Degenerate two-boundary centralizer algebras

Zajj Daugherty

University of Wisconsin, Madison

May 3, 2010

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k, Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k, Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k,
Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

$\mathrm{GL}_n(\mathbb{C})$	acts on $V^{\otimes k}$	diagonally
S_k	acts on $V^{\otimes k}$	by place permutations

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k,
Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

$\mathrm{GL}_n(\mathbb{C})$	acts on $V^{\otimes k}$	diagonally
S_k	acts on $V^{\otimes k}$	by place permutations

These actions commute!

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k,
Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

$\mathrm{GL}_n(\mathbb{C})$	acts on $V^{\otimes k}$	diagonally
S_k	acts on $V^{\otimes k}$	by place permutations

These actions commute!

For example: $g\ (1\ 2) \cdot (v_1 \otimes v_2)$

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k, Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

$\mathrm{GL}_n(\mathbb{C})$	acts on $V^{\otimes k}$	diagonally
S_k	acts on $V^{\otimes k}$	by place permutations

These actions commute!

For example: $g\ (1\ 2) \cdot (v_1 \otimes v_2) = g \cdot (v_2 \otimes v_1)$

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k,
Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

$\mathrm{GL}_n(\mathbb{C})$	acts on $V^{\otimes k}$	diagonally
S_k	acts on $V^{\otimes k}$	by place permutations

These actions commute!

For example:
$$\begin{aligned} g\ (1\ 2) \cdot (v_1 \otimes v_2) &= g \cdot (v_2 \otimes v_1) \\ &= gv_2 \otimes gv_1 \end{aligned}$$

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k, Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

$\mathrm{GL}_n(\mathbb{C})$	acts on $V^{\otimes k}$	diagonally
S_k	acts on $V^{\otimes k}$	by place permutations

These actions commute!

For example:

$$\begin{aligned} g\ (1\ 2) \cdot (v_1 \otimes v_2) &= g \cdot (v_2 \otimes v_1) \\ &= gv_2 \otimes gv_1 \\ &= (1\ 2) \cdot (gv_1 \otimes gv_2) \end{aligned}$$

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Knowing the representation theory of the symmetric group S_k, Schur (1901) produced simple $\mathrm{GL}_n(\mathbb{C})$-modules as follows:

Let $V = \mathbb{C}^n$, and denote $V \otimes V \otimes \cdots \otimes V = V^{\otimes k}$.

$\mathrm{GL}_n(\mathbb{C})$	acts on $V^{\otimes k}$	diagonally
S_k	acts on $V^{\otimes k}$	by place permutations

These actions commute!

For example:

$$\begin{aligned} g\ (1\ 2) \cdot (v_1 \otimes v_2) &= g \cdot (v_2 \otimes v_1) \\ &= gv_2 \otimes gv_1 \\ &= (1\ 2) \cdot (gv_1 \otimes gv_2) \\ &= (1\ 2)\ g \cdot (v_1 \otimes v_2) \end{aligned}$$

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Big deal:

For $n \geq k$, the *centralizer* of the action of $\mathrm{GL}_n(\mathbb{C})$ on $V^{\otimes k}$ in $\mathrm{End}(V^{\otimes k})$ is

$$\mathrm{End}_{\mathrm{GL}_n(\mathbb{C})}(V^{\otimes k}) \cong \mathbb{C}S_k.$$

Warm-up with Schur-Weyl duality

Goal: Find simple modules.

Big deal:

For $n \geq k$, the *centralizer* of the action of $\mathrm{GL}_n(\mathbb{C})$ on $V^{\otimes k}$ in $\mathrm{End}(V^{\otimes k})$ is

$$\mathrm{End}_{\mathrm{GL}_n(\mathbb{C})}(V^{\otimes k}) \cong \mathbb{C}S_k.$$

Bigger deal:

Centralizer relationship produces

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ ht(\lambda) \leq n}} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

where

G^λ	are distinct irreducible	GL_n-modules
S^λ	are distinct irreducible	S_k-modules

The set up

Let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra.

e.g. $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$.

Let M, N, and V be finite dimensional simple $\mathfrak{g}$-modules.

Our goal:

Understand $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.

(the set of endomorphisms which commute with the action of $\mathfrak{g}$)

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

1. If $M = N = L(0)$ and

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

1. If $M = N = L(0)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives $\mathbb{C}S_k$ modules (Schur, 1901);

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

1. If $M = N = L(0)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives $\mathbb{C}S_k$ modules (Schur, 1901);
   - $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, this gives Brauer algebra modules (Brauer, 1937);

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

1. If $M = N = L(0)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives $\mathbb{C}S_k$ modules (Schur, 1901);
   - $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, this gives Brauer algebra modules (Brauer, 1937);
2. If $M = L(0)$ and $N = L(\lambda)$ and

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

1. If $M = N = L(0)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives $\mathbb{C}S_k$ modules (Schur, 1901);
   - $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, this gives Brauer algebra modules (Brauer, 1937);
2. If $M = L(0)$ and $N = L(\lambda)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives graded Hecke algebra of type A modules (Arikawa & Suzuki, 1998);

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

1. If $M = N = L(0)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives $\mathbb{C}S_k$ modules (Schur, 1901);
   - $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, this gives Brauer algebra modules (Brauer, 1937);
2. If $M = L(0)$ and $N = L(\lambda)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives graded Hecke algebra of type A modules (Arikawa & Suzuki, 1998);
   - $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, this gives degenerate affine Wenzl algebra modules (Nazarov, 1996).

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Let $L(\lambda)$ be the finite dim'l irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1) = L(\square)$ (the first fundamental weight).

1. If $M = N = L(0)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives $\mathbb{C}S_k$ modules (Schur, 1901);
   - $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, this gives Brauer algebra modules (Brauer, 1937);
2. If $M = L(0)$ and $N = L(\lambda)$ and
   - $\mathfrak{g} = \mathfrak{sl}_n$, this gives graded Hecke algebra of type A modules (Arikawa & Suzuki, 1998);
   - $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, this gives degenerate affine Wenzl algebra modules (Nazarov, 1996).

Quantized versions yield standard and affine type A Hecke and Birman-Murakami-Wenzl algebra modules (Orellana & Ram, 2007)

First big question:

Is there an algebra which has centralizers $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients?

Definition

The *degenerate two-boundary braid group* $\mathcal{G}_k$ is the $\mathbb{C}$-algebra generated by

$$\mathbb{C}S_k = \mathbb{C}\left\langle t_i \;\middle|\; \begin{array}{cc} i = 1, \dots k & \\ t_i^2 = 1 & \\ t_i t_j = t_j t_i & |i-j| > 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} & \end{array} \right\rangle$$

$$\mathbb{C}[z_0, z_1, \dots, z_k],\ \mathbb{C}[y_1, \dots, y_k],\ \mathbb{C}[x_1, \dots, x_k]$$

and relations...

Definition

The *degenerate two-boundary braid group* $\mathcal{G}_k$ is the $\mathbb{C}$-algebra generated by

$$\mathbb{C}S_k = \mathbb{C}\left\langle t_i \;\middle|\; \begin{array}{cc} i = 1, \dots k & \\ t_i^2 = 1 & \\ t_i t_j = t_j t_i & |i-j| > 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} & \end{array} \right\rangle$$

$$\mathbb{C}[z_0, z_1, \dots, z_k],\ \mathbb{C}[y_1, \dots, y_k],\ \mathbb{C}[x_1, \dots, x_k]$$

and relations...

$t_i x_j = x_j t_i, \quad t_i y_j = y_j t_i, \quad t_i z_j = z_j t_i, \quad$ for $j \neq i, i+1$
$(z_0 + \cdots + z_i) x_j = x_j (z_0 + \cdots + z_i), \quad (z_0 + \cdots + z_i) y_j = y_j (z_0 + \cdots + z_i), \quad$ for $i \geq j$
$t_i(x_i + x_{i+1}) = (x_i + x_{i+1})t_i, \quad t_i(y_i + y_{i+1}) = (y_i + y_{i+1})t_i, \quad$ for $1 \leq i \leq k-1$
$(t_i t_{i+1})(x_{i+1} - t_i x_i t_i)(t_{i+i} t_i) = x_{i+2} - t_{i+1} x_{i+1} t_{i+1}$
$(t_i t_{i+1})(y_{i+1} - t_i y_i t_i)(t_{i+i} t_i) = y_{i+2} - t_{i+1} y_{i+1} t_{i+1} \quad$ for $1 \leq i \leq k-2$,
$x_{i+1} - t_i x_i t_i = y_{i+1} - t_i y_i t_i \quad$ for $1 \leq i \leq k-1$,
$z_i = x_i + y_i - m_i, \quad 1 \leq i \leq k$,
where if $m_{i,j} = \begin{cases} x_{i+1} - t_i x_i t_i & \text{if } j = i+1, \\ (i+1\ j) m_{i,i+1} (i+1\ j) & \text{if } j \neq i, i+1, \end{cases}$ then $m_1 = 0$, $m_i = \sum_{1<j<i} m_{i,j}$.

Definition

The *degenerate two-boundary braid group* $\mathcal{G}_k$ is the $\mathbb{C}$-algebra generated by

$$\mathbb{C}S_k = \mathbb{C}\left\langle t_i \;\middle|\; \begin{matrix} i = 1, \dots k & \\ t_i^2 = 1 & \\ t_i t_j = t_j t_i & |i-j| > 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} & \end{matrix} \right\rangle$$

$$\mathbb{C}[z_0, z_1, \dots, z_k],\ \mathbb{C}[y_1, \dots, y_k],\ \mathbb{C}[x_1, \dots, x_k]$$

and relations twisting the four factors together...

$\mathcal{G}_k$ contains three images of the graded braid group:

$$\frac{\mathbb{C}[z_1, \dots, z_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[y_1, \dots, y_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}S_k}{\sim}$$

and

$$z_i = x_i + y_i - \textit{lower terms},$$

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$\mathbb{C}S_k \qquad \text{permutes factors of } V^{\otimes k},$$

$$\mathbb{C}[x_1, \ldots, x_k]$$

$$\mathbb{C}[y_1, \ldots, y_k]$$

$$\mathbb{C}[z_1, \ldots, z_k]$$

$$z_0$$

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$\begin{aligned} \mathbb{C}S_k &\qquad \text{permutes factors of } V^{\otimes k}, \\ \mathbb{C}[x_1, \dots, x_k] &\qquad \text{acts on } M \text{ and } V^{\otimes k}, \\ \mathbb{C}[y_1, \dots, y_k] & \\ \mathbb{C}[z_1, \dots, z_k] & \\ z_0 & \end{aligned}$$

by nested central elements of $\mathcal{U}\mathfrak{g}$.

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$
\begin{array}{rl}
\mathbb{C}S_k & \text{permutes factors of } V^{\otimes k}, \\
\mathbb{C}[x_1, \ldots, x_k] & \text{acts on } M \text{ and } V^{\otimes k}, \\
\mathbb{C}[y_1, \ldots, y_k] & \text{acts on } N \text{ and } V^{\otimes k}, \\
\mathbb{C}[z_1, \ldots, z_k] & \\
z_0 &
\end{array}
$$

by nested central elements of $\mathcal{U}\mathfrak{g}$.

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$
\begin{aligned}
\mathbb{C}S_k &\qquad \text{permutes factors of } V^{\otimes k}, \\
\mathbb{C}[x_1, \ldots, x_k] &\qquad \text{acts on } M \text{ and } V^{\otimes k}, \\
\mathbb{C}[y_1, \ldots, y_k] &\qquad \text{acts on } N \text{ and } V^{\otimes k}, \\
\mathbb{C}[z_1, \ldots, z_k] &\qquad \text{acts on } M \otimes N \text{ together and } V^{\otimes k}, \\
z_0 &
\end{aligned}
$$

by nested central elements of $\mathcal{U}\mathfrak{g}$.

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$
\begin{aligned}
\mathbb{C}S_k &\qquad \text{permutes factors of } V^{\otimes k}, \\
\mathbb{C}[x_1, \ldots, x_k] &\qquad \text{acts on } M \text{ and } V^{\otimes k}, \\
\mathbb{C}[y_1, \ldots, y_k] &\qquad \text{acts on } N \text{ and } V^{\otimes k}, \\
\mathbb{C}[z_1, \ldots, z_k] &\qquad \text{acts on } M \otimes N \text{ together and } V^{\otimes k}, \\
z_0 &\qquad \text{acts on } M \otimes N \text{ alone},
\end{aligned}
$$

by nested central elements of $\mathcal{U}\mathfrak{g}$.

Let $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be the trace form:

$\langle x, y \rangle = \mathrm{Tr}(xy),$ where x and y are viewed in a defining rep of $\mathfrak{g}$.

Let $\{b\}$ be a basis of $\mathfrak{g}$ and $\{b^*\}$ the dual basis wrt $\langle , \rangle$.

Let $\kappa = \sum_b bb^*$.

κ is the *Casimir invariant* and is central in $\mathcal{U}\mathfrak{g}$.

Theorem (D.)

Define $\Phi\colon \mathcal{G}_k \to \mathrm{End}(M \otimes N \otimes V^{\otimes k})$

$$
\begin{aligned}
\Phi(t_j) &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(j-1)} \otimes t_1 \otimes \mathrm{id}_V^{\otimes(k-j-1)},\\
\Phi(x_j) &= \tfrac{1}{2}(\kappa|_{M\otimes V^{\otimes j}} - \kappa|_{M\otimes V^{\otimes j-1}}),\\
\Phi(y_j) &= \tfrac{1}{2}(\kappa|_{N\otimes V^{\otimes j}} - \kappa|_{N\otimes V^{\otimes j-1}}),\\
\Phi(z_j) &= \tfrac{1}{2}(\kappa|_{M\otimes N\otimes V^{\otimes j}} - \kappa|_{M\otimes N\otimes V^{\otimes j-1}} + \kappa|_V),\\
\Phi(z_0) &= \tfrac{1}{2}(\kappa|_{M\otimes N} - \kappa|_M - \kappa|_N),
\end{aligned}
$$

where $t_1 \cdot (v_{i_1} \otimes v_{i_2}) = v_{i_2} \otimes v_{i_1}$.

Theorem (D.)

Define $\Phi\colon \mathcal{G}_k \to \mathrm{End}(M \otimes N \otimes V^{\otimes k})$

$$\begin{aligned}
\Phi(t_j) &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(j-1)} \otimes t_1 \otimes \mathrm{id}_V^{\otimes(k-j-1)}, \\
\Phi(x_j) &= \tfrac{1}{2}(\kappa|_{M\otimes V^{\otimes j}} - \kappa|_{M\otimes V^{\otimes j-1}}), \\
\Phi(y_j) &= \tfrac{1}{2}(\kappa|_{N\otimes V^{\otimes j}} - \kappa|_{N\otimes V^{\otimes j-1}}), \\
\Phi(z_j) &= \tfrac{1}{2}(\kappa|_{M\otimes N\otimes V^{\otimes j}} - \kappa|_{M\otimes N\otimes V^{\otimes j-1}} + \kappa|_V), \\
\Phi(z_0) &= \tfrac{1}{2}(\kappa|_{M\otimes N} - \kappa|_M - \kappa|_N),
\end{aligned}$$

where $t_1 \cdot (v_{i_1} \otimes v_{i_2}) = v_{i_2} \otimes v_{i_1}$.
Then Φ *is a representation of* $\mathcal{G}_k$ *which commutes with the action of* $\mathfrak{g}$.

An Example:

Is there an algebra which has centralizers
$\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients
when $\mathfrak{g}$ is of type A?

Definition

Fix $a, b, p, q \in \mathbb{Z}_{>0}$.

The *degenerate extended two-boundary Hecke algebra* $\mathcal{H}_k^{\mathrm{ext}}$ is the quotient of the degenerate two-boundary braid group by the relations

$$\begin{aligned} t_i x_i &= x_{i+1} t_i - 1, \\ t_i y_i &= y_{i+1} t_i - 1, \qquad i = 1, \ldots, k-1. \\ t_i z_i &= z_{i+1} t_i - 1, \end{aligned}$$

$$(x_1 - a)(x_1 + p) = 0 \qquad (y_1 - b)(y_1 + q) = 0.$$

Definition

Fix $a, b, p, q \in \mathbb{Z}_{>0}$.

The *degenerate extended two-boundary Hecke algebra* $\mathcal{H}_k^{\mathrm{ext}}$ is the quotient of the degenerate two-boundary braid group by the relations

$$\begin{aligned} t_i x_i &= x_{i+1} t_i - 1, \\ t_i y_i &= y_{i+1} t_i - 1, \qquad i = 1, \ldots, k-1. \\ t_i z_i &= z_{i+1} t_i - 1, \end{aligned}$$

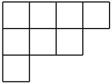
$$(x_1 - a)(x_1 + p) = 0 \qquad (y_1 - b)(y_1 + q) = 0.$$

The *degenerate two-boundary Hecke algebra* $\mathcal{H}_k$ is the subalgebra of $\mathcal{H}_k^{\mathrm{ext}}$ generated by

$$x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k, t_1, \ldots, t_{k-1}.$$

(everything but z_0... we'll come back to this.)

A *partition* is a collections of boxes:

$\lambda =$ 

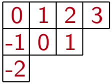
A *partition* is a collections of boxes:

$$\lambda = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} -2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

If a box B is in row i and column j, then the *content* of B is

$$c(B) = j - i.$$

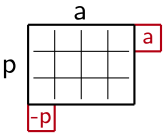
A *partition* is a collections of boxes:

$\lambda =$ 

If a box B is in row i and column j, then the *content* of B is

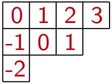
$$c(B) = j - i.$$

If $\lambda = (a^p)$ is rectangular, there are exactly two "addable" boxes:

$(a^p) =$ 

A *partition* is a collections of boxes:

$$\lambda =$$

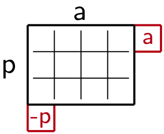


If a box B is in row i and column j, then the *content* of B is

$$c(B) = j - i.$$

If $\lambda = (a^p)$ is rectangular, there are exactly two "addable" boxes:

$$(a^p) =$$



(recall relations $(x_1 - a)(x_1 + p) = 0$ and $(y_1 - b)(y_1 + q) = 0$)

Theorem (D.)

Fix $k < n$ non-neg. integers.
Let $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$, $N = L((b^q))$, and $V = L((1^1))$.

Theorem (D.)

Fix $k < n$ non-neg. integers.
Let $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$, $N = L((b^q))$, and $V = L((1^1))$.

(1) *Φ is a rep. of $\mathcal{H}_k^{\mathrm{ext}}$ which commutes with the $\mathfrak{g}$-action, so*

$$\Phi(\mathcal{H}_k^{\mathrm{ext}}) \subseteq \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

Theorem (D.)

Fix $k < n$ non-neg. integers.
Let $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$, $N = L((b^q))$, and $V = L((1^1))$.

(1) Φ *is a rep. of $\mathcal{H}_k^{\mathrm{ext}}$ which commutes with the $\mathfrak{g}$-action, so*

$$\Phi(\mathcal{H}_k^{\mathrm{ext}}) \subseteq \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

(2) *For small cases,*

$$\Phi(\mathcal{H}_k^{\mathrm{ext}}) = \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

Theorem (D.)

Fix $k < n$ non-neg. integers.
Let $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$, $N = L((b^q))$, and $V = L((1^1))$.

(1) *Φ is a rep. of $\mathcal{H}_k^{\text{ext}}$ which commutes with the $\mathfrak{g}$-action, so*

$$\Phi(\mathcal{H}_k^{\text{ext}}) \subseteq \text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

(2) *For small cases,*

$$\Phi(\mathcal{H}_k^{\text{ext}}) = \text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

Remark

(1) When Φ is not surjective, the image differs by a portion of the action of the center of $\mathcal{U}\mathfrak{g}$ on $M \otimes N$.
(2) Same results for $\mathfrak{g} = \mathfrak{sl}_n$ and a shift of Φ.

Let $M = L((a^p))$ and $N = L((b^q))$. Then

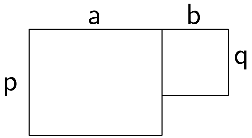
$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

where Λ is the following set of partitions: (Okada)

Let $M = L((a^p))$ and $N = L((b^q))$. Then

$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

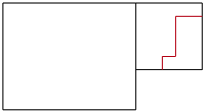
where Λ is the following set of partitions: (Okada)



Let $M = L((a^p))$ and $N = L((b^q))$. Then

$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda)$$ (multiplicity one!)

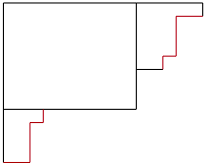
where Λ is the following set of partitions: (Okada)



Let $M = L((a^p))$ and $N = L((b^q))$. Then

$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda)$$ (multiplicity one!)

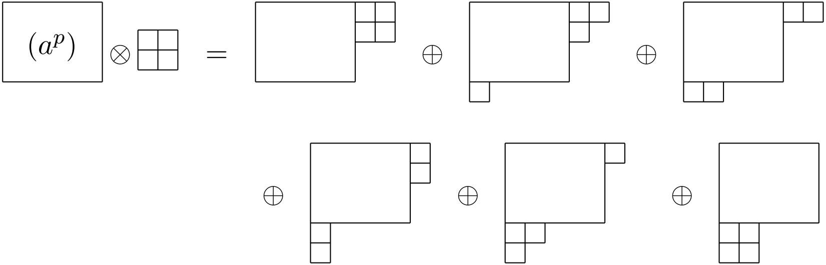
where Λ is the following set of partitions: (Okada)



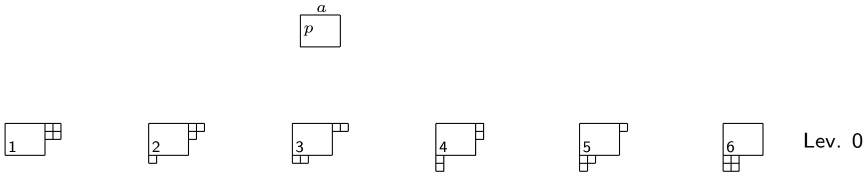
Let $M = L((a^p))$ and $N = L((b^q))$. Then

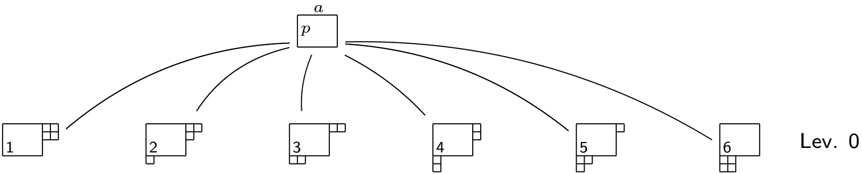
$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

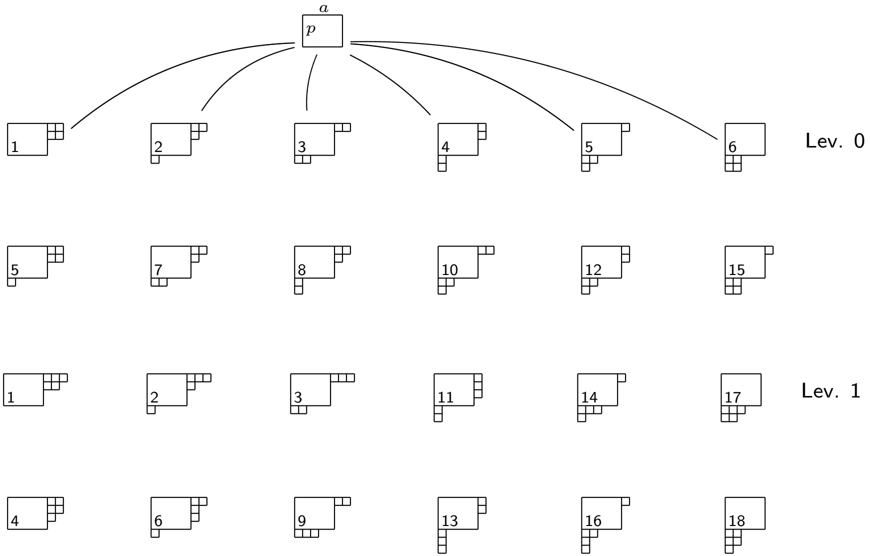
where Λ is the following set of partitions... (Okada)

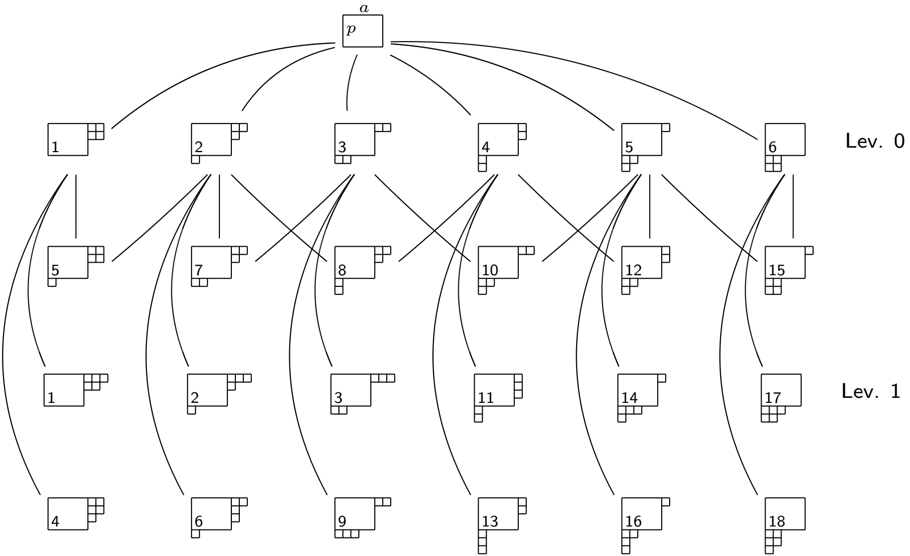



a
p

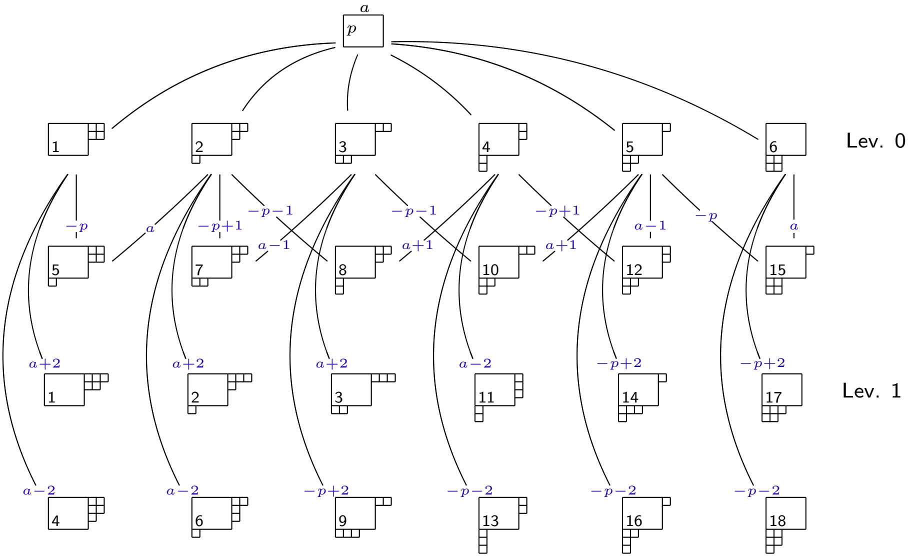

a
p
1
2
3
4
5
6
Lev. 0


a
p
1
2
3
4
5
6
Lev. 0

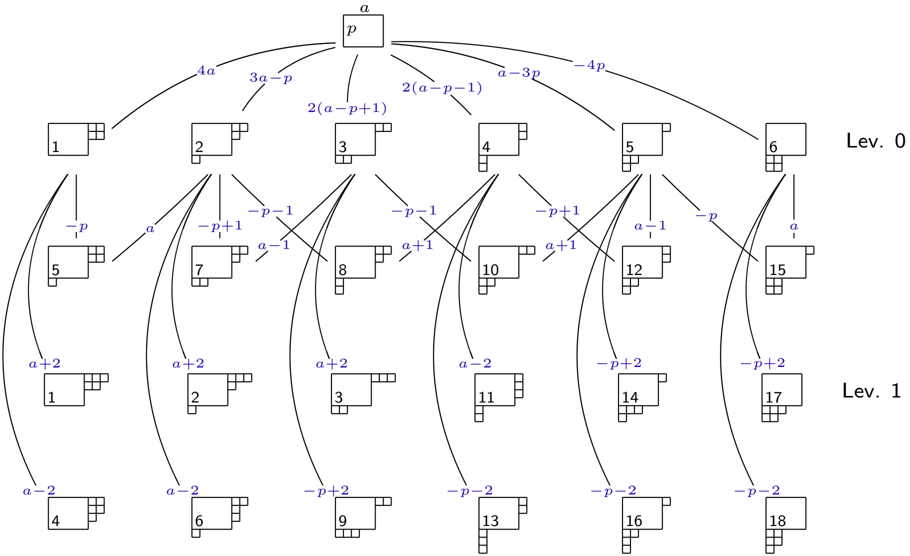

a
p
1
2
3
4
5
6
Lev. 0
5
7
8
10
12
15
1
2
3
11
14
17
Lev. 1
4
6
9
13
16
18



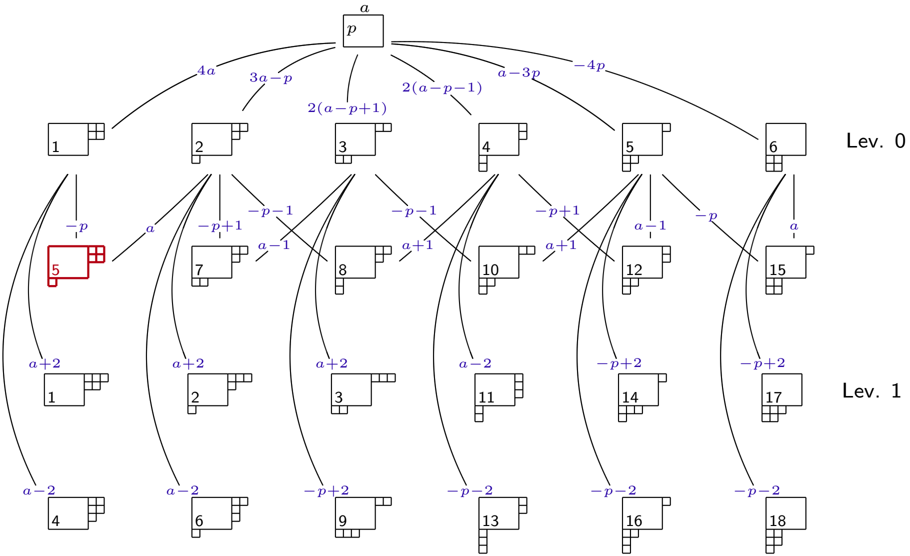
a
p
1
2
3
4
5
6
Lev. 0
5
7
8
10
12
15
1
2
3
11
14
17
Lev. 1
4
6
9
13
16
18



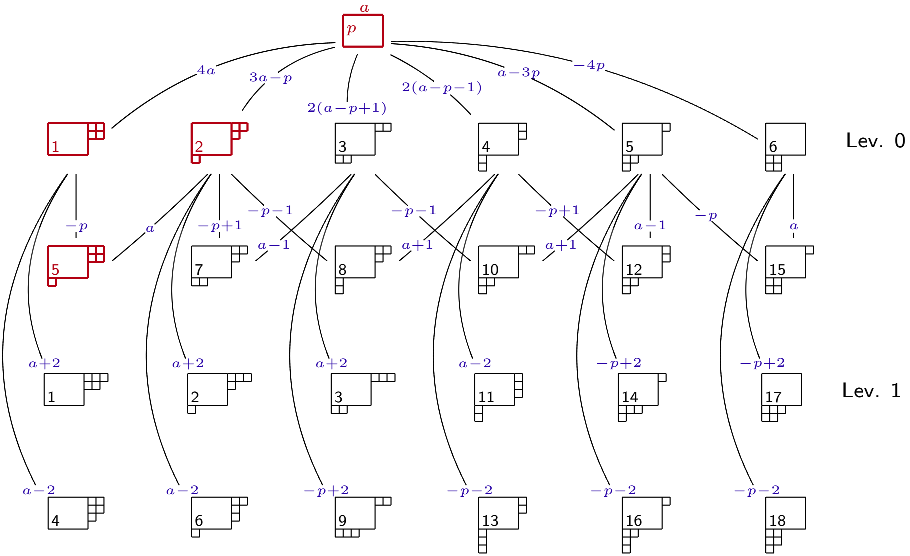
a
p
1
2
3
4
5
6
Lev. 0
−p
a
−p+1
−p−1
a−1
−p−1
a+1
−p+1
a+1
a−1
−p
a
5
7
8
10
12
15
a+2
a+2
a+2
a−2
−p+2
−p+2
1
2
3
11
14
17
Lev. 1
a−2
a−2
−p+2
−p−2
−p−2
−p−2
4
6
9
13
16
18



a
p
4a
3a−p
2(a−p+1)
2(a−p−1)
a−3p
−4p
1
2
3
4
5
6
Lev. 0
−p
a
−p+1
−p−1
a−1
−p−1
a+1
−p+1
a+1
a−1
−p
a
5
7
8
10
12
15
a+2
a+2
a+2
a−2
−p+2
−p+2
1
2
3
11
14
17
Lev. 1
a−2
a−2
−p+2
−p−2
−p−2
−p−2
4
6
9
13
16
18

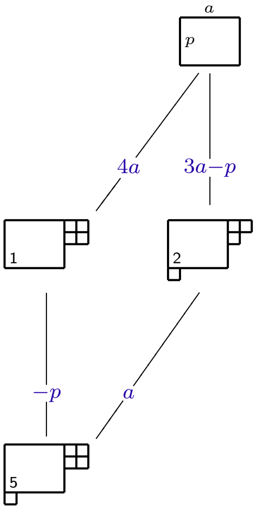


a
p
4a
3a − p
2(a − p + 1)
2(a − p − 1)
a − 3p
−4p
1
2
3
4
5
6
Lev. 0
−p
a
−p + 1
−p − 1
a − 1
−p − 1
a + 1
−p + 1
a + 1
a − 1
−p
a
5
7
8
10
12
15
a + 2
a + 2
a + 2
a − 2
−p + 2
−p + 2
1
2
3
11
14
17
Lev. 1
a − 2
a − 2
−p + 2
−p − 2
−p − 2
−p − 2
4
6
9
13
16
18



a
p
4a
3a−p
2(a−p+1)
2(a−p−1)
a−3p
−4p
1
2
3
4
5
6
Lev. 0
−p
a
−p+1
−p−1
a−1
−p−1
a+1
−p+1
a+1
a−1
−p
a
5
7
8
10
12
15
a+2
a+2
a+2
a−2
−p+2
−p+2
1
2
3
11
14
17
Lev. 1
a−2
a−2
−p+2
−p−2
−p−2
−p−2
4
6
9
13
16
18

A two-dimensional $\mathcal{H}_1^{\mathrm{ext}}$-module:



$$z_0 = \begin{pmatrix} 4a & 0 \\ 0 & 3a-p \end{pmatrix}$$

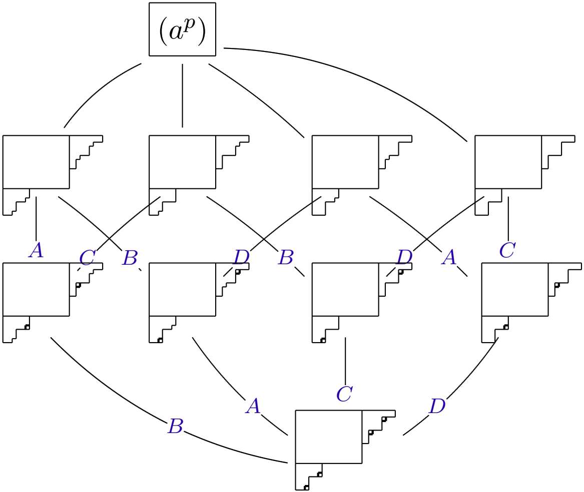
$$z_1 = \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$x_1 \sim \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$y_1 \sim \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

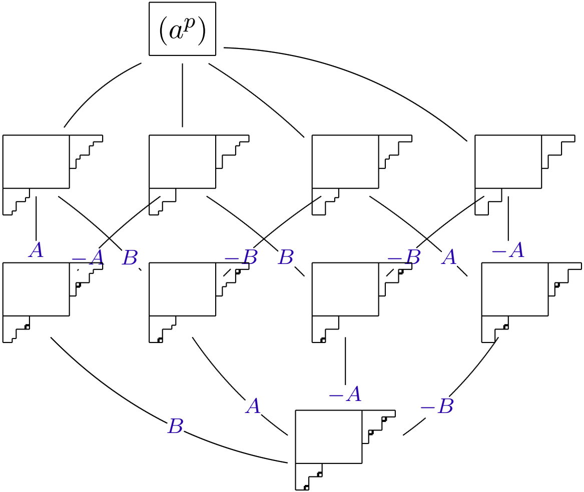
(formulas x_1, y_1, z_1, z_0 all given in terms of contents of added boxes)

An eight-dimensional $\mathcal{H}_2$-module:



where $C = -A + (a - p + b - q)$ and $D = -B + (a - p + b - q)$

An eight-dimensional $\mathcal{H}_2$-module:



Shift! Label edges by action of $z_1 - \frac{1}{2}(a-p+b-q)$ and $z_2 - \frac{1}{2}(a-p+b-q))$

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$\mathcal{H}_k$ is presented by generators

$$x_1, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i - j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$(x_1 - a)(x_1 + p) = 0, \qquad x_1(t_1 x_1 t_1 + t_1) = (t_1 x_1 t_1 + t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$x_1 w_i = w_i x_1 \qquad \text{and} \qquad x_1 t_i = t_i x_1, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$x_1 w_1 = -w_1 x_1 + (a - p) w_1 + w_1^2 + \left(\tfrac{(a+p)^2 - (b+q)^2}{4} \right).$$

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$\mathcal{H}_k$ is presented by generators

$$x_1, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i - j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$(x_1 - a)(x_1 + p) = 0, \qquad x_1(t_1 x_1 t_1 + t_1) = (t_1 x_1 t_1 + t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$x_1 w_i = w_i x_1 \qquad \text{and} \qquad x_1 t_i = t_i x_1, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$x_1 w_1 = -w_1 x_1 + (a - p) w_1 + w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right).$$

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \tfrac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \tfrac{2}{a+p} \left(w_1^2 + \left(\tfrac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

$$\text{where } t_0 = \tfrac{1}{a+p}(2x_1 - (a-p)).$$

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \tfrac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \tfrac{2}{a+p} \left(w_1^2 + \left(\tfrac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

where $t_0 = \frac{1}{a+p}(2x_1 - (a-p))$.

$\mathcal{H}_k$ is presented by generators

$$t_0, t_1, \dots, t_{k-1}, w_1, \dots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \tfrac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \dots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \tfrac{2}{a+p} \left(w_1^2 + \left(\tfrac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

$$\text{where } t_0 = \tfrac{1}{a+p}(2x_1 - (a-p)).$$

$\mathcal{H}_k$ is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \tfrac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \tfrac{2}{a+p} \left(w_1^2 + \left(\tfrac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

$$\text{where } t_0 = \tfrac{1}{a+p}(2x_1 - (a-p)).$$

The graded Hecke algebra of type C is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \tfrac{2}{(a+p)}\left(t_1 t_0 - t_0 t_1\right)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \tfrac{2}{a+p}\left(w_1^2 + \left(\tfrac{(a+p)^2 - (b+q)^2}{4}\right)\right)$$

$$\text{where } t_0 = \tfrac{1}{a+p}(2x_1 - (a-p)).$$

The graded Hecke algebra of type C is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \tfrac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + \tfrac{2}{a+p} \left(w_1^2 + \left(\tfrac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

$$\text{where } t_0 = \tfrac{1}{a+p} (2x_1 - (a-p)).$$

The graded Hecke algebra of type C is presented by generators

$$t_0, t_1, \dots, t_{k-1}, w_1, \dots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \frac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \dots, k,$$

and

$$t_0 w_1 = -w_1 t_0 + c \, \frac{2}{a+p} \left(w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

where $t_0 = \frac{1}{a+p}(2x_1 - (a-p))$.

The graded Hecke algebra of type C is presented by generators

$$t_0, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| >, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$t_0^2 = 1, \qquad t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 + \frac{2}{(a+p)} (t_1 t_0 - t_0 t_1)$$

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \quad t_{s_i} w_j = w_j t_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$t_0 w_i = w_i t_0 \qquad \text{and} \qquad t_0 t_i = t_i t_0, \quad \text{for } i \geq 2,$$

$$w_i w_j = w_j w_i, \qquad \text{for } i, j = 0, \ldots, k,$$

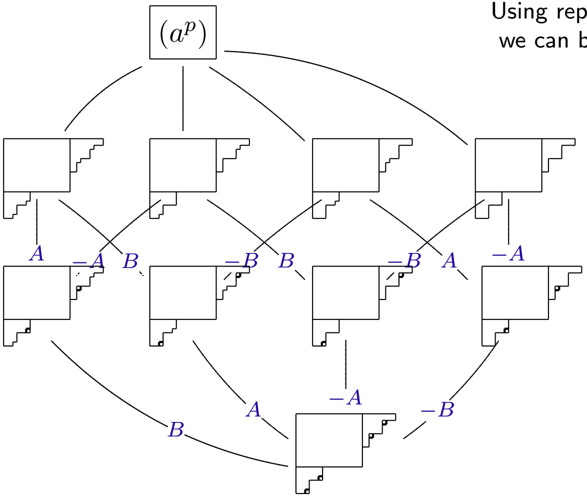
and

$$t_0 w_1 = -w_1 t_0 + c \, \frac{2}{a+p} \left(w_1^2 + \left(\frac{(a+p)^2 - (b+q)^2}{4} \right) \right)$$

$$\in \mathcal{Z}(\mathcal{H}_1)$$

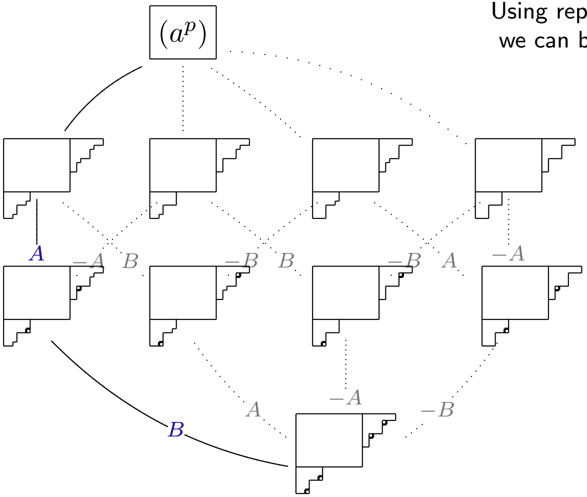
where $t_0 = \frac{1}{a+p}(2x_1 - (a-p))$.

An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

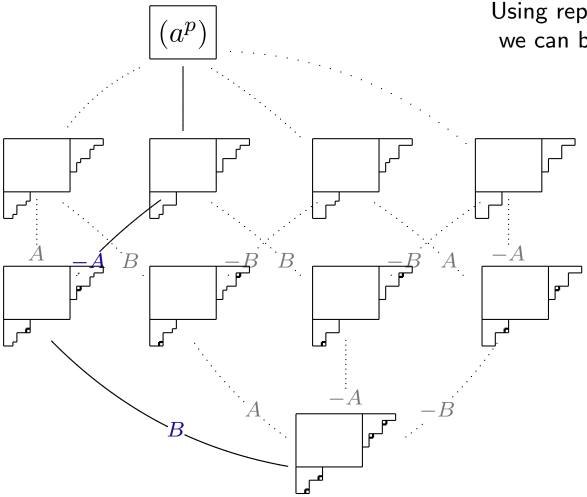
An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

$1 \qquad (A, B)$

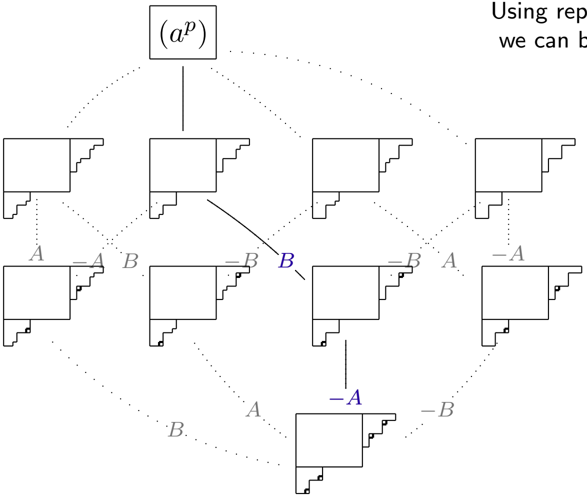
An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$

An eight-dimensional $\mathcal{H}_2$-module:

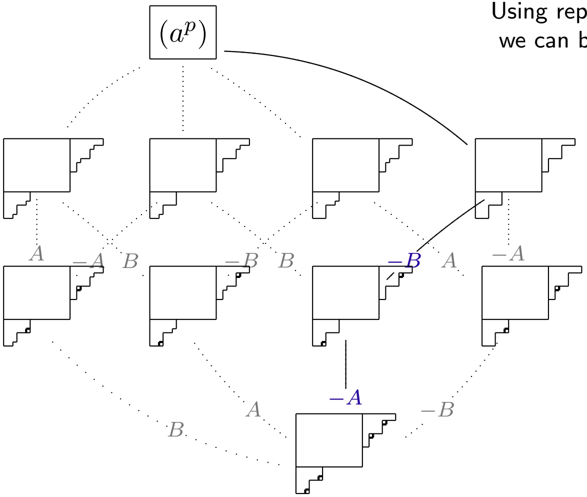


Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$

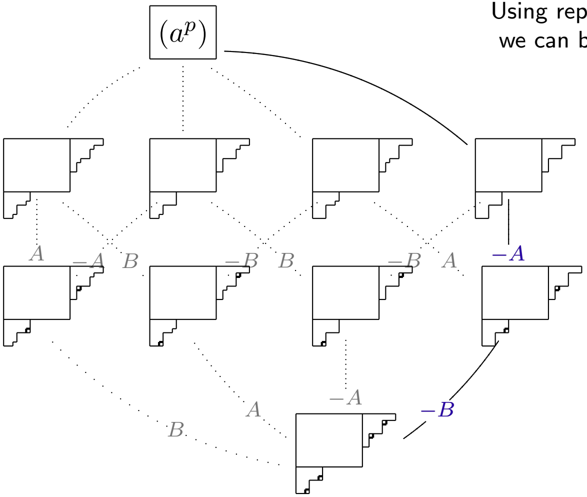
An eight-dimensional $\mathcal{H}_2$-module:

Using representation from before, we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1



1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$

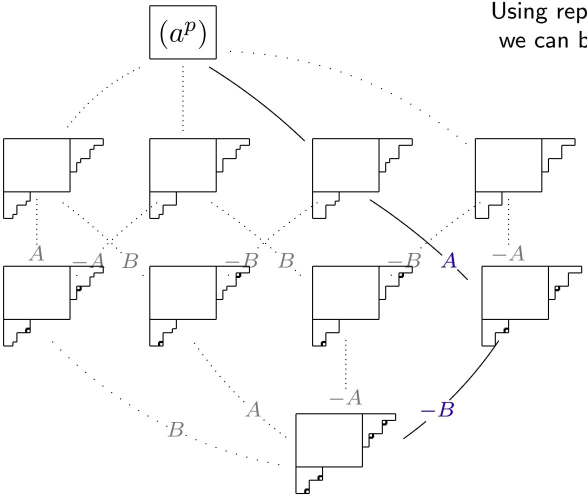
An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$

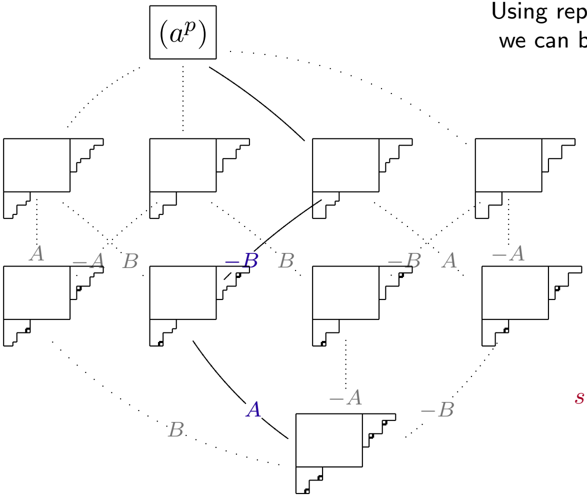
An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before, we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$

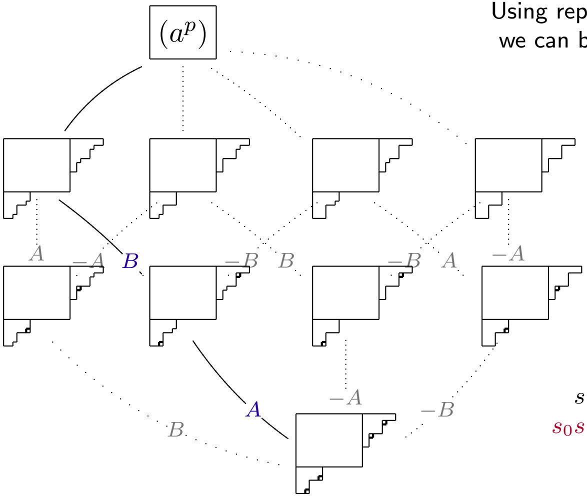
An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$
$s_1 s_0 s_1 s_0 s_1 s_0$	$(-B, A)$

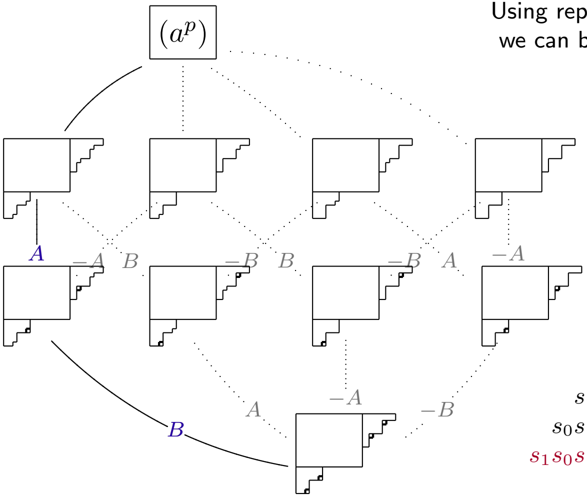
An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$
$s_1 s_0 s_1 s_0 s_1 s_0$	$(-B, A)$
$s_0 s_1 s_0 s_1 s_0 s_1 s_0$	(B, A)

An eight-dimensional $\mathcal{H}_2$-module:



Using representation from before,
we can build operators from $\mathcal{H}_2$:
s_0 changes level 0
s_1 changes level 1

1	(A, B)
s_0	$(-A, B)$
$s_1 s_0$	$(B, -A)$
$s_0 s_1 s_0$	$(-B, -A)$
$s_1 s_0 s_1 s_0$	$(-A, -B)$
$s_0 s_1 s_0 s_1 s_0$	$(A, -B)$
$s_1 s_0 s_1 s_0 s_1 s_0$	$(-B, A)$
$s_0 s_1 s_0 s_1 s_0 s_1 s_0$	(B, A)
$s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0$	(A, B)

Up next for $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

1. When $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, and M and N are rectangular, we get the degenerate (extended) two-boundary Hecke algebra.
   1. Quantized versions yield two-boundary Hecke algebras.
   2. What is a good basis? What is the center? How does the center act?
   3. Develop the combinatorics: cool dimension formulas? familiar tableaux games?
   4. What exactly *is* the correspondence to type C?
2. When $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, and M and N are rectangular, study the the (degenerate and nondegenerate) two-boundary BMW algebras.

References

[OR] R. Orellana and A. Ram, *Affine braids, Markov traces and the category $\mathcal{O}$*, Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces Mumbai 2004, V.B. Mehta ed., Tata Institute of Fundamental Research, Narosa Publishing House, Amer. Math. Soc. (2007) 423-473.

[GN] J. de Gier and A. Nichols, *The two-boundary Temperley-Lieb algebra*, J. Algebra **321** (2009) 11321167.

In preparation:

[Dau] Z. Daugherty, *Degenerate two-boundary centralizer algebras*

[DRV] Z. Daugherty, A. Ram, R. Virk, *Affine and graded BMW algebras*