Degenerate two-boundary centralizer algebras

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Background

Let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra.

e.g. $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$.

Let M, N, and V be finite dimensional simple $\mathfrak{g}$-modules.

Goal:
Understand $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.
(the set of endomorphisms which commute with the action of $\mathfrak{g}$)

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Fix $k < n$ integers.
Let $L(\lambda)$ be the f.d. irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1)$.

1. If $\mathfrak{g} = \mathfrak{sl}_n$ and
   - $M = N = L(0)$, this gives $\mathbb{C}S_k$;
   - $M = L(0)$ and $N = L(\lambda)$, this gives is a quotient of the graded Hecke algebra of type A;
2. If $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$ and
   - $M = N = L(0)$, this gives the Brauer algebra;
   - $M = L(0)$ and $N = L(\lambda)$, this gives a quotient of the degenerate affine Wenzl algebra.

Quantized versions yield standard and affine type A Hecke and Birman-Murakami-Wenzl algebras.

Big question:

Is there an algebra which has centralizers
$\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients?

Definition

The *degenerate two-boundary braid group* $\mathcal{G}_k$ is the $\mathbb{C}$-algebra generated by

$$\mathbb{C}S_k = \mathbb{C}\left\langle t_i \;\middle|\; \begin{array}{cc} i = 1, \ldots k & \\ t_i^2 = 1 & \\ t_i t_j = t_j t_i & |i-j| > 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} & \end{array} \right\rangle$$

$$\mathbb{C}[z_0, z_1, \ldots, z_k],\ \mathbb{C}[y_1, \ldots, y_k],\ \mathbb{C}[x_1, \ldots, x_k]$$

and relations twisting the four factors together...

$\mathcal{G}_k$ contains three images of the graded braid group:

$$\frac{\mathbb{C}[z_1, \ldots, z_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[y_1, \ldots, y_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[x_1, \ldots, x_k] \otimes \mathbb{C}S_k}{\sim}$$

and

$$z_i = x_i + y_i - \textit{lower terms},$$

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$\begin{aligned}
\mathbb{C}S_k & \qquad \text{permutes factors of } V^{\otimes k}, \\
\mathbb{C}[x_1, \ldots, x_k] & \qquad \text{acts on } M \text{ and } V^{\otimes k}, \\
\mathbb{C}[y_1, \ldots, y_k] & \qquad \text{acts on } N \text{ and } V^{\otimes k}, \\
\mathbb{C}[z_1, \ldots, z_k] & \qquad \text{acts on } M \otimes N \text{ together and } V^{\otimes k}, \\
z_0 & \qquad \text{acts on } M \otimes N \text{ alone},
\end{aligned}$$

by nested central elements of $\mathcal{U}\mathfrak{g}$.

Let $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be the trace form:

$\langle x, y \rangle = \mathrm{Tr}(xy)$, where x and y are viewed in a defining rep of $\mathfrak{g}$.

Let $\{b\}$ be a basis of $\mathfrak{g}$ and $\{b^*\}$ the dual basis wrt $\langle , \rangle$.

Let $\kappa = \sum_b bb^*$.

κ is the *Casimir invariant* and is central in $\mathcal{U}\mathfrak{g}$.

Theorem (D.)

Define $\Phi\colon \mathcal{G}_k \to \mathrm{End}(M \otimes N \otimes V^{\otimes k})$

$$\begin{aligned}
\Phi(t_j) &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(j-1)} \otimes t_1 \otimes \mathrm{id}_V^{\otimes(k-j-1)}, \\
\Phi(x_j) &= \tfrac{1}{2}(\kappa|_{M\otimes V^{\otimes j}} - \kappa|_{M\otimes V^{\otimes j-1}}), \\
\Phi(y_j) &= \tfrac{1}{2}(\kappa|_{N\otimes V^{\otimes j}} - \kappa|_{N\otimes V^{\otimes j-1}}), \\
\Phi(z_j) &= \tfrac{1}{2}(\kappa|_{M\otimes N\otimes V^{\otimes j}} - \kappa|_{M\otimes N\otimes V^{\otimes j-1}} + \kappa|_V), \\
\Phi(z_0) &= \tfrac{1}{2}(\kappa|_{M\otimes N} - \kappa|_M - \kappa|_N),
\end{aligned}$$

where $t_1 \cdot (v_{i_1} \otimes v_{i_2}) = v_{i_2} \otimes v_{i_1}$.
Then Φ *is a representation of* $\mathcal{G}_k$ *which commutes with the action of* $\mathfrak{g}$.

An Example:

Is there an algebra which has centralizers

$\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients

when $\mathfrak{g}$ is of type A?

Definition

Fix $a, b, p, q \in \mathbb{Z}_{>0}$.

The *degenerate extended two-boundary Hecke algebra* $\mathcal{H}_k^{\text{ext}}$ is the quotient of the degenerate two-boundary braid group by the relations

$$\begin{aligned} t_i x_i &= x_{i+1} t_i - 1, \\ t_i y_i &= y_{i+1} t_i - 1, \qquad i = 1, \ldots, k-1. \\ t_i z_i &= z_{i+1} t_i - 1, \end{aligned}$$

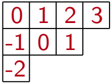
$$(x_1 - a)(x_1 + p) = 0 \qquad (y_1 - b)(y_1 + q) = 0.$$

The *degenerate two-boundary Hecke algebra* $\mathcal{H}_k$ is the subalgebra of $\mathcal{H}_k^{\text{ext}}$ generated by

$$x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k, t_1, \ldots, t_{k-1}.$$

(everything but z_0 . . . we'll come back to this.)

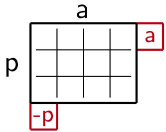
A *partition* is a collections of boxes:

$\lambda =$ 

If a box B is in row i and column j, then the *content* of B is

$$c(B) = j - i.$$

If $\lambda = (a^p)$ is rectangular, there are exactly two "addable" boxes:

$(a^p) =$ 

(recall relations $(x_1 - a)(x_1 + p) = 0$ and $(y_1 - b)(y_1 + q) = 0$)

Theorem (D.)

Fix $k < n$ non-neg. integers.
Let $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$, $N = L((b^q))$, and $V = L((1^1))$.

(1) Φ *is a rep. of $\mathcal{H}_k^{\text{ext}}$ which commutes with the $\mathfrak{g}$-action, so*

$$\Phi(\mathcal{H}_k^{\text{ext}}) \subseteq \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

(2) *For small cases,*

$$\Phi(\mathcal{H}_k^{\text{ext}}) = \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

Remark

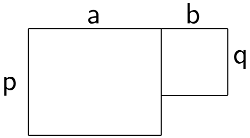
(1) When Φ is not surjective, the image differs by a portion of the action of the center of $\mathcal{U}\mathfrak{g}$ on $M \otimes N$.

(2) Same results for $\mathfrak{g} = \mathfrak{sl}_n$ and a shift of Φ.

Let $M = L((a^p))$ and $N = L((b^q))$. Then

$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda)$$ (multiplicity one!)

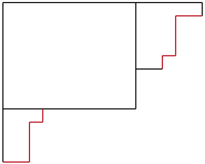
where Λ is the following set of partitions:... (Okata)



Let $M = L((a^p))$ and $N = L((b^q))$. Then

$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

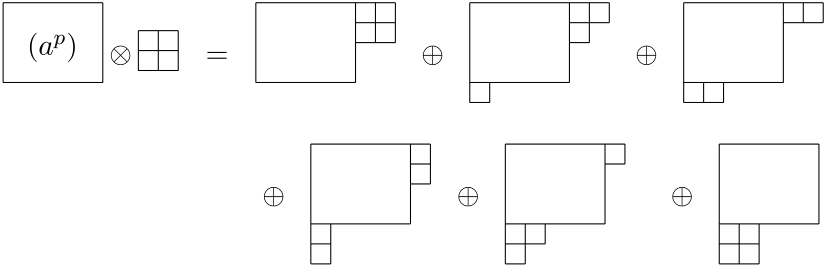
where Λ is the following set of partitions:... (Okata)

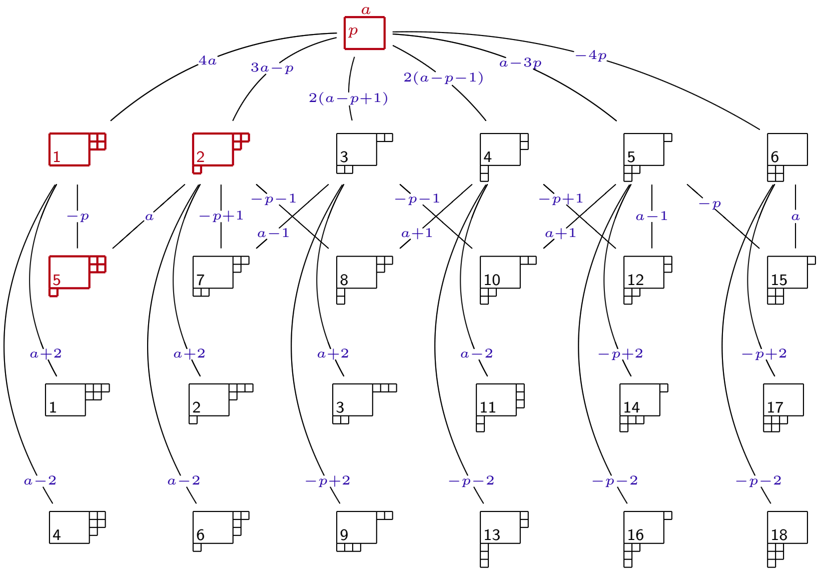


Let $M = L((a^p))$ and $N = L((b^q))$. Then

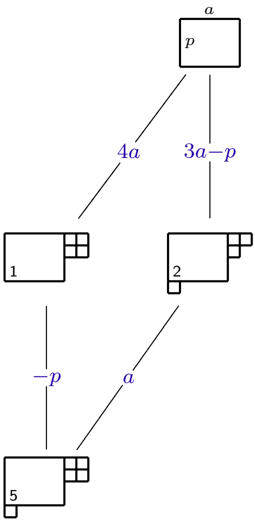
$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

where Λ is the following set of partitions:... (Okata)




a
p
4a
3a−p
2(a−p+1)
2(a−p−1)
a−3p
−4p
1
2
3
4
5
6
−p
a
−p+1
−p−1
a−1
−p−1
a+1
−p+1
a+1
a−1
−p
a
5
7
8
10
12
15
a+2
a+2
a+2
a−2
−p+2
−p+2
1
2
3
11
14
17
a−2
a−2
−p+2
−p−2
−p−2
−p−2
4
6
9
13
16
18

A two-dimensional $\mathcal{H}_1^{\mathrm{ext}}$-module:



$$z_0 = \begin{pmatrix} 4a & 0 \\ 0 & 3a-p \end{pmatrix}$$

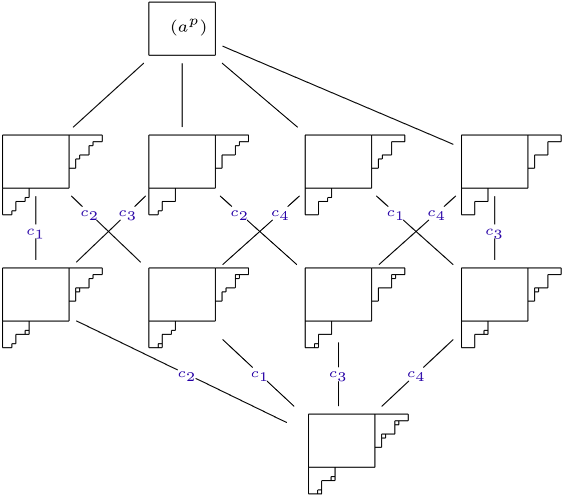
$$z_1 = \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$x_1 \sim \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$y_1 \sim \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

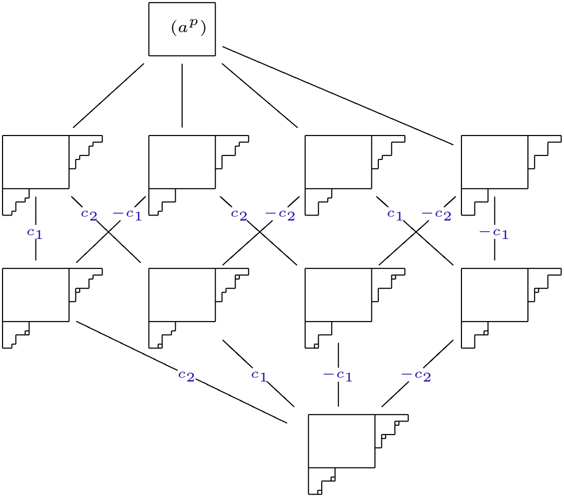
(formulas x_1, y_1, z_1, z_0 all given in terms of contents of added boxes)

An eight-dimensional $\mathcal{H}_2$-module:



where $c_3 = -c_1 + (a - p + b - q)$ and $c_4 = -c_2 + (a - p + b - q)$

An eight-dimensional $\mathcal{H}_2$-module:



Shift! Label edges by action of $z_1 - \frac{1}{2}(a-p+b-q)$ and $z_2 - \frac{1}{2}(a-p+b-q))$

Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$.

$\mathcal{H}_k$ is presented by generators

$$x_1, t_1, \ldots, t_{k-1}, w_1, \ldots, w_k,$$

and relations

$$\begin{gathered} \cdots \\ t_i w_i = w_{i+1} t_i - 1, \quad i = 1, \ldots, k-1, \\ x_1 w_1 = -w_1 x_1 + (a-p) w_1 + w_1^2 + \tfrac{1}{4}(a+p+b+q)(a+p-(b+q)), \\ x_1 w_1 = -w_1 x_1 + (a-p) w_1 + \underbrace{w_1^2 + \tfrac{1}{4}(a+p+b+q)(a+p-(b+q))}_{\text{central in } \mathcal{H}_1}, \end{gathered}$$

The type C graded Hecke algebra is presented by generators

$$x_1, t_1, \ldots, t_{k-1}, \varepsilon_1, \ldots, \varepsilon_k,$$

and relations

$$\begin{gathered} \cdots \text{(similar)} \cdots \\ t_i \varepsilon_i = \varepsilon_{i+1} t_i + c, \quad i = 1, \ldots, k-1, \\ x_1 \varepsilon_1 = -\varepsilon_1 x_1 + (a-p)\varepsilon_1 + \tfrac{1}{2}c(a-p) \end{gathered}$$

Punchline: In the quantized versions, the two-boundary Hecke algebra appears to be isomorphic to the type C affine Hecke algebra. Similarities are appearing suggestively in degenerate versions, where some computations are easier.

More examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Fix $k < n$ integers.
Let $L(\lambda)$ be the f.d. irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1)$.

1. When $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, and M and N are rectangular, we get the degenerate (extended) two-boundary Hecke algebra.
   (explored in thesis)
2. When $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, and M and N are rectangular, we get the *degenerate two-boundary Brauer algebra*.
   (future work)

Quantized versions should yield two-boundary Hecke and BMW algebras.

Two-boundary centralizer algebras are yielding familiar objects.

References

[OR] R. Orellana and A. Ram, *Affine braids, Markov traces and the category $\mathcal{O}$*, Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces Mumbai 2004, V.B. Mehta ed., Tata Institute of Fundamental Research, Narosa Publishing House, Amer. Math. Soc. (2007) 423-473.

[GN] J. de Gier and A. Nichols, *The two-boundary Temperley-Lieb algebra*, J. Algebra **321** (2009) 11321167.

In preparation:

[Dau] Z. Daugherty, *Degenerate two-boundary centralizer algebras*

[DRV] Z. Daugherty, A. Ram, R. Virk, *Affine and graded BMW algebras*

find me at...

http://www.math.wisc.edu/∼daughert/