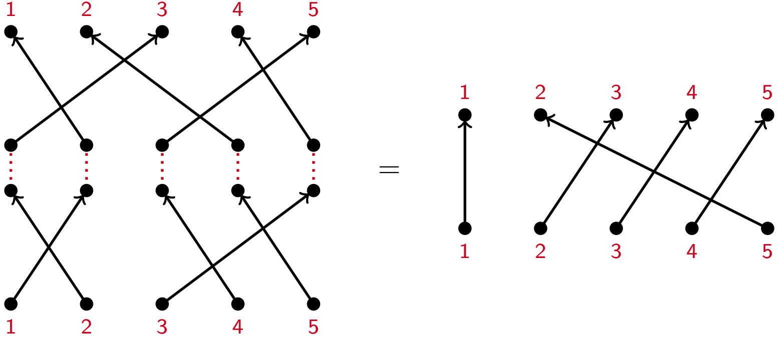


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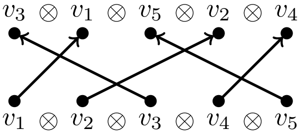

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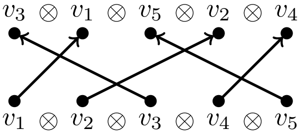
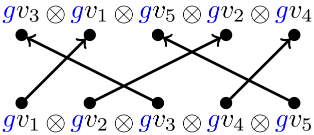



3. These actions commute!

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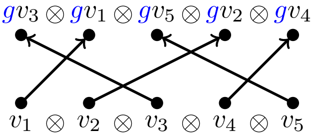

# Modules (vector spaces, with the group or algebra acting as matrices)

Simple $S_k$-modules are in bijection with partitions, $\lambda \vdash k$

$$\lambda = \ydiagram{4,3,1} \qquad \begin{array}{l} 4 \\ +3 \\ +1 \end{array}$$

(a collection of boxes piled up and to the left)

So, for example,

$$S^{\ydiagram{3}} \qquad S^{\ydiagram{2,1}} \qquad \text{and} \qquad S^{\ydiagram{1,1,1}}$$

are the simple $S_3$-modules (up to isomorphism).

# Schur-Weyl duality and centralizer algebras

**Classical example:** $S_k$ and $\mathrm{GL}_n$ have commuting actions on $(\mathbb{C}^n)^{\otimes k}$.
Even better, if $k \leq n$,

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\substack{\text{(all linear maps that} \\ \text{commute with } \mathrm{GL}_n\text{)}}} = \mathbb{C}S_k.$$

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## Why this is exciting:

Centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} L(\lambda) \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

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For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \left(L(\ydiagram{3}) \otimes S^{\ydiagram{3}}\right) \oplus \left(L(\ydiagram{2,1}) \otimes S^{\ydiagram{2,1}}\right) \oplus \left(L(\ydiagram{1,1,1}) \otimes S^{\ydiagram{1,1,1}}\right)$$

# Switching roles: the partition algebra

Let $V$ be the permutation representation of $S_n$.

$n \times n$ *matrices with 1's and 0's* i.e. $\sigma \cdot v_i = v_{\sigma(i)}$

Now let $S_n$ act diagonally on $V^{\otimes k}$:

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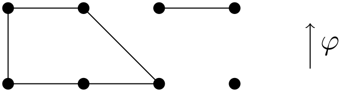

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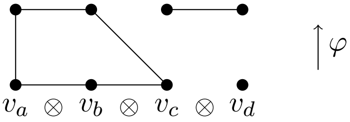

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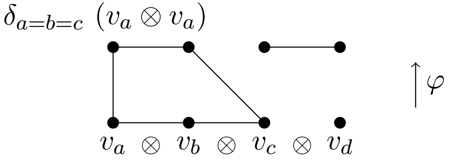

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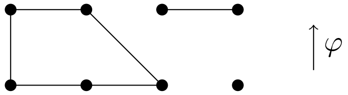
$$\delta_{a=b=c}\ (v_a \otimes v_a) \otimes \Big( \textstyle\sum_{i=1}^n v_i \otimes v_i \Big)$$




$$v_a \otimes v_b \otimes v_c \otimes v_d$$

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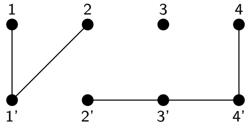

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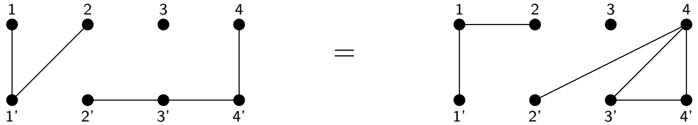


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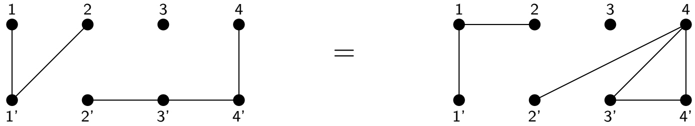


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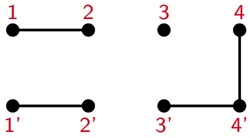

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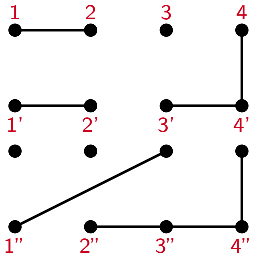

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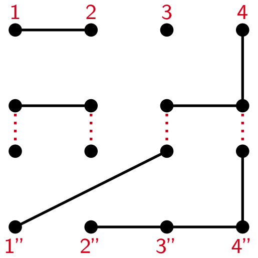

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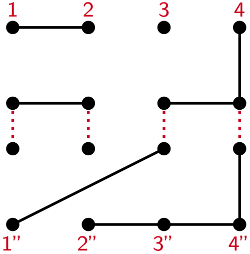


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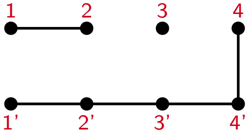

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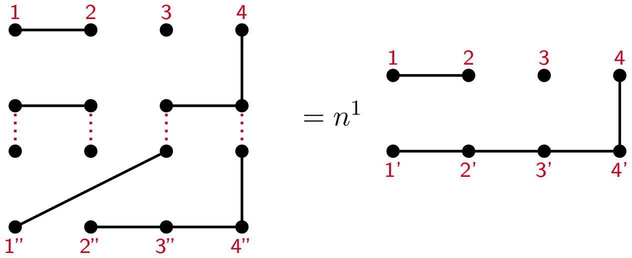


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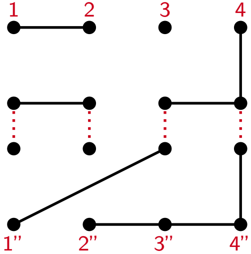


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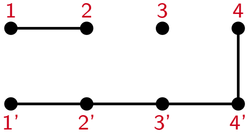


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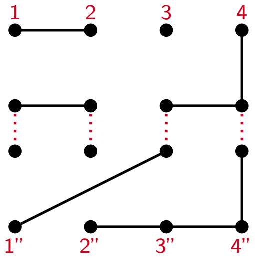


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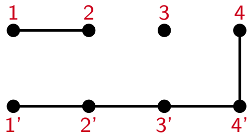


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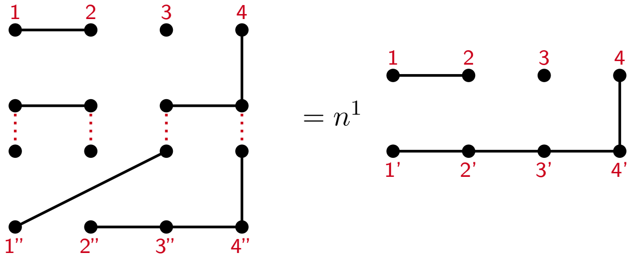


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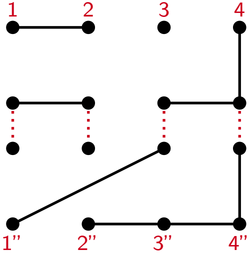


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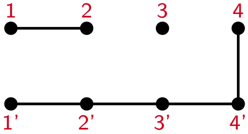


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($*$) $\mathrm{End}_{S_n}(V^{\otimes k}) = P_k(n)$ for $n \geq 2k$.

Problem: $V$ is not irreducible!

$$\begin{aligned} V &= \mathbb{C}\{v_1, \ldots, v_n\} \\ W &= \mathbb{C}\{w_2, \ldots, w_n\} && \text{where } w_i = v_i - v_1, \\ T &= \mathbb{C}v, && \text{where } v = v_1 + \cdots + v_n. \end{aligned}$$

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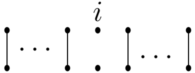
Let $p_i =$  Then $p_i : V^{\otimes k} \to V^{\otimes i-1} \otimes T \otimes V^{\otimes k-i}$.

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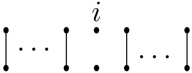
Let $p_i =$  Then $p_i : V^{\otimes k} \to V^{\otimes i-1} \otimes T \otimes V^{\otimes k-i}$.

Any diagram $d$ an isolated vertex satisfies $d = p_i d'$ or $d = d' p_i$.

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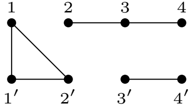
Goal: Express $\bar{d}$ in terms of $[d']$'s.

If $X$ is a set of vertices, the isolation of $d$ (at $X$) is $d_X$ , the diagram constructed from $d$ by isolating all vertices in $X$.

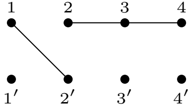
For example, if $X = \{1', 4'\}$ and

$d =$




then

$d_X =$




We can also place an order on diagrams, where $d' \leq d$ if $d'$ is a refinement of $d$. In particular, $d_X \leq d$.

Define the quasi-partition algebra as $QP_k(n) = \mathrm{End}_{S_n}(W^{\otimes k})$.
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*If $d \in \mathcal{D}$ then*

$$\bar{d} = [d] + \sum_{X \subseteq [k] \cup [k']} c_X [d_X],$$

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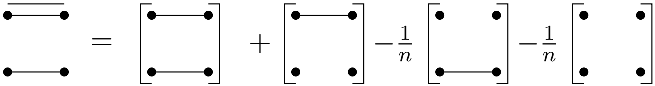

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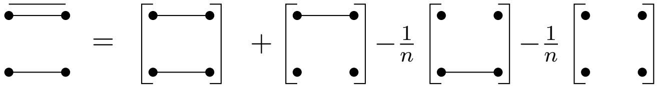


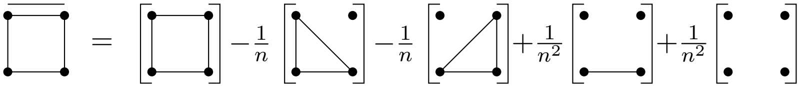

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*$QP_k(n)$ has basis $\{\bar{d} \mid d \in \mathcal{D}\}$, and thus has dimension*

$$\sum_{j=1}^{2k} (-1)^{j-1} B(2k-j) + 1, \qquad \text{where } B(r) \text{ is the Bell number.}$$

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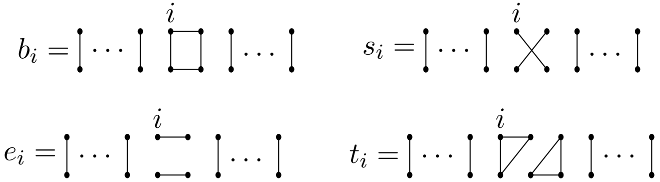
## Corollary

*If $d_1, d_2 \in \mathcal{D}$,*

$$\bar{d}_1 \bar{d}_2 = \sum_{d \leq d_1 d_2} c_d \bar{d}.$$

*In particular, if $d_1 d_2 \notin \mathcal{D}$, then $\bar{d}_1 \bar{d}_2 = 0$.*

So *functionally*, $QP_k(n)$ is a subalgebra of $P_k(n-1)$.
It's generated by projections of



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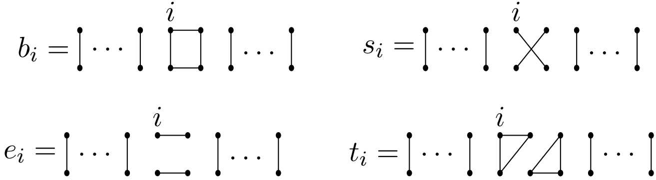


With relations that look like

| in $P_k(n-1)$ : | in $QP_k(n)$ : |
|---|---|
| $s_i^2 = 1$ | $\bar{s}_i^2 = 1$ |
| $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ | $\bar{s}_i \bar{s}_{i+1} \bar{s}_i = \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}$ |
| $e_i^2 = (n-1)e_i$ | $\bar{e}_i^2 = (n-1)\bar{e}_i$ |
| $b_i^2 = b_i$ | $\bar{b}_i^2 = \frac{n-2}{n}\bar{b}_i + \frac{1}{n^2}\bar{e}_i$ |

# Representation theory

Tensoring rule for $W = S^{(n-1,1)}$

$$S^\lambda \otimes W = c(\lambda) S^\lambda \oplus \bigoplus_{\mu \in \Lambda} S^\mu$$

where $\Lambda$ is the set of partitions gotten from $\lambda$ by moving any corner box to another place, and $c(\lambda) = \#$ corner boxes $-1$.

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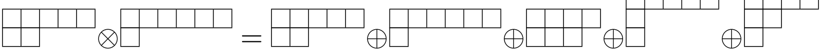

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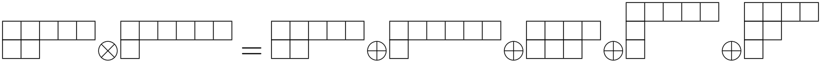
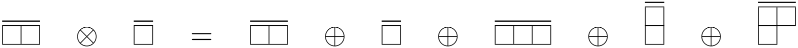


Assume $n >> 1$. We can forget the top row:



# Bratteli diagram for $QP_k(n)$

$\square$

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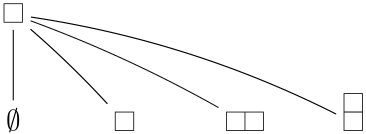

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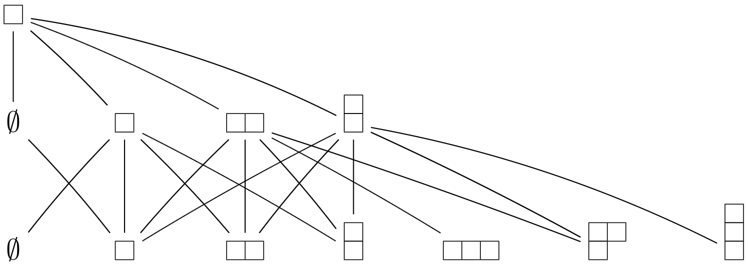

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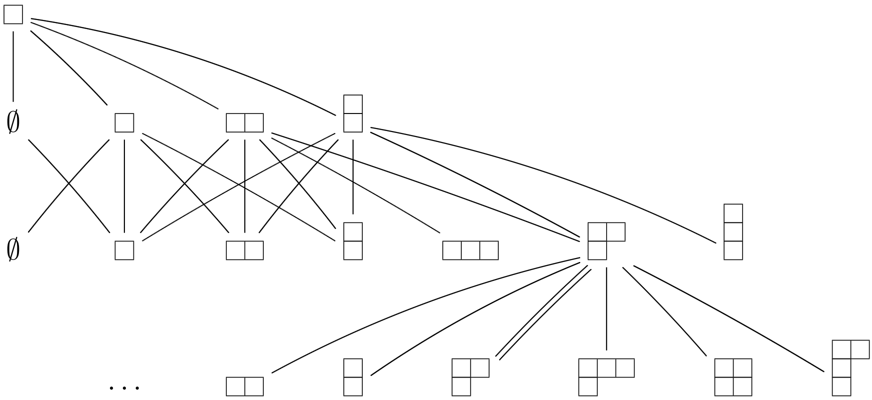

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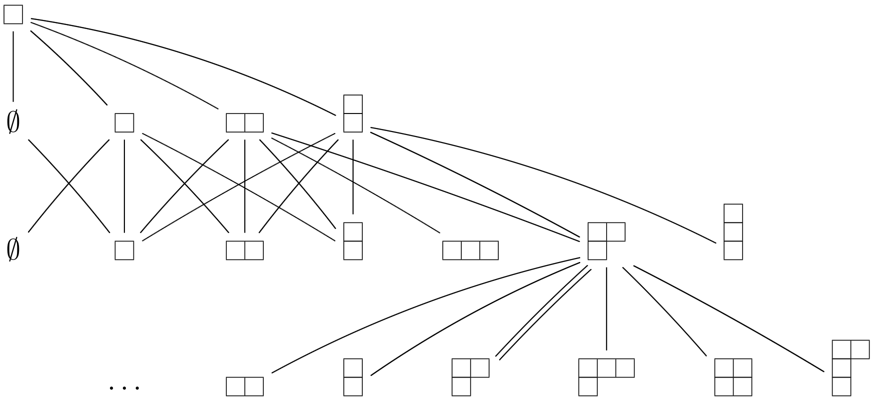

(Dimensions expressed explicitly in terms of numbers of standard tableaux and Bell numbers.)