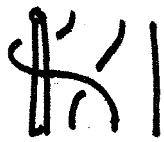
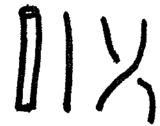
Definitions

The affine braid group B_k is the group of braids on k strands in a space with one puncture:

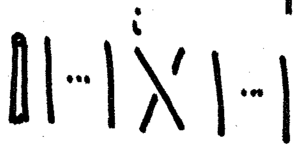
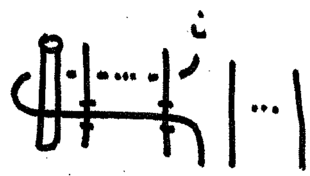
$b_1 =$  $\in B_3$, $b_2 =$ 

puncture or "flag pole"

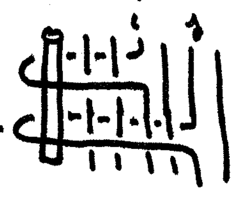
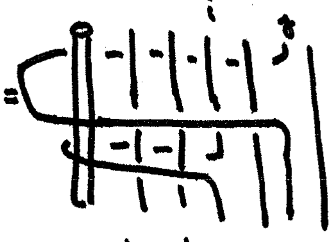
with mult done by stacking:

$b_1 b_2 =$ 

B_k is generated by

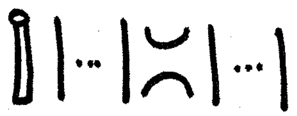
$T_i =$  $i = 1, \dots, k-1$ & $Y_i =$  $i = 1, \dots, k$

notice:

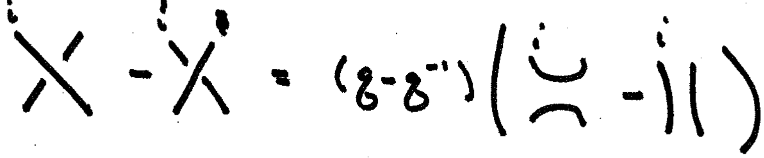
$Y_i Y_j =$  $=$  $= Y_j Y_i$

So $\mathbb{C}[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$ is a subalgebra of $\mathbb{C}B_k$, (group algebra w/ complex coefs)

Fix $q, z \in \mathbb{C}^*$ and let

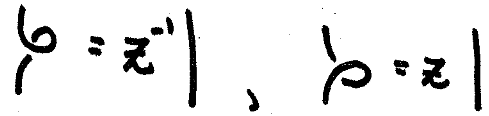
$E_i =$  $\in \mathbb{C}B_k$ $\quad i = 1, \dots, k$

be defined by

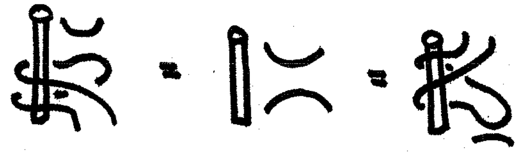


(skein relation, Kauffman poly)

Defn. The affine Birman-Murakami-Wenzel (BMW) algebra W_k is the quotient of $\mathbb{C}B_k$ by rels

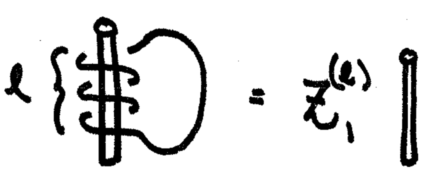


(Reidemeister 1 for ribbons)



& R2 & R3

and



where $Z_1^{(l)} \in \mathbb{C}$ for $l \in \mathbb{Z}$

can wrap backwards!

note: $Z_1^{(l)}$ form infinite family of constants. Many choices give $W_k = 0$! $Z_1^{(0)} = \frac{z - z^{-1}}{q - q^{-1}} + 1$

Examples:

① $\mathbb{C}\langle T_i, E_i, i = 1, \dots, k-1\rangle$ = BMW algebra, [Birman-Wenzl '89, Murakami '87, Kauffman polys.

② $W_k / \langle E_i = 0, i = 1, \dots, k-1\rangle \cong$ Affine Hecke algebra of type A (GL_n) [Lusztig ~89

③ $W_k / \langle (Y_1 - u_1) \cdots (Y_1 - u_r) = 0\rangle$ = Cyclotomic BMW algebra

(Affine braids: Orellana & Ram 2004)

Degenerate versions

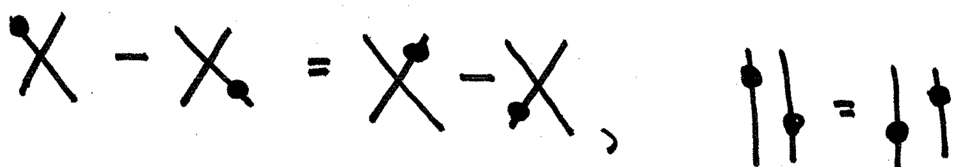
for now think: degeneration = flatten and take log of everything

multiplication $\longrightarrow$ addition

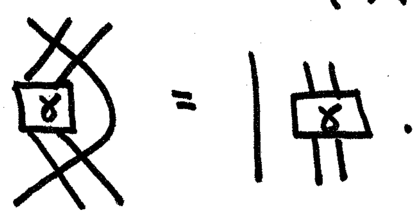
The degenerate affine braid group $\mathbb{B}_k$ is the algebra / $\mathbb{C}$ of "dotted" permutations of k:

$b =$  $\in \mathbb{B}_3$ (mult by stacking)

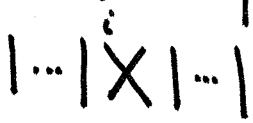
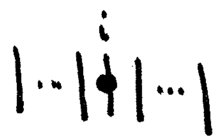
w/ relations



and if $\gamma_{i,i+1} = -($  $)$ then

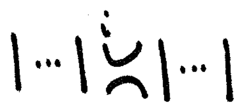


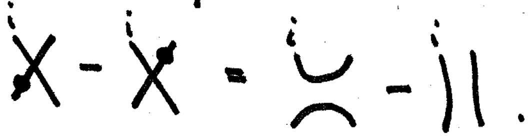
$\mathbb{B}_k$ is generated by

$t_i =$  $i = 1, \dots, k-1$ and $x_i =$  $i = 1, \dots, k$

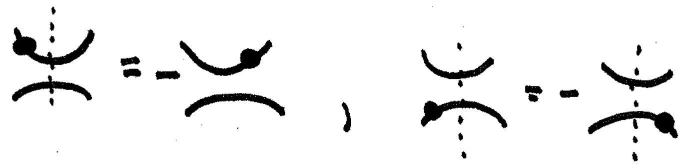
Sym grp $S_k = \left\langle t_i \;\middle|\; \begin{array}{l} t_i^2 = 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ t_i t_j = t_j t_i \quad |i-j| > 1 \end{array} \right\rangle$, $\quad \mathbb{C}[x_1, \dots, x_k] \subset \mathbb{B}_k$

$\uparrow$ sub alg

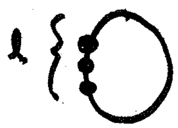
Fix $\epsilon = \pm 1$, and
let $e_i =$  $i = 1, \dots, k-1$
be defined by



Defn. The degenerate affine BMW algebra $\mathbb{W}_k$ is the quotient of $\mathcal{B}_k$ by the rels

$\varphi = \epsilon \,|$, 

and

 $= z_1^{(l)}$ where $z_1^{(l)} \in \mathbb{C}$ for $l = 0, 1, 2, \dots$ (no backwards wrapping!)

Again, lots of choices of $z_1^{(l)}$'s give $\mathbb{W}_k = 0$.

Introduced by Nazarov (~96) studying Jucys-Murphy ops on irreps of Brauer algs. ("Youngs orthog form for Brauers' ...")

Examples

① $\mathbb{C}\langle t_i, e_i, \; i = 1, \dots, k-1 \rangle$ = Brauer algebra

② $\mathbb{W}_k / \langle e_i = 0, \; i = 1, \dots, k-1 \rangle \cong$ graded Hecke algebra of type A (Lusztig ~89)

③ $\mathbb{W}_k / \langle (X_1 - u_1) \cdots (X_1 - u_r) = 0 \rangle$ = cyclotomic deg aff BMW

Studied by Ariki - Mathas - Rui ~2006
what $z_1^{(l)}$'s work w/ what $\bar{u}$'s?

Where else do these appear?

Crash course in Schur-Weyl duality:

If A is a nice algebra

M is an A-module (think: $A \subseteq \mathrm{End}(M)$)

$B = \mathrm{End}_A(M) = \{ b \in \mathrm{End}(M) \mid ab = ba \ \forall a \in A \}$

"centralizer"

Then the irred. A-mods in M are paired w/ irred. B-mods in M in a way which gives info about multiplicities, dimensions, etc.

Fav. example: A "=" $\mathbb{C}S_k$, $M = \underbrace{\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k}$, B "=" $\mathbb{C}Gl_n(\mathbb{C})$

↑ irreps indexed by partitions of k ⟹ by varying k, we learned irreps of Gl_n indexed by part'ns of k too.

Back to BMW:

let $\mathfrak{g}$ be the Lie algebra $sp_{2n}(\mathbb{C})$, $so_{2n}(\mathbb{C})$, or $so_{2n+1}(\mathbb{C})$

$$A = \begin{cases} ① \ U\mathfrak{g} & \text{the enveloping alg of } \mathfrak{g} \\ ② \ U_q\mathfrak{g} & \text{the D-J quantum grp of } \mathfrak{g} \end{cases}$$

$$M = L(\lambda) \otimes \underbrace{L(\omega_1) \otimes \cdots \otimes L(\omega_1)}_{k}$$

$L(\lambda)$ ↑ highest wt mod; $L(\omega_1)$ ↖ first fund rep of A

$$B \text{ "}\simeq\text{" } \begin{cases} ① \ \mathbb{W}_k \\ ② \ W_k \end{cases}$$

ie there's an action of B which commutes and...

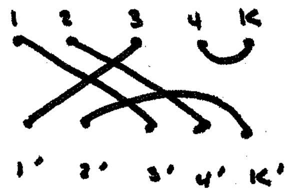
(Other sense of degeneration.)

Check time

5

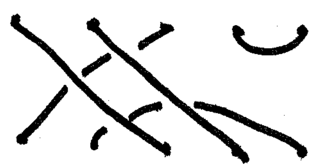
Bases:

A **Brauer diagram** on k vertices is

$d =$ 

$= \{(1,3'), (2,4'), (3,1'), (4,k), (2',k')\}$

Each diagram d has lots of elts in BMW that flatten to d. Systematically pick one, call it T_d.

ex. $T_d =$ 

Also, for each d, pick a node from each edge (ex $1, 2, 3, 4, 2'$)

Call top nodes $i_1, \dots, i_\ell$ $\quad (1, 2, 3, 4)$

bot nodes $i_{\ell-k}, \dots, i_k$ $\quad (2')$

Thm

① [Nazarov] $\mathbb{W}_k$ has basis

$$\left\{ x_{i_1}^{n_1} \cdots x_{i_\ell}^{n_\ell}\, d\, x_{i_{k-\ell}}^{n_{k-\ell}} \cdots x_{i_k}^{n_k} \;\middle|\; \begin{array}{l} d \text{ digrams} \\ n_j \in \mathbb{Z}_{\geq 0} \end{array} \right\}$$

② [Goodman-Mosley] W_k has basis

$$\left\{ Y_{i_1}^{n_1} \cdots Y_{i_\ell}^{n_\ell}\, T_d\, Y_{i_{k-\ell}}^{n_{k-\ell}} \cdots Y_{i_k}^{n_k} \;\middle|\; \begin{array}{l} d \text{ diagram} \\ n_j \in \mathbb{Z} \end{array} \right\}$$

Central elements:

some history:

The center of the affine Hecke alg $(H_k \cong W_k/_{E_i=0})$ is

$$Z(H_k) = \mathbb{C}[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]^{S_k}$$

(symmetric laurent polys) [Bernstein - Zelevinsky, Lusztig ~83]

and the center of the graded Hecke alg $(\mathbb{H}_k \cong \mathbb{W}_k/_{e_i=0})$ is

$$Z(\mathbb{H}_k) = \mathbb{C}[x_1, \dots x_k]^{S_k}$$

(symmetric polys in x's) [Lusztig ~89]

Thm (Daugherty - Ram - Virk)

① $Z(\mathbb{W}_k) = \{ p \in \mathbb{C}[x_1, \dots, x_k]^{S_k} \mid p(x_1, -x_1, x_3, \dots, x_k) = p(0, 0, x_3, \dots, x_k) \}$

$\downarrow$ exp

② $Z(W_k) = \{ p \in \mathbb{C}[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]^{S_k} \mid p(Y_1, Y_1^{-1}, Y_3, \dots, Y_k) = p(1, 1, Y_3, \dots, Y_k) \}$

pf Use basis to

1. eliminate crossings & horiz edges $(T_i/t_i \ \& \ E_i/e_i)$
2. show symmetry
3. give cancellation properties.

examples

① let $p_i = x_1^i + \cdots + x_k^i$ be the i^{th} power sum

Then for ℓ odd, since $x_1^\ell + (-x_1)^\ell = 0$

$$p_\ell(x_2 \to -x_1) = x_3^\ell + \cdots + x_k^\ell$$

so $p_\ell \in Z(\mathbb{W}_k)$. So $\mathbb{C}[p_1, p_3, \ldots] \subseteq Z(\mathbb{W}_k)$

② let $p_i^- = p_i(y\text{'s}) - p_{-i}(y\text{'s}) = y_1^i + \cdots + y_k^i - (y_1^{-i} + \cdots + y_k^{-i})$

Then $p_i^-(y_2 \to y_1^{-1}) = y_1^i + y_1^{-i} + y_3^i + \cdots + y_k^i$

$$- (y_1^{-i} + y_1^i + y_3^{-i} + \cdots + y_k^{-i})$$

so $p_i^- \in Z(\mathcal{W}_k)$

Also $e_k = y_1 \cdots y_k \in Z(\mathcal{W}_k)$

k^{th} elementary pol

So $\mathbb{C}[e_k, p_1^-, p_2^-, \ldots] \subseteq Z(\mathcal{W}_k)$

<u>Cor 1</u> $\underline{\mathbb{C}[p_1, p_3, \ldots] = Z(\mathbb{W}_k)}$

pf. • Claimed by Nazarov, w/out proof.

• In Stanley EC2 in infinitely many variables

• For finite variables, Pragacz used Schur Q-functions to show all sym functions w/ "Q-cancellation" ($x_2 \to -x_1$ $\equiv$ $x_1, x_2 \to 0$) are ~~[illegible]~~ Q-fns ~~[illegible]~~ for strict partitions and so sit in $\mathbb{C}[p_1, p_3, \ldots]$.

<u>Cor 2</u>* $\mathbb{C}[e_k, p_1^-, p_2^-, \ldots] = Z(\mathcal{W}_k)$

pf. we've done infinitely many variables (ie $\deg(p) < k$)