Two-boundary centralizer algebras

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Background

Let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra.

e.g. $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$.

Let M, N, and V be finite dimensional simple $\mathfrak{g}$-modules.

Goal:

Understand $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.

(the set of endomorphisms which commute with the action of $\mathfrak{g}$)

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Fix $k < n$ integers.
Let $L(\lambda)$ be the f.d. irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1)$.

1. If $\mathfrak{g} = \mathfrak{sl}_n$ and
   - $M = N = L(0)$, this gives $\mathbb{C}S_k$;
   - $M = L(0)$ and $N = L(\lambda)$, this gives is a quotient of the graded Hecke algebra of type A;
2. If $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$ and
   - $M = N = L(0)$, this gives the Brauer algebra;
   - $M = L(0)$ and $N = L(\lambda)$, this gives a quotient of the degenerate affine Wenzl algebra.

Quantized versions yield standard and affine type A Hecke and Birman-Murakami-Wenzl algebras.

Big question:

Is there an algebra which has centralizers $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients?

Definition

The *two-boundary graded braid group* $\mathcal{G}_k$ is the $\mathbb{C}$-algebra generated by

$$\mathbb{C}S_k = \mathbb{C}\left\langle s_i \;\middle|\; \begin{array}{cc} i = 1, \dots k & \\ s_i^2 = 1 & \\ s_i s_j = s_j s_i & |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \end{array} \right\rangle$$

$$\mathbb{C}[z_0, z_1, \dots, z_k],\ \mathbb{C}[y_1, \dots, y_k],\ \mathbb{C}[x_1, \dots, x_k]$$

and relations...

Representations of $\mathcal{G}_k$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$\begin{aligned}
\mathbb{C}S_k &\qquad \text{permutes factors of } V^{\otimes k}, \\
\mathbb{C}[x_1, \ldots, x_k] &\qquad \text{acts on } M \text{ and } V^{\otimes k}, \\
\mathbb{C}[y_1, \ldots, y_k] &\qquad \text{acts on } N \text{ and } V^{\otimes k}, \\
\mathbb{C}[z_1, \ldots, z_k] &\qquad \text{acts on } M \otimes N \text{ together and } V^{\otimes k}, \\
z_0 &\qquad \text{acts on } M \otimes N \text{ alone},
\end{aligned}$$

by nested central elements of $\mathcal{U}\mathfrak{g}$.

Let $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be the trace form:

$\langle x, y \rangle = \mathrm{Tr}(xy)$, where x and y are viewed in a defining rep of $\mathfrak{g}$.

Let $\{b\}$ be a basis of $\mathfrak{g}$ and $\{b^*\}$ the dual basis wrt $\langle , \rangle$.

Let $\kappa = \sum_b bb^*$.

κ is the *Casimir invariant* and is central in $\mathcal{U}\mathfrak{g}$.

Theorem (D.)

Define $\Phi\colon \mathcal{G}_k \to \mathrm{End}(M \otimes N \otimes V^{\otimes k})$

$$
\begin{aligned}
\Phi(s_j) &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(j-1)} \otimes s_1 \otimes \mathrm{id}_V^{\otimes(k-j-1)}, \\
\Phi(x_j) &= \tfrac{1}{2}(\kappa|_{M\otimes V^{\otimes j}} - \kappa|_{M\otimes V^{\otimes j-1}}), \\
\Phi(y_j) &= \tfrac{1}{2}(\kappa|_{N\otimes V^{\otimes j}} - \kappa|_{N\otimes V^{\otimes j-1}}), \\
\Phi(z_j) &= \tfrac{1}{2}(\kappa|_{M\otimes N\otimes V^{\otimes j}} - \kappa|_{M\otimes N\otimes V^{\otimes j-1}} + \kappa|_V), \\
\Phi(z_0) &= \tfrac{1}{2}(\kappa|_{M\otimes N} - \kappa|_M - \kappa|_N),
\end{aligned}
$$

where $s_1 \cdot (v_{i_1} \otimes v_{i_2}) = v_{i_2} \otimes v_{i_1}$.
Then Φ *is a representation of* $\mathcal{G}_k$ *which commutes with the action of* $\mathfrak{g}$.

An Example:

Is there an algebra which has centralizers
$\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients
when $\mathfrak{g}$ is of type A?

Definition

Fix $a, b, p, q \in \mathbb{Z}_{>0}$.

The *extended two-boundary graded Hecke algebra* $\mathcal{H}_k^{\text{ext}}$ is the quotient of the two-boundary graded braid group by the relations

$$\begin{array}{rcl} t_{s_i} x_i & = & x_{i+1} t_{s_i} - 1, \\ t_{s_i} y_i & = & y_{i+1} t_{s_i} - 1, \qquad i = 1, \dots, k-1. \\ t_{s_i} z_i & = & z_{i+1} t_{s_i} - 1, \end{array}$$

$$(x_1 - a)(x_1 + p) = 0 \qquad (y_1 - b)(y_1 + q) = 0.$$

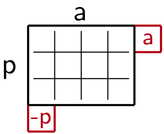
A *partition* is a collections of boxes:

$$\lambda = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} -2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

If a box B is in row i and column j, then the *content* of B is

$$c(B) = j - i.$$

If $\lambda = (a^p)$ is rectangular, there are exactly two "addable" boxes:

$(a^p) =$


(recall relations $(x_1 - a)(x_1 + p) = 0$ and $(y_1 - b)(y_1 + q) = 0$)

Theorem (D.)

Fix $k < n$ non-neg. integers.
Let $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$, $N = L((b^q))$, and $V = L((1^1))$.

(1) *Φ is a rep. of $\mathcal{H}_k^{ext}$ which commutes with the $\mathfrak{g}$-action, so*

$$\Phi(\mathcal{H}_k^{ext}) \subseteq \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

(2) *For suitable choices of a, b, p, q,*

$$\Phi(\mathcal{H}_k^{ext}) = \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

Remark

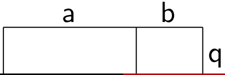
(1) When Φ is not surjective, the image differs by a portion of the action of the center of $\mathcal{U}\mathfrak{g}$ on $M \otimes N$.

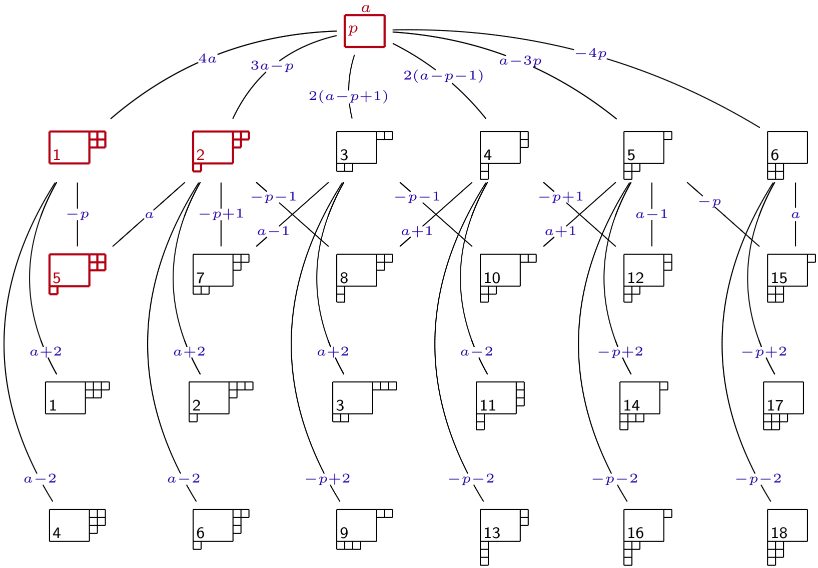
(2) Same theorem for $\mathfrak{g} = \mathfrak{sl}_n$ and a shift of Φ.

Let $M = L((a^p))$ and $N = L((b^q))$. Then

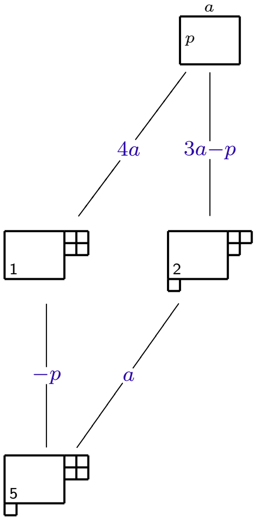
$$M \otimes N = \bigoplus_{\lambda \in \Lambda} L(\lambda) \qquad \text{(multiplicity one!)}$$

where Λ is the set of partitions:... (Okata)




a
p
4a
3a − p
2(a − p + 1)
2(a − p − 1)
a − 3p
−4p
1
2
3
4
5
6
−p
a
−p + 1
−p − 1
a − 1
−p − 1
a + 1
−p + 1
a + 1
a − 1
−p
a
5
7
8
10
12
15
a + 2
a + 2
a + 2
a − 2
−p + 2
−p + 2
1
2
3
11
14
17
a − 2
a − 2
−p + 2
−p − 2
−p − 2
−p − 2
4
6
9
13
16
18

A two-dimensional $\mathcal{H}_1^{\text{ext}}$-module:



$$z_0 = \begin{pmatrix} 4a & 0 \\ 0 & 3a-p \end{pmatrix}$$

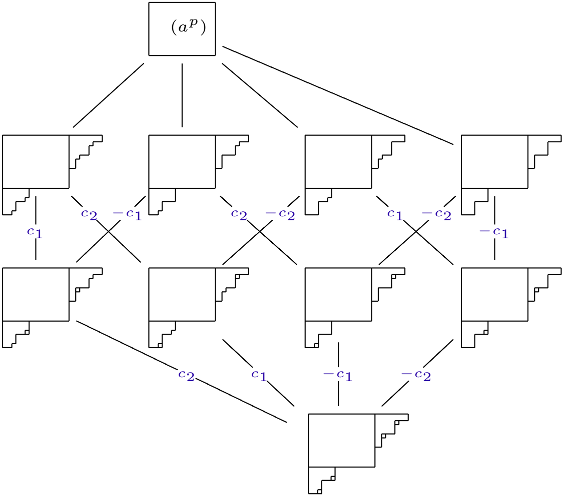
$$z_1 = \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$x_1 \sim \begin{pmatrix} -p & 0 \\ 0 & a \end{pmatrix}$$

$$y_1 \sim \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

(formulas x_1, y_1, z_1, z_0 all given in terms of contents of added boxes)

An eight-dimentional $\mathcal{H}_2^{\text{ext}}$-module:



(Labeling edges by action of $z_1 - \frac{1}{2}(a - p + b - q)$)

More examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Fix $k < n$ integers.
Let $L(\lambda)$ be the f.d. irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1)$.

1. When $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, and M and N are rectangular, we get the (extended) two-boundary graded Hecke algebra.
(explored in thesis)
2. When $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, and M and N are rectangular, we get the *two-boundary graded Brauer algebra*.
(future work)

Quantized versions should yield two-boundary affine Hecke and BMW algebras.
Striking: The two-boundary affine Hecke algebra is isomorphic to the type C affine Hecke algebra. Similarities also appear suggestively in graded versions.

References

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[GN] J. de Gier and A. Nichols, *The two-boundary Temperley-Lieb algebra*, J. Algebra **321** (2009) 11321167.

In preparation:

[Dau] Z. Daugherty, *Two-boundary graded centralizer algebras*

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find me at...

http://www.math.wisc.edu/∼daughert/