Two-boundary Hecke algebras and the graded Hecke algebra of type C

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Warm-up with Schur-Weyl duality

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For $n \geq k$, the *centralizer* of the action of $\mathrm{GL}_n(\mathbb{C})$ on $V^{\otimes k}$ in $\mathrm{End}(V^{\otimes k})$ is

$$\mathrm{End}_{\mathrm{GL}_n(\mathbb{C})}(V^{\otimes k}) \cong \mathbb{C}S_k.$$

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Bigger deal:

Centralizer relationship produces

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ ht(\lambda) \leq n}} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

where G^λ are distinct irreducible GL_n-modules
S^λ are distinct irreducible S_k-modules

The set up

Let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra.

e.g. $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$.

Let M, N, and V be finite dimensional simple $\mathfrak{g}$-modules.

Our goal:

Understand $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.
(the set of endomorphisms which commute with the action of $\mathfrak{g}$)

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.

Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

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Quantized versions yield standard and affine type A Hecke and Birman-Murakami-Wenzl algebra modules (Orellana & Ram, 2007)

First big question:

Is there an algebra which has centralizers $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients?

Definition

The *degenerate two-boundary braid group* $\mathcal{G}_k$ is the $\mathbb{C}$-algebra generated by

$$\mathbb{C}S_k = \mathbb{C}\left\langle t_i \;\middle|\; \begin{array}{cc} i = 1, \dots k & \\ t_i^2 = 1 & \\ t_i t_j = t_j t_i & |i-j| > 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} & \end{array} \right\rangle$$

$$\mathbb{C}[z_0, z_1, \dots, z_k],\ \mathbb{C}[y_1, \dots, y_k],\ \mathbb{C}[x_1, \dots, x_k]$$

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and relations...

$t_i x_j = x_j t_i, \quad t_i y_j = y_j t_i, \quad t_i z_j = z_j t_i,$ for $j \neq i, i+1$
$(z_0 + \cdots + z_i)\, x_j = x_j\,(z_0 + \cdots + z_i), \quad (z_0 + \cdots + z_i)\, y_j = y_j\,(z_0 + \cdots + z_i),$ for $i \geq j$
$t_i(x_i + x_{i+1}) = (x_i + x_{i+1})t_i, \quad t_i(y_i + y_{i+1}) = (y_i + y_{i+1})t_i,$ for $1 \leq i \leq k-1$
$(t_i t_{i+1})\left(x_{i+1} - t_i x_i t_i\right)(t_{+i} t_i) = x_{i+2} - t_{i+1} x_{i+1} t_{i+1}$
$(t_i t_{i+1})\left(y_{i+1} - t_i y_i t_i\right)(t_{+i} t_i) = y_{i+2} - t_{i+1} y_{i+1} t_{i+1}$ for $1 \leq i \leq k-2$,
$x_{i+1} - t_i x_i t_i = y_{i+1} - t_i y_i t_i$ for $1 \leq i \leq k-1$,
$z_i = x_i + y_i - m_i, \quad 1 \leq i \leq k,$
where if $m_{i,j} = \begin{cases} x_{i+1} - t_i x_i t_i & \text{if } j = i+1, \\ (i+1\ j) m_{i,i+1} (i+1\ j) & \text{if } j \neq i, i+1, \end{cases}$ then $m_1 = 0$, $m_i = \sum_{1<j<i} m_{i,j}$.

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$$\mathbb{C}[z_0, z_1, \dots, z_k],\; \mathbb{C}[y_1, \dots, y_k],\; \mathbb{C}[x_1, \dots, x_k]$$

and relations twisting the four factors together...

$\mathcal{G}_k$ contains three images of the graded braid group:

$$\frac{\mathbb{C}[z_1, \dots, z_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[y_1, \dots, y_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}S_k}{\sim}$$

and

$$z_i = x_i + y_i - \textit{lower terms}.$$

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by nested central elements of $\mathcal{U}\mathfrak{g}$.

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Theorem (D.)

Φ is a representation of $\mathcal{G}_k$ which commutes with the action of $\mathfrak{g}$.

An Example:

Is there an algebra which has centralizers $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients when $\mathfrak{g}$ is of type A?

Definition

Fix $a, b, p, q \in \mathbb{Z}_{>0}$.
The *degenerate extended two-boundary Hecke algebra* $\mathcal{H}_k^{\text{ext}}$ is the quotient of the degenerate two-boundary braid group by the relations

$$\begin{aligned} t_i x_i &= x_{i+1} t_i - 1, \\ t_i y_i &= y_{i+1} t_i - 1, \qquad i = 1, \ldots, k-1. \\ t_i z_i &= z_{i+1} t_i - 1, \end{aligned}$$

$$(x_1 - a)(x_1 + p) = 0 \qquad (y_1 - b)(y_1 + q) = 0.$$

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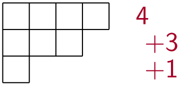
The *degenerate two-boundary Hecke algebra* $\mathcal{H}_k$ is the subalgebra of $\mathcal{H}_k^{\text{ext}}$ generated by

$$x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k, t_1, \ldots, t_{k-1}.$$

(everything but z_0... we'll come back to this.)

A *partition* is a collection of boxes:

$\lambda =$ 

A *partition* is a collection of boxes:

$$\lambda = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} -2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

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$$c(B) = j - i.$$

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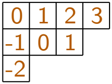
$\lambda =$ 

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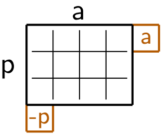
$(a^p) =$ 

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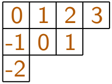
$\lambda =$ 

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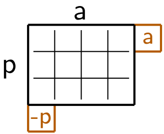
$(a^p) =$ 

(recall relations $(x_1 - a)(x_1 + p) = 0$ and $(y_1 - b)(y_1 + q) = 0$)

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Fix $k < n$ non-neg. integers.
Let $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$, $N = L((b^q))$, and $V = L((1^1))$.

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(1) *Φ is a rep. of $\mathcal{H}_k^{\text{ext}}$ which commutes with the $\mathfrak{g}$-action, so*

$$\Phi(\mathcal{H}_k^{\text{ext}}) \subseteq \text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

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Remark

(1) When Φ is not surjective, the image differs by a portion of the action of the center of $\mathcal{U}\mathfrak{g}$ on $M \otimes N$.

(2) Same results for $\mathfrak{g} = \mathfrak{sl}_n$ and a shift of Φ.

Let $M = L((a^p))$ and $N = L((b^q))$. Then

$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

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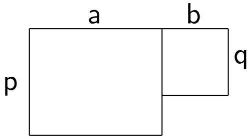
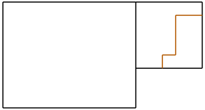


Let $M = L((a^p))$ and $N = L((b^q))$. Then

$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

where Λ is the following set of partitions: (Okada)



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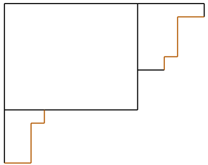

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$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

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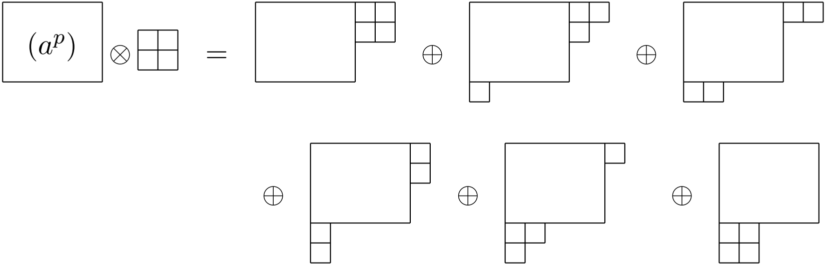
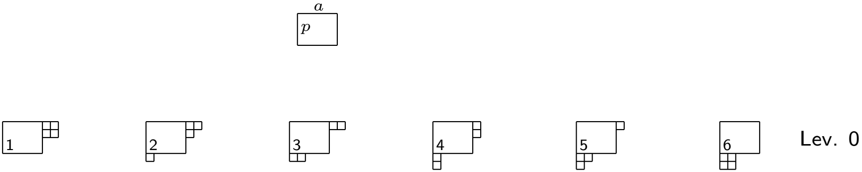
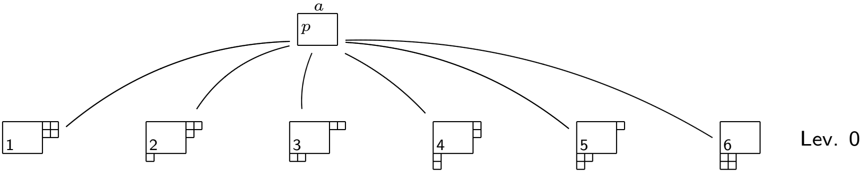
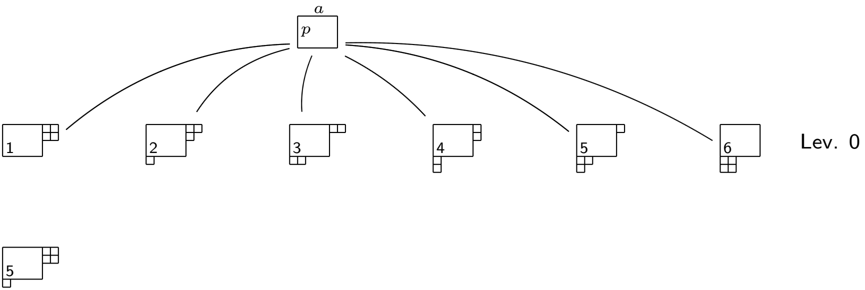
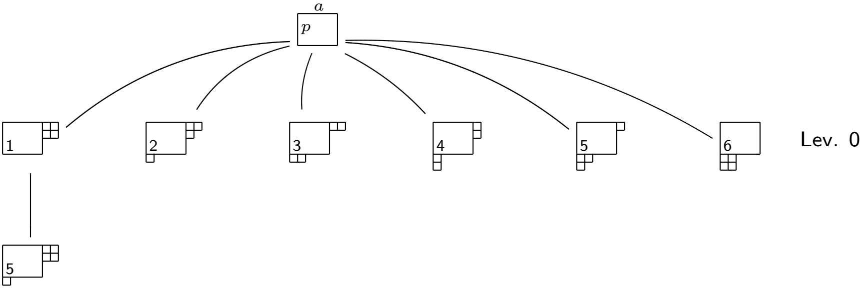


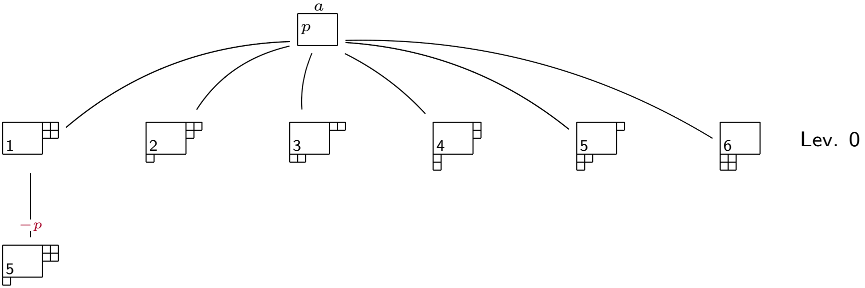

a
p

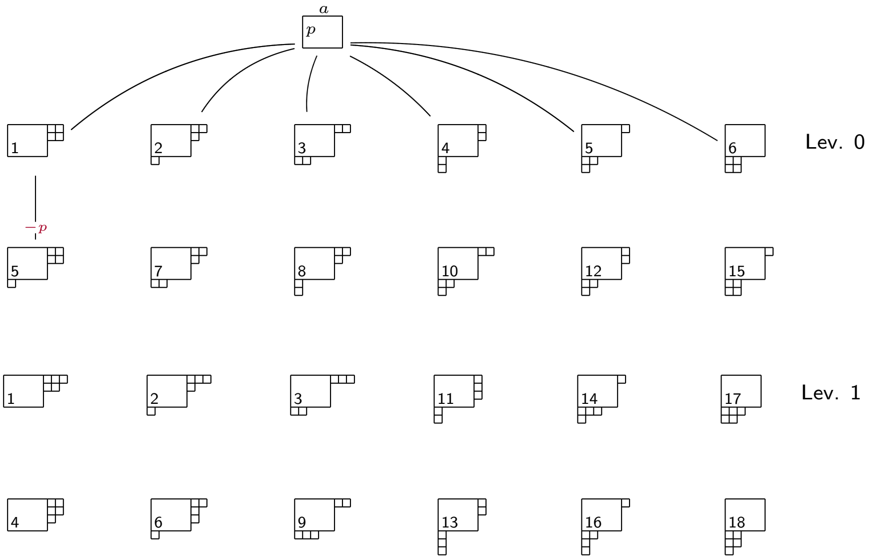

a
p
1
2
3
4
5
6
Lev. 0

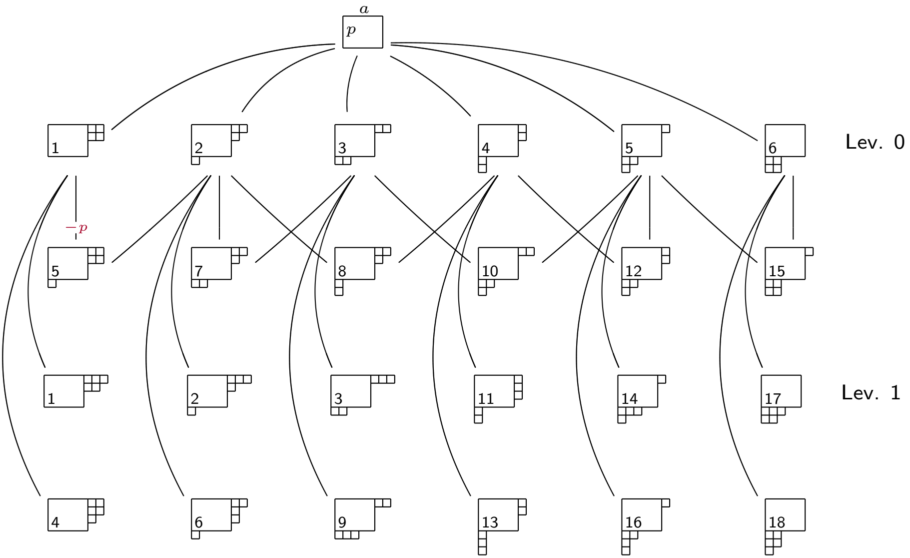

a
p
1
2
3
4
5
6
Lev. 0

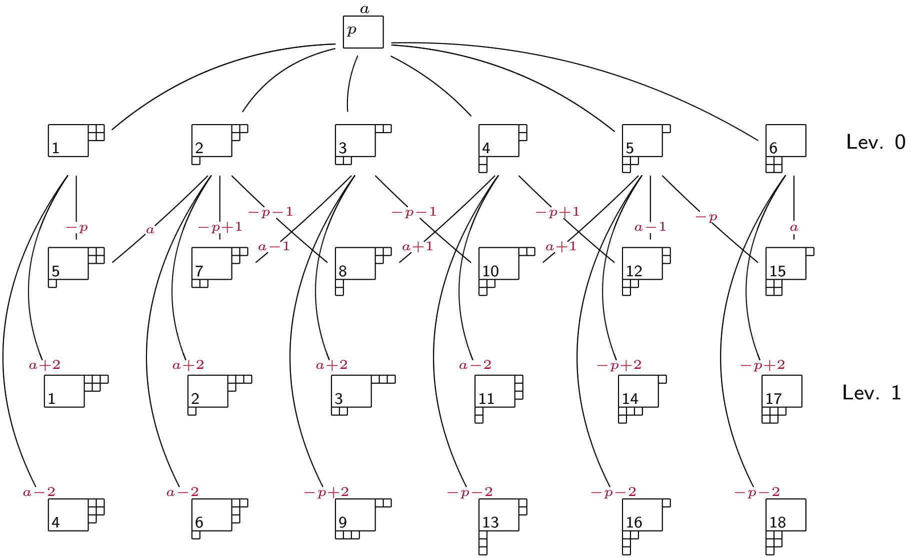

a
p
1
2
3
4
5
6
Lev. 0
5

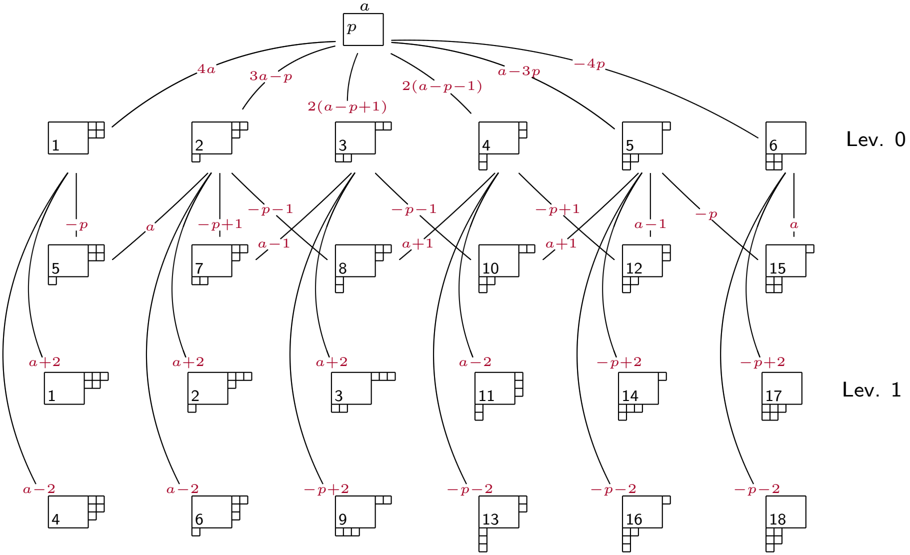

a
p
1
2
3
4
5
6
Lev. 0
5

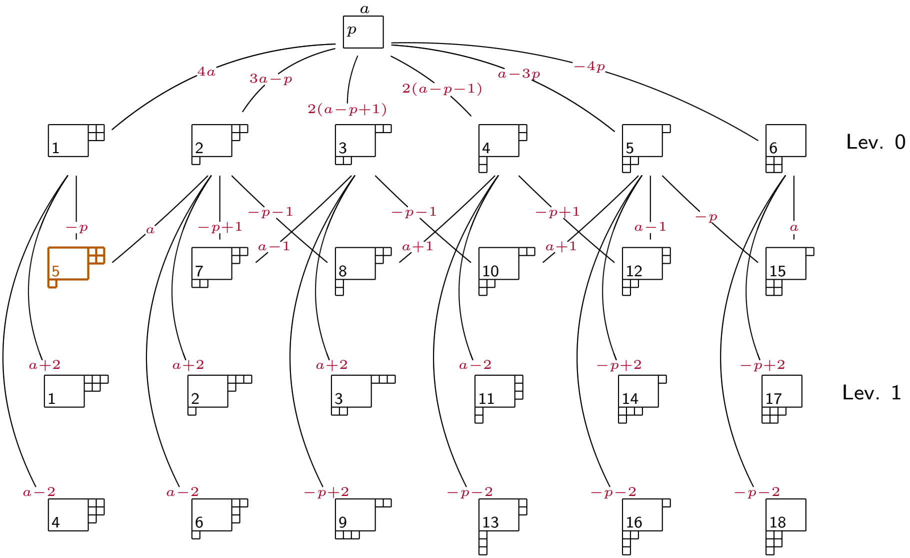

a
p
1
2
3
4
5
6
Lev. 0
−p
5

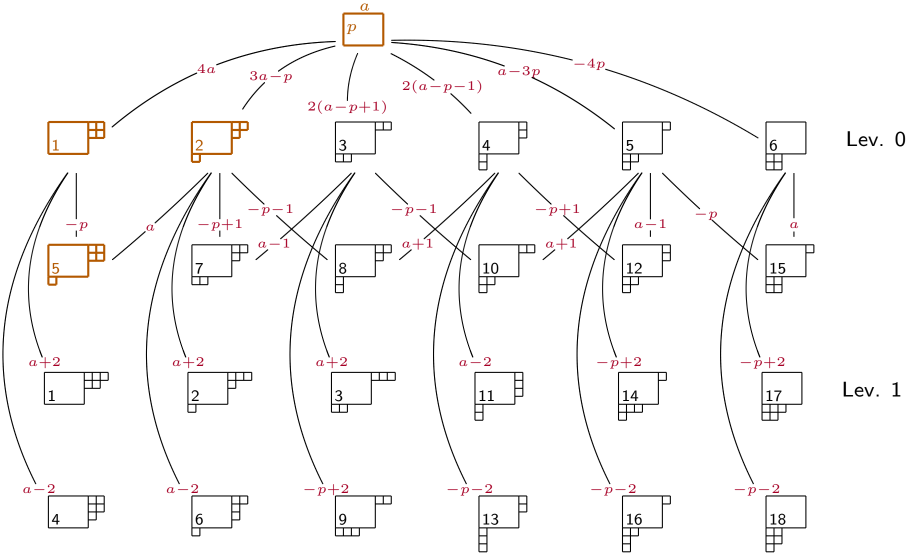

a
p
1
2
3
4
5
6
Lev. 0
−p
5
7
8
10
12
15
1
2
3
11
14
17
Lev. 1
4
6
9
13
16
18


a
p
1
2
3
4
5
6
Lev. 0
−p
5
7
8
10
12
15
1
2
3
11
14
17
Lev. 1
4
6
9
13
16
18


a
p
1
2
3
4
5
6
Lev. 0
−p
a
−p+1
−p−1
a−1
−p−1
a+1
−p+1
a+1
a−1
−p
a
5
7
8
10
12
15
a+2
a+2
a+2
a−2
−p+2
−p+2
1
2
3
11
14
17
Lev. 1
a−2
a−2
−p+2
−p−2
−p−2
−p−2
4
6
9
13
16
18

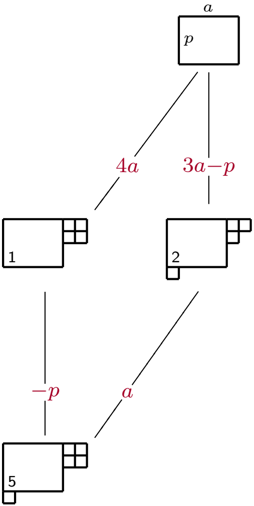

a
p
4a
3a − p
2(a − p + 1)
2(a − p − 1)
a − 3p
−4p
1
2
3
4
5
6
Lev. 0
−p
a
−p + 1
−p − 1
a − 1
−p − 1
a + 1
−p + 1
a + 1
a − 1
−p
a
5
7
8
10
12
15
a + 2
a + 2
a + 2
a − 2
−p + 2
−p + 2
1
2
3
11
14
17
Lev. 1
a − 2
a − 2
−p + 2
−p − 2
−p − 2
−p − 2
4
6
9
13
16
18


a
p
4a
3a − p
2(a − p + 1)
2(a − p − 1)
a − 3p
−4p
1
2
3
4
5
6
Lev. 0
−p
a
−p + 1
−p − 1
a − 1
−p − 1
a + 1
−p + 1
a + 1
a − 1
−p
a
5
7
8
10
12
15
a + 2
a + 2
a + 2
a − 2
−p + 2
−p + 2
1
2
3
11
14
17
Lev. 1
a − 2
a − 2
−p + 2
−p − 2
−p − 2
−p − 2
4
6
9
13
16
18


a
p
4a
3a−p
2(a−p+1)
2(a−p−1)
a−3p
−4p
Lev. 0
Lev. 1
−p
a
−p+1
−p−1
a−1
a+1
−p+1
a−1
−p
a+2
a−2
−p+2
−p−2

A two-dimensional $\mathcal{H}_1^{\mathrm{ext}}$-module:

$$\mathcal{H}_1^{\mathrm{ext}} = \mathbb{C}\langle z_0, z_1, x_1, y_1 \rangle$$



$$z_0 = \begin{pmatrix} 4a & 0 \\ 0 & 3a-p \end{pmatrix}$$

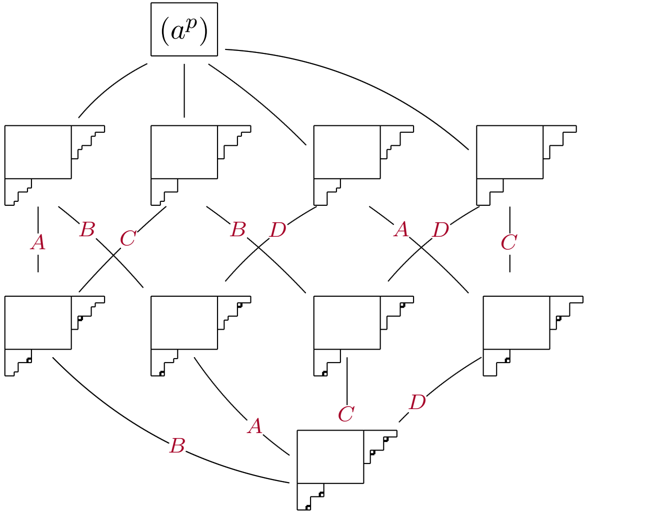
$$z_1 = \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$x_1 \sim \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$y_1 \sim \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

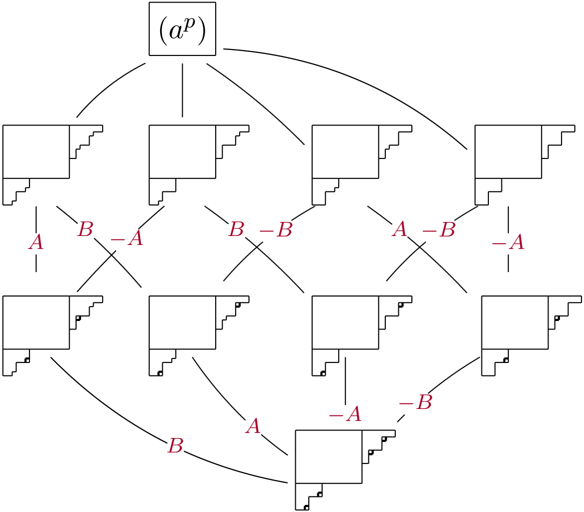
(formulas x_1, y_1, z_1, z_0 all given in terms of contents of added boxes)

An eight-dimensional $\mathcal{H}_2$-module:



where $C = -A + (a - p + b - q)$ and $D = -B + (a - p + b - q)$

An eight-dimensional $\mathcal{H}_2$-module:

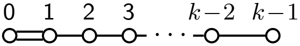


Shift! Label edges by action of $z_1 - \frac{1}{2}(a-p+b-q)$ and $z_2 - \frac{1}{2}(a-p+b-q))$

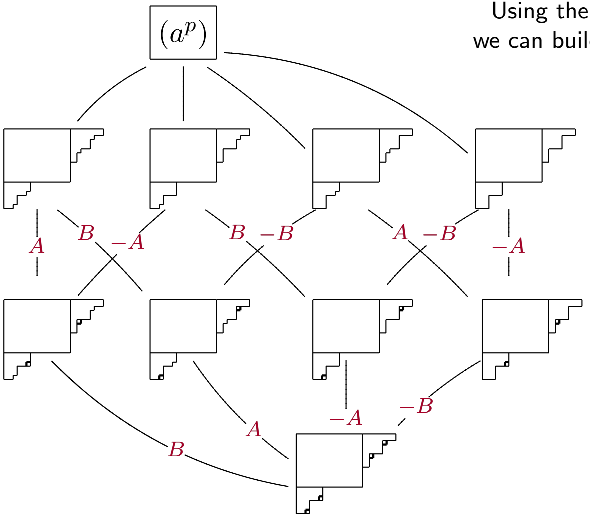
The type C Weyl group W is generated by s_0, s_1, ..., s_{k-1}, with relations

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ for } |i-j| > 1, \quad s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \qquad \text{and}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{for } i = 1, \ldots, k-2.$$



An eight-dimensional $\mathcal{H}_2$-module:

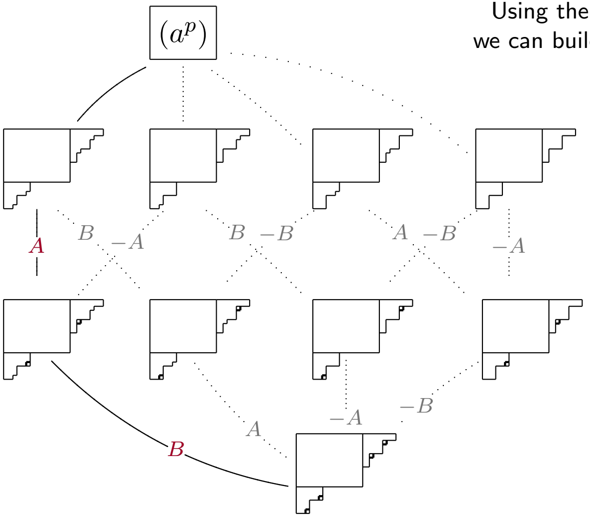


Using the same representation, we can build operators from $\mathcal{H}_2$:

s_0 changes level 0

s_1 changes level 1

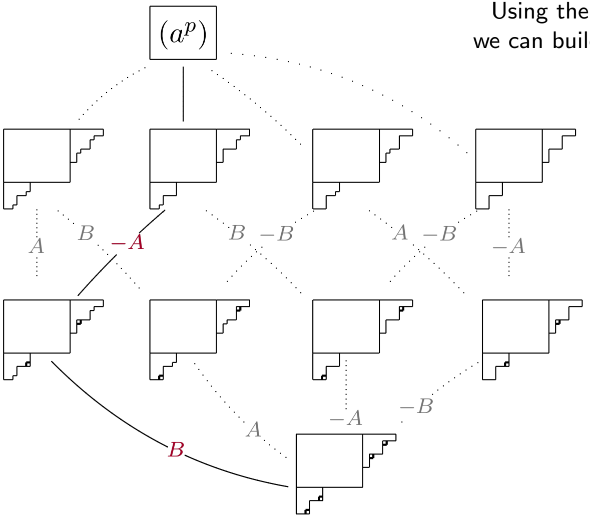
An eight-dimensional $\mathcal{H}_2$-module:



Using the same representation, we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

$1 \qquad (A, B)$

An eight-dimensional $\mathcal{H}_2$-module:



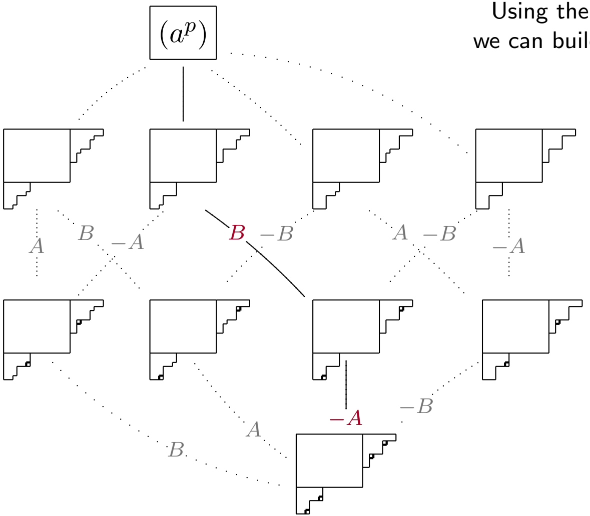
Using the same representation, we can build operators from $\mathcal{H}_2$:

s_0 changes level 0

s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$

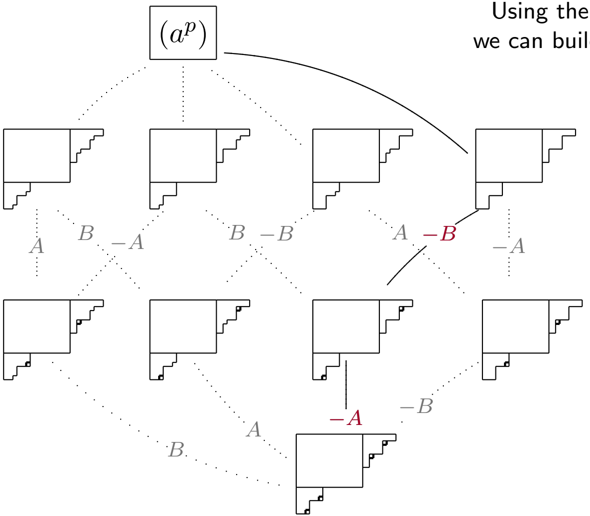
An eight-dimensional $\mathcal{H}_2$-module:



Using the same representation, we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$

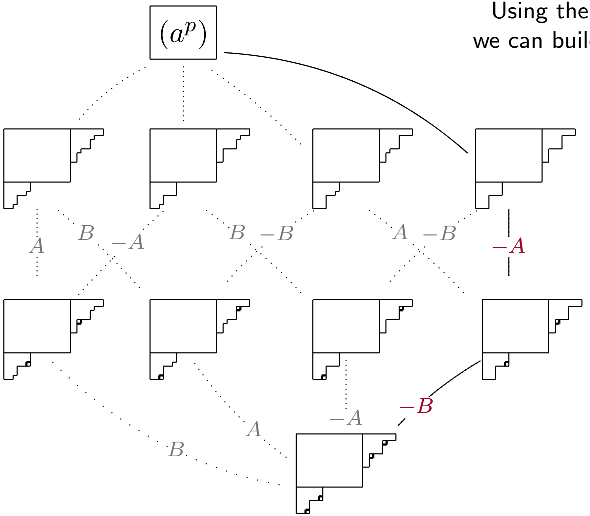
An eight-dimensional $\mathcal{H}_2$-module:



Using the same representation,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$

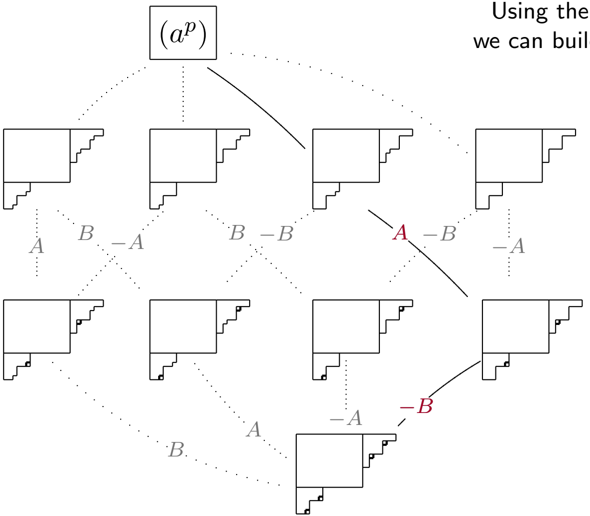
An eight-dimensional $\mathcal{H}_2$-module:



Using the same representation, we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$

An eight-dimensional $\mathcal{H}_2$-module:



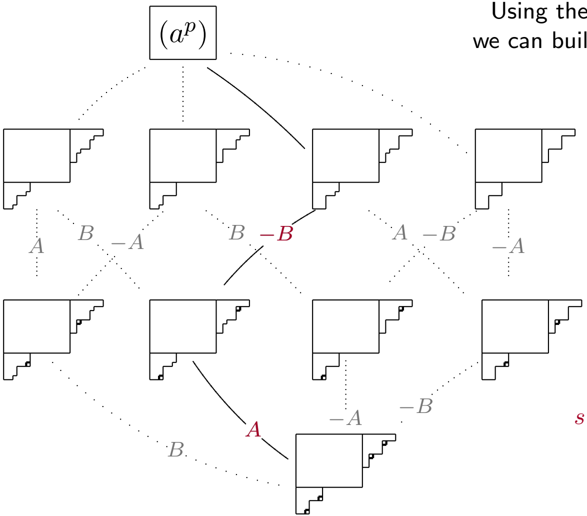
Using the same representation, we can build operators from $\mathcal{H}_2$:

s_0 changes level 0

s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$

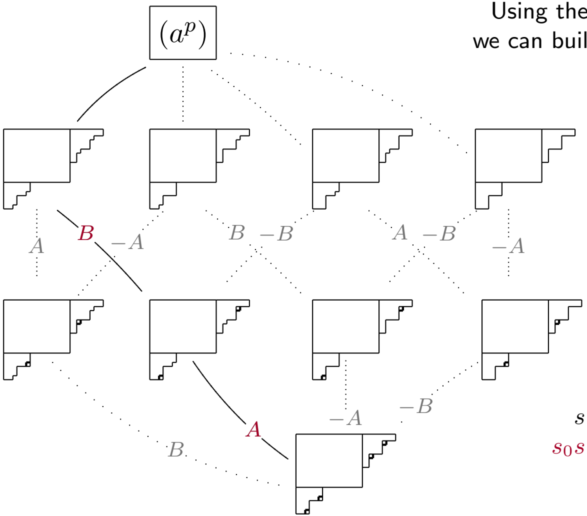
An eight-dimensional $\mathcal{H}_2$-module:



Using the same representation,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$
$s_1 s_0 s_1 s_0 s_1 s_0$	$(-B, A)$

An eight-dimensional $\mathcal{H}_2$-module:



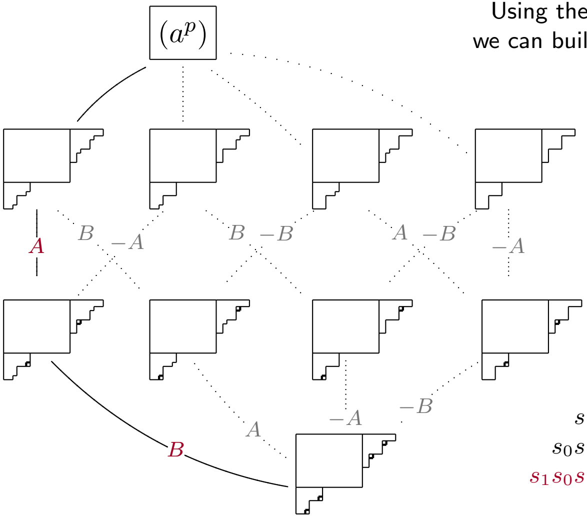
Using the same representation, we can build operators from $\mathcal{H}_2$:

s_0 changes level 0

s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$
$s_1 s_0 s_1 s_0 s_1 s_0$	$(-B, A)$
$s_0 s_1 s_0 s_1 s_0 s_1 s_0$	(B, A)

An eight-dimensional $\mathcal{H}_2$-module:



Using the same representation, we can build operators from $\mathcal{H}_2$:

s_0 changes level 0

s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$
$s_1 s_0 s_1 s_0 s_1 s_0$	$(-B, A)$
$s_0 s_1 s_0 s_1 s_0 s_1 s_0$	(B, A)
$s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0$	(A, B)

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$\mathcal{H}_k$ is presented by generators

$$x_1, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i - j| > 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$(x_1 - a)(x_1 + p) = 0, \qquad x_1(t_1 x_1 t_1 + t_1) = (t_1 x_1 t_1 + t_1) x_1$$

$$t_i w_i = w_{i+1} t_i - 1, \quad t_i w_j = w_j t_i, \qquad \text{for } j \neq i, i+1,$$

$$x_1 w_i = w_i x_1 \qquad \text{and} \qquad x_1 t_i = t_i x_1, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$x_1 w_1 = -w_1 x_1 + (a - p) w_1 + w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right).$$

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$\mathcal{H}_k$ is presented by generators

$$x_1, t_1, \dots, t_{k-1}, w_1, \dots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| > 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$(x_1 - a)(x_1 + p) = 0, \qquad x_1(t_1 x_1 t_1 + t_1) = (t_1 x_1 t_1 + t_1) x_1$$

$$t_i w_i = w_{i+1} t_i - 1, \quad t_i w_j = w_j t_i, \qquad \text{for } j \neq i, i+1,$$

$$x_1 w_i = w_i x_1 \qquad \text{and} \qquad x_1 t_i = t_i x_1, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \dots, k,$$

and

$$x_1 w_1 = -w_1 x_1 + (a - p) w_1 + w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right).$$

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$\mathcal{H}_k$ is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| > 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \frac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_i w_i = w_{i+1} t_i - 1, \quad t_i w_j = w_j t_i, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \frac{2}{a+p} \left(w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

where $t_0 = \frac{1}{a+p}(2x_1 - (a-p))$.

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$\mathcal{H}_k$ is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| > 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \frac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_i w_i = w_{i+1} t_i - 1, \quad t_i w_j = w_j t_i, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \ \frac{2}{a+p} \left(w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

$$\text{where } t_0 = \frac{1}{a+p}(2x_1 - (a-p)).$$

Let $w_i = z_i - \frac{1}{2}(a-p+b-q)$.

The graded Hecke algebra of type C is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| > 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \frac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_i w_i = w_{i+1} t_i - 1, \quad t_i w_j = w_j t_i, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \frac{2}{a+p} \left(w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

$$\text{where } t_0 = \frac{1}{a+p}(2x_1 - (a-p)).$$

Let $w_i = z_i - \frac{1}{2}(a-p+b-q)$.

The graded Hecke algebra of type C is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j|>1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \frac{2}{(a+p)}\left(t_1 t_0 - t_0 t_1\right)$$

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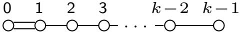
Recall: The type C Weyl group W is generated by $s_0, s_1, \ldots, s_{k-1}$, with relations



For each element $\mathrm{w} \in W$, fix a preferred word $\mathrm{w} = s_{i_1} s_{i_2} \cdots$ of minimal length. Then let $t_{\mathrm{w}} = t_{i_1} t_{i_2} \cdots$.

Let $w^\lambda = w_1^{\lambda_1} \cdots w_k^{\lambda_k}$, where $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$.

Recall: The type C Weyl group W is generated by s_0, s_1, ..., s_{k-1},
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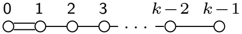

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Fact (by definition)

The graded Hecke algebra of type C
has basis

$$\{w^\lambda \, t_{\mathrm{w}} \mid \mathrm{w} \in W, \lambda \in \mathbb{Z}^k_{\geq 0}\}$$

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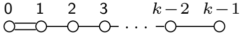

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Monomials in the w's — *Elements of the reflection group*

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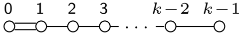

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Theorem (D.)

The degenerate two-boundary Hecke algebra $\mathcal{H}_k$ has basis

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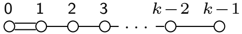
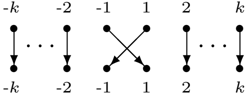
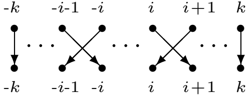


W permutes $\{-k, \ldots, -1, 1, \ldots, k\}$ by

$s_0 =$ 

$s_i =$ 

Define $w_{-i} = -w_i$. Then W acts on monomials w^λ by

$$\mathrm{w} \cdot (w_1^{\lambda_1} w_2^{\lambda_2} \cdots w_k^{\lambda_k}) = w_{\mathrm{w}(1)}^{\lambda_1} w_{\mathrm{w}(2)}^{\lambda_2} \cdots w_{\mathrm{w}(k)}^{\lambda_k}$$

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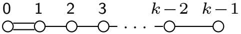

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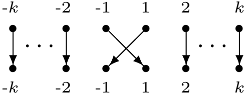
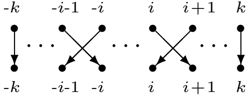
$s_0 =$ 

$s_i =$ 

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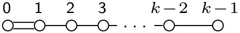
Theorem (Lusztig)

The graded Hecke algebra of type C
has center

$$\mathbb{C}[w_1, \ldots, w_k]^W,$$

symmetric polynomials in the w's with respect to the action of W.

Recall: The type C Weyl group W is generated by $s_0, s_1, \ldots, s_{k-1}$, with relations



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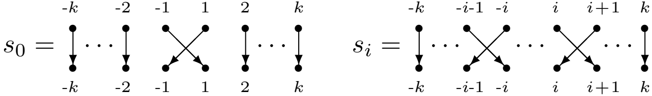

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The degenerate two-boundary Hecke algebra $\mathcal{H}_k$ has center

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Up next for $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

1. When $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, and M and N are rectangular, we get the degenerate (extended) two-boundary Hecke algebra.
    1. What are the intertwining operators? What is the correspondence between type C Hecke modules and $\mathcal{H}_k$ modules?
    2. How does the center act?
    3. Develop the combinatorics: cool dimension formulas? familiar tableaux games?
    4. Quantized versions yield two-boundary Hecke algebras.
2. When $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, and M and N are rectangular, study the the (degenerate and nondegenerate) two-boundary BMW algebras.

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[OR] R. Orellana and A. Ram, *Affine braids, Markov traces and the category* $\mathcal{O}$, Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces Mumbai 2004, V.B. Mehta ed., Tata Institute of Fundamental Research, Narosa Publishing House, Amer. Math. Soc. (2007) 423-473.

In preparation:

[Da2] Z. Daugherty, *Centralizer properties of the graded Hecke algebra of type C*

[DRV] Z. Daugherty, A. Ram, R. Virk, *Affine and graded BMW algebras*