The centers of the affine BMW algebra and its degenerate version

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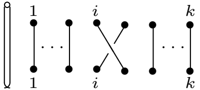
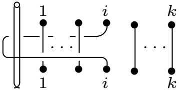
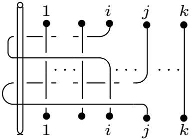
Definition

The *affine braid group* B_k is generated by

$T_i =$ 

, $Y_i =$ 

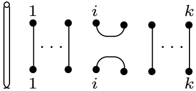
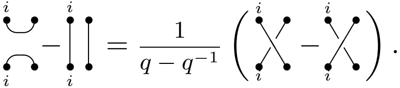
with multiplication given by concatenation, and braids behaving as they should. In particular,

$Y_iY_j =$ 

$= Y_jY_i.$

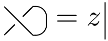
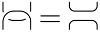
So $\mathbb{C}[\underbrace{Y_1^{\pm 1}, \dots, Y_k^{\pm 1}}_{\text{(can wrap backwards)}}]$ is a big commutative subalgebra of $\mathbb{C}B_k$!

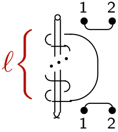
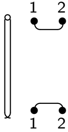
Fix $q \in \mathbb{C}^*$. Let

$E_i =$  be defined by .

Definition

Fix constants $z \in \mathbb{C}$, and $Z_\ell \in \mathbb{C}$, $\ell = 0, \pm 1, \pm 2, \ldots$.
The *affine Birman-Murakami-Wenzl (BMW) algebra* W_k is generated by $\mathbb{C}B_k$ with relations...

(many which amount to Ribbon R1:  and R2: )

and $E_1 Y_1^\ell E_1 =$  $= Z_\ell$ 

(because the loop should be central!)

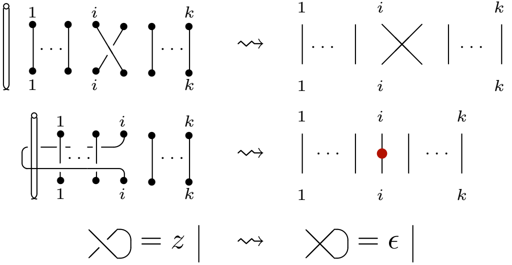
Degenerate versions

For our purposes, think Flatten!

$$q \rightsquigarrow 1$$

$$z \rightsquigarrow \pm 1$$

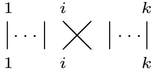
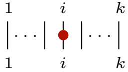
$$\text{tangle} \rightsquigarrow \log(\text{tangle})$$



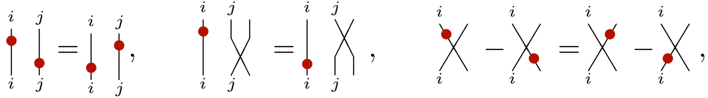
Philosophy: Algebraic properties (representations, centers, combinatorics) should look similar, and some computations are easier after degeneration.

Definition

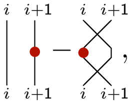
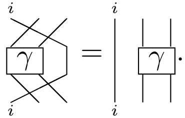
The *degenerate affine braid algebra* $\mathcal{B}_k$ is generated over $\mathbb{C}$ by

$t_i =$  and $y_i =$  ,

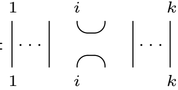
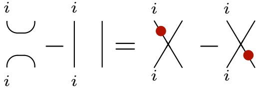
with multiplication given by concatenation, permutations behaving as they should, and relations



and

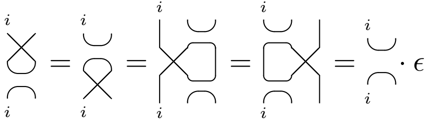
if $\gamma_{i,i+1} =$  , then 

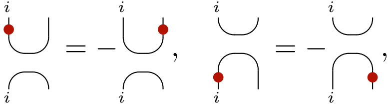
Let

$e_i =$  $\in \mathcal{B}_k$ be defined by .

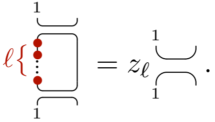
Definition

Fix constants $\epsilon = \pm 1$, and $z_\ell \in \mathbb{C}$, $\ell = 0, 1, 2, \ldots$. The *degenerate affine BMW algebra* $\mathcal{W}_k$ is generated by $\mathcal{B}_k$ with additional relations





and



Recall

$$\mathbb{C}[y_1, y_2, \ldots, y_k] = \left\{ \begin{array}{c} \text{polynomials in} \\ \text{dotted vertical strands} \end{array} \right\}$$

is a big commutative subalgebra of $\mathcal{W}_k$ and

$$\mathbb{C}[Y_1^{\pm 1}, Y_2^{\pm 1}, \ldots, Y_k^{\pm 1}] = \left\{ \begin{array}{c} \text{Laurent polynomials in} \\ \text{wrapping around the pole} \end{array} \right\}$$

is a big commutative subalgebra of W_k.

The symmetric group S_k acts on polynomials in k variables by permuting the variables, and we write

$$\mathbb{C}[y_1, y_2, \ldots, y_k]^{S_k} \quad \text{and} \quad \mathbb{C}[Y_1^{\pm 1}, Y_2^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k}$$

to mean the (Laurent) polynomials which are symmetric in the y_i's and Y_i's, respectively.

The graded Hecke algebra of type A is $\mathcal{H}_k = \mathcal{W}_k / \langle e_i = 0 \rangle$

Theorem (Lusztig, '89)

The center of $\mathcal{H}_k$ *is* $\mathbb{C}[y_1, y_2, \ldots, y_k]^{S_k}$.

Punchline: So we expect that the center of $\mathcal{W}_k$ is a subalgebra of the symmetric polynomials!

The affine Hecke algebra of type A is $H_k = W_k / \langle E_i = 0 \rangle$

Theorem (Bernstein-Zelevinsky, Lusztig '83)

The center of $\mathcal{H}_k$ *is* $\mathbb{C}[Y_1^{\pm 1}, Y_2^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k}$.

Punchline: So we expect that the center of W_k is a subalgebra of the symmetric Laurent polynomials!

Theorem (Daugherty, Ram, Virk)

1. *The center of the degenerate affine BMW algebra is*

$$\{p \in \mathbb{C}[y_1, y_2, \ldots, y_k]^{S_k} \mid \underbrace{p(y_1, -y_1, y_3, \ldots, y_k) = p(0, 0, y_3, \ldots, y_k)}_{\text{“}Q\text{-cancellation”}}\}$$

$\downarrow$ *exp* $\downarrow$

2. *The center of the affine BMW algebra is*

$$\{p \in \mathbb{C}[Y_1^{\pm 1}, Y_2^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} \mid p(Y_1, Y_1^{-1}, Y_3, \ldots, Y_k) = p(1, 1, Y_3, \ldots, Y_k)\}$$

$$\mathcal{R}_k = \{p \in \mathbb{C}[y_1, y_2, \ldots, y_k]^{S_k} \mid p(y_1, -y_1, y_3, \ldots, y_k) = p(0, 0, y_3, \ldots, y_k)\}$$

Notice: $p_i = y_1^i + y_2^i + \cdots + y_k^i$ is in this ring when i is odd.

Nazarov observed $Z(\mathcal{W}_k) = \mathbb{C}[p_1, p_3, \ldots]$ without proof.

Theorem (Pracacz, '91)

$$\mathcal{R}_k = \mathbb{C}\langle \text{ Schur Q-functions } \rangle = \mathbb{C}[p_1, p_3, \ldots]$$

Interesting connections:

1. Pragacz: $\mathcal{R}_k$ appears as the cohomology of orthogonal and symplectic Grassmannians.
2. Lam: $\mathbb{Z}[p_1, p_3, \ldots]$ appears as the cohomology of the loop Grassmannian for the symplectic group.
3. The induction in Pragacz depends on the same symmetric function which appears in studying polynomial quotients (Ariki, Mathas, Rui) and central recursions (Nazarov) of $\mathcal{W}_k$.

$$R_k = \{p \in \mathbb{C}[Y_1^{\pm 1}, Y_2^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} \mid p(Y_1, Y_1^{-1}, Y_3, \ldots, Y_k) = p(1, 1, Y_3, \ldots, Y_k)\}$$

1. Is there a nice analog for Schur Q-functions which satisfy this cancellation property?

(We'll look at the analogous functions showing up in central recursions for affine BMW)

2. Notice:
$$P_i^- = p_i - p_{-i} = Y_1^i + Y_2^i + \cdots + Y_k^i - \left(Y_1^{-i} + Y_2^{-i} + \cdots + Y_k^{-i}\right)$$

and $\mathcal{E}_k = Y_1 Y_2 \cdots Y_k$ are in this ring.

Is $R_k = \mathbb{C}[\mathcal{E}_k, P_1^-, P_2^-, \ldots]$?

(True in infinitely many variables. Can we learn from 1?)

3. Does the nice analog speak to K-theory?

For more:

[Na] M. Nazarov, *Youngs Orthogonal Form for Brauers Centralizer Algebra*, (1996).

[OR] R. Orellana and A. Ram, *Affine braids, Markov traces and the category $\mathcal{O}$*, 2007.

[Pr] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials*, 1991.

In preparation:

[DRV] Z. Daugherty, A. Ram, R. Virk, *Affine and graded BMW algebras*

http://ms.unimelb.edu.au/∼ram/notes.html
http://www.stolaf.edu/people/daugherz/

Thank you!