Building two boundary diagram algebras

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Background

Let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra.

e.g. $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$.

Let M, N, and V be finite dimensional simple $\mathfrak{g}$-modules.

Goal:
Understand $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.
(the set of endomorphisms which commute with the action of $\mathfrak{g}$)

Examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Fix $k < n$ integers.
Let $L(\lambda)$ be the f.d. irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1)$.

1. If $\mathfrak{g} = \mathfrak{sl}_n$ and
   - $M = N = L(0)$, this gives $\mathbb{C}S_k$;
   - $M = L(0)$ and $N = L(\lambda)$, this gives is a quotient of the graded Hecke algebra of type A;
2. If $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$ and
   - $M = N = L(0)$, this gives the Brauer algebra;
   - $M = L(0)$ and $N = L(\lambda)$, this gives a quotient of the degenerate affine Wenzl algebra.

Quantized versions yield standard and affine type A Hecke and Birman-Murakami-Wenzl algebras.

Big question:

Is there an algebra which has centralizers $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ as quotients?

Definition

The *two boundary graded braid group* $\mathcal{G}_k$ is the $\mathbb{C}$-algebra generated by

$$\mathbb{C}S_k = \mathbb{C}\left\langle s_i \;\middle|\; \begin{array}{cc} i = 1, \ldots k & \\ s_i^2 = 1 & \\ s_i s_j = s_j s_i & |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \end{array} \right\rangle$$

$$\mathbb{C}[z_0, z_1, \ldots, z_k],\; \mathbb{C}[y_1, \ldots, y_k],\; \mathbb{C}[x_1, \ldots, x_k]$$

and relations twisting the four factors together...

$\mathcal{G}_k$ contains three images of the graded braid group:

$$\frac{\mathbb{C}[z_1, \ldots, z_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[y_1, \ldots, y_k] \otimes \mathbb{C}S_k}{\sim} \cong \frac{\mathbb{C}[x_1, \ldots, x_k] \otimes \mathbb{C}S_k}{\sim}$$

and

$$z_i = x_i + y_i - \textit{lower terms},$$

We'll define an action of $\mathcal{G}_k$ on $M \otimes N \otimes V^{\otimes k}$:

$$\begin{aligned}
\mathbb{C}S_k &\qquad \text{permutes factors of } V^{\otimes k}, \\
\mathbb{C}[x_1, \ldots, x_k] &\qquad \text{acts on } M \text{ and } V^{\otimes k}, \\
\mathbb{C}[y_1, \ldots, y_k] &\qquad \text{acts on } N \text{ and } V^{\otimes k}, \\
\mathbb{C}[z_1, \ldots, z_k] &\qquad \text{acts on } M \otimes N \text{ together and } V^{\otimes k}, \\
z_0 &\qquad \text{acts on } M \otimes N \text{ alone},
\end{aligned}$$

by nested central elements of $\mathcal{U}\mathfrak{g}$.

Let $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be the trace form:

$\langle x, y \rangle = \mathrm{Tr}(xy),$ where x and y are viewed in a defining rep of $\mathfrak{g}$.

Let $\{b\}$ be a basis of $\mathfrak{g}$ and $\{b^*\}$ the dual basis wrt $\langle , \rangle$.

Let $\kappa = \sum_b bb^*$.

κ is the *Casimir invariant* and is central in $\mathcal{U}\mathfrak{g}$.

Theorem (D)

Define $\Phi\colon \mathcal{G}_k \to \mathrm{End}(M \otimes N \otimes V^{\otimes k})$

$$\begin{aligned}
\Phi(s_j) &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(j-1)} \otimes s_1 \otimes \mathrm{id}_V^{\otimes(k-j-1)}, \\
\Phi(x_j) &= \tfrac{1}{2}(\kappa|_{M\otimes V^{\otimes j}} - \kappa|_{M\otimes V^{\otimes j-1}}), \\
\Phi(y_j) &= \tfrac{1}{2}(\kappa|_{N\otimes V^{\otimes j}} - \kappa|_{N\otimes V^{\otimes j-1}}), \\
\Phi(z_j) &= \tfrac{1}{2}(\kappa|_{M\otimes N\otimes V^{\otimes j}} - \kappa|_{M\otimes N\otimes V^{\otimes j-1}} + \kappa|_V), \\
\Phi(z_0) &= \tfrac{1}{2}(\kappa|_{M\otimes N} - \kappa|_M - \kappa|_N),
\end{aligned}$$

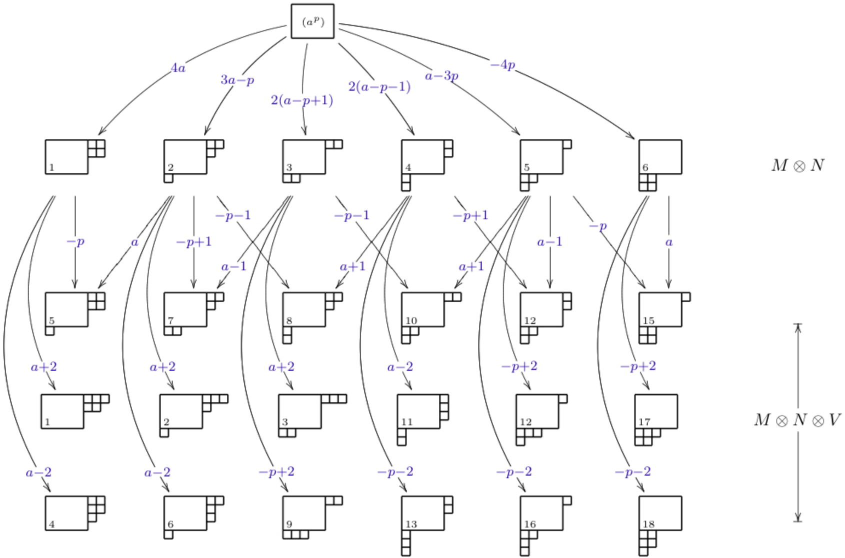
where $s_1 \cdot (v_{i_1} \otimes v_{i_2}) = v_{i_2} \otimes v_{i_1}$.
Then Φ *is a representation of* $\mathcal{G}_k$ *which commutes with the action of* $\mathfrak{g}$.

More examples of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$

Fix $k < n$ integers.
Let $L(\lambda)$ be the f.d. irreducible $\mathfrak{g}$-module of highest weight λ.
Let $V = L(\omega_1)$.

1. When $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, and M and N are rectangular, we get the *two boundary graded Hecke algebra*.
(explored in thesis)
2. When $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, and M and N are rectangular, we get the *two boundary graded Brauer algebra*.
(future work)

Quantized versions should yield two boundary affine Hecke and BMW algebras.
Striking: The two boundary affine Hecke algebra is isomorphic to the type C affine Hecke algebra. Similarities also appear suggestively in graded versions.


(a^p)
$4a$
$3a-p$
$2(a-p+1)$
$2(a-p-1)$
$a-3p$
$-4p$
1
2
3
4
5
6
$M \otimes N$
$-p$
a
$-p+1$
$-p-1$
$a-1$
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$M \otimes N \otimes V$
$a-2$
$a-2$
$-p+2$
$-p-2$
$-p-2$
$-p-2$
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In preparation:

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find me at...

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