Degenerate two-boundary centralizer algebras

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Abstract

Diagram algebras (e.g. graded braid groups, Hecke algebras, Brauer algebras) arise as tensor power centralizer algebras, algebras of commuting operators for a Lie algebra action on a tensor space. This work explores centralizers of the action of a complex reductive Lie algebra $\mathfrak{g}$ on tensor space of the form $M \otimes N \otimes V^{\otimes k}$. We define the degenerate two-boundary braid group $\mathcal{G}_k$ and show that centralizer algebras contain quotients of this algebra in a general setting. As an example, we study in detail the combinatorics of special cases corresponding to Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ and modules M and N indexed by rectangular partitions. For this setting, we define the degenerate two-boundary Hecke algebra $\mathcal{H}_k^{\text{ext}}$ as a quotient of $\mathcal{G}_k$, and show that a quotient of $\mathcal{H}_k^{\text{ext}}$ is isomorphic to a large subalgebra of the centralizer. We further study the representation theory of $\mathcal{H}_k^{\text{ext}}$ to find that the seminormal representations are indexed by a known family of partitions. The bases for the resulting modules are given by paths in a lattice of partitions, and the action of $\mathcal{H}_k^{\text{ext}}$ is given by combinatorial formulas.

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Contents

Chapter 1

Introduction

This work is in the construction and study of algebras that are very similar in nature to the group algebra of the symmetric group. Early work of Frobenius laid the groundwork for studying groups using character theory. He specifically outlined the rich and beautiful structure of the character theory of symmetric group through the use of combinatorial tools. Schur's thesis then provided the link between the combinatorics developed for the symmetric group S_k and similar phenomena appearing in the character theory of the general linear group $\mathrm{GL}_n(\mathbb{C})$. He brought these two groups together using the insight that the action of $\mathrm{GL}_n(\mathbb{C})$ on tensor space $(\mathbb{C}^n)^{\otimes k}$ centralizes the action of S_k, and that this relationship could be used to produce irreducible modules. This phenomenon is now known as *Schur-Weyl duality*—the general statement is that if A is the full centralizer of the action of a semisimple algebra B on a B-module M, then M decomposes into direct irreducible summands

$$M \cong \bigoplus_{\lambda} A^{\lambda} \otimes B^{\lambda}$$

as an (A, B)-bimodule, where A^{λ} and B^{λ} are distinct irreducible modules for A and B, respectively. This means that the representation theory of A is "determined" by the representation theory of B, and vice versa.

The centralizer property stimulated many advances in the development of *tensor power centralizer algebras*, algebras of operators which preserve symmetries in a tensor space. Striking examples include:

1. the *Brauer algebras* in [Br] centralize the action of symplectic and orthogonal groups on tensor space $(\mathbb{C}^n)^{\otimes k}$;

2. the *graded Hecke algebra of type A* centralizes the action of $\mathfrak{sl}_n$ on $L(\lambda) \otimes (\mathbb{C}^n)^{\otimes k}$, where $L(\lambda)$ is the irreducible $\mathfrak{sl}_n$ module indexed by a partition λ (see [AS]);

3. the *degenerate affine Wenzl algebra* in [Naz] centralizes the action of symplectic and orthogonal groups on $L(\lambda) \otimes (\mathbb{C}^n)^{\otimes k}$.

The paper of Orellana and Ram [OR] provides a unified approach to studying tensor power centralizer algebras, including the *affine and cyclotomic Hecke and Birman-Murakami-Wenzl algebras.*

Recent work in the study of loop models and spin chains in statistical mechanics uncovered yet another potential use of Schur-Weyl duality in [GN]. Specifically, a connection was discovered between the two-boundary Temperley-Lieb algebra and a quotient of the affine Hecke algebra of type C. Since the Temperley-Lieb algebra is the centralizer of the quantum group $\mathcal{U}_q\mathfrak{sl}_2$ on tensor space $M \otimes N \otimes (\mathbb{C}^2)^{\otimes k}$, where M and N are simple $\mathcal{U}_q\mathfrak{sl}_2$-modules, this connection opened the community's eyes to the possibility of constructing affine Hecke algebra type C modules explicitly using Schur-Weyl duality tools.

In Section 3, we begin the study of the centralizer of $\mathfrak{g}$ on $M \otimes N \otimes V^{\otimes k}$, where $\mathfrak{g}$ is a finite dimensional complex reductive Lie algebra and M, N, and V are finite dimensional irreducible $\mathfrak{g}$-modules. The new definition is that of the *degenerate two-boundary braid group* $\mathcal{G}_k$, an associative algebra over the complex numbers. The structure of $\mathcal{G}_k$ is

$$\mathbb{C}[z_0, z_1, \ldots, z_k] \otimes \mathbb{C}[y_1, \ldots, y_k] \otimes \mathbb{C}[x_1, \ldots, x_k] \otimes \mathbb{C}S_k,$$

with relations twisting the polynomial rings and the symmetric group together. The first main theorem, Theorem 3.3, is that $\mathcal{G}_k$ acts on $M \otimes N \otimes V^{\otimes k}$ and that this action commutes with the action of $\mathfrak{g}$. In many examples, this action will produce a large subalgebra of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.

In Section 4, we consider the cases where $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, $M = L((a^p))$ (the finite dimensional irreducible $\mathfrak{g}$-module indexed by the rectangular partition with p parts of length a), $N = ((b^q))$, and V is the first fundamental representation. In this case, we identify a large subalgebra of the centralizer as a quotient of $\mathcal{G}_k$. Theorem 4.6 states that the representations given in Theorem 3.3 and Corollary 3.5 factor through this quotient. We call this the *extended degenerate two-boundary Hecke algebra* $\mathcal{H}_k^{\mathrm{ext}}$, as it is a generalization of the graded Hecke algebra of type A and is related to the two-boundary Hecke algebra.

We further study the representation theory of $\mathcal{H}_k^{\mathrm{ext}}$ in Sections 4.2 and 4.3, and find that the seminormal representations are indexed by partitions which index $\mathfrak{g}$-submodules of $M \otimes N \otimes V^{\otimes k}$. Using the combinatorics of Young tableaux, we describe these representations explicitly in Section 4.2 and Theorem 4.15. The basis for the resulting modules are given by paths in a lattice of partitions, and the action of $\mathcal{H}_k^{\mathrm{ext}}$ is given in terms of contents of boxes in those partitions.

One subalgebra of $\mathcal{H}_k^{\mathrm{ext}}$, the *degenerate two-boundary Hecke algebra* $\mathcal{H}_k$, is of particular interest as it is strikingly similar to the graded Hecke algebra of type C. This observation opens a new door to studying representations of type C Hecke algebras using Schur-Weyl duality techniques.

Chapter 2

Preliminaries

2.1 Structure

Good sources for the background provided here on Lie algebras include [GW], [Hum] and [Ser].

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A *weight* is an element of $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$, the dual space of $\mathfrak{h}$. If M is a $\mathfrak{g}$-module and $\mu \in \mathfrak{h}^*$, the *μ-weight space* of M is

$$M_\mu = \{m \in M \mid hm = \mu(h)m, \text{ for all } h \in \mathfrak{h}\}. \tag{2.1}$$

Furthermore, if M is finite dimensional, then M decomposes into weight spaces as $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$.

The vector space $\mathfrak{g}$ is a $\mathfrak{g}$-module under the adjoint action:

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g} \qquad \text{given by} \qquad \mathrm{ad}_x(y) = [x, y].$$

Under the adjoint action, $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [h, x] = 0 \text{ for all } h \in \mathfrak{g}\} = \mathfrak{h}$, and $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right), \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{g}\},$$

where $R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$ is the set of *roots*.

The set R is the root system of $\mathfrak{g}$, whose structure is explored in depth in [Hum, Ch. 8-10] and [Ser, Ch. 5]. In particular, the roots of $\mathfrak{g}$ span $\mathfrak{h}^*$, and we choose a basis $\{\alpha_1, \ldots, \alpha_n\} \subseteq R$ for $\mathfrak{h}^*$, called the set of *simple roots* (see [Hum, Ch.10, §1], [Ser, Ch.5, §8]). This basis is chosen to have the property that it generates R by integral linear combinations which have either all non-negative or all non-positive coefficients. Let R^+ be the set of *positive roots*, the roots which are non-negative combinations of the simple roots, and let $R^- = \{-\alpha \mid \alpha \in R^+\}$ be the set of *negative roots*, so $R = R^+ \cup R^-$. This fixes the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \text{where} \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha. \tag{2.2}$$

The *highest weight module* $L(\lambda)$ is a $\mathfrak{g}$-module generated by a nonzero highest weight vector v_λ^+ of weight λ, i.e.

$$hv_\lambda^+ = \lambda(h)v_\lambda^+ \quad \text{and} \quad xv_\lambda^+ = 0 \quad \text{for } x \in \mathfrak{n}^+, h \in \mathfrak{h}.$$

The *Killing form* is the symmetric bilinear form $\mathcal{K}(,):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ defined by

$$\mathcal{K}(x,y)=\mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y).$$

The properties of this form are explored in [Hum, Ch. 5]. A *trace form* is the symmetric bilinear form $\langle,\rangle:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ associated to an injective representation ϕ of $\mathfrak{g}$, defined by

$$\langle x,y\rangle=\mathrm{Tr}(\phi(x)\phi(y)),$$

When $\mathfrak{g}$ is simple, it follows from Schur's lemma that the trace form differs from the Killing form by a constant. For the purposes of this exposition, we concentrate primarily on types $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ and define ϕ explicitly in Section 2.2, and

$$\mathcal{K}(x,y)=2n\langle x,y\rangle \qquad \text{when } \mathfrak{g}=\mathfrak{sl}_n.$$

However, if $\mathfrak{g}=\mathfrak{gl}_n$, $\mathcal{K}(,)$ is degenerate on the center but $\langle,\rangle$ may be nondegenerate. In fact, for the choice of ϕ in Section 2.2, $\langle,\rangle$ is nondegenerate on $\mathfrak{g}$ and $\mathfrak{h}$. The trace form, like $\mathcal{K}(,)$, is also ad-invariant, meaning that $\langle[x,y],z\rangle=-\langle y,[x,z]\rangle$ for $x,y,z\in\mathfrak{g}$.

As a result of these properties of the trace form, the map

$$\begin{array}{ccc}\mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ h & \mapsto & \langle h,\cdot\rangle \\ h_\mu & \mapsto & \mu\end{array} \qquad \text{is an isomorphism,} \tag{2.3}$$

where h_μ is the unique element of $\mathfrak{h}$ such that

$$\langle h_\mu,h\rangle=\mu(h) \quad \text{for all } h\in\mathfrak{h}. \tag{2.4}$$

Extending notation, we define a form $\langle,\rangle:\mathfrak{h}^*\otimes\mathfrak{h}^*\to\mathbb{C}$ by

$$\langle\lambda,\mu\rangle=\langle h_\lambda,h_\mu\rangle. \tag{2.5}$$

Then $\langle,\rangle:\mathfrak{h}^*\otimes\mathfrak{h}^*\to\mathbb{C}$ is also symmetric, bilinear, and nondegenerate on $\mathfrak{h}^*$, and

$$\langle\lambda,\mu\rangle=\mu(h_\lambda)=\lambda(h_\mu). \tag{2.6}$$

2.2 Type A Lie algebras and their weights

All finite dimensional simple Lie algebras over $\mathbb{C}$ are either members of the four infinite families $\{A_n \mid n\geq 1\}$, $\{B_n \mid n\geq 3\}$, $\{C_n \mid n\geq 2\}$, $\{D_n \mid n\geq 4\}$ (the *classical Lie algebras*), or are one of the five exceptional Lie algebras G_2, F_4, E_6, E_7, and E_8. A semisimple Lie algebra is a direct sum of simple Lie algebras, and a reductive Lie algebra is the sum of a semisimple Lie algebra and an abelian Lie algebra. In this work, we concentrate predominantly on the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$. While type A strictly refers to the family $\mathfrak{sl}_n$, $\mathfrak{gl}_n$ exhibits a similar combinatorial structure, and so we study both in tandem.

Let $V = \mathbb{C}^{n+1}$ with orthonormal basis $\{v_1, \ldots, v_{n+1}\}$. Let $E_{ij} \in \mathrm{End}(V)$ be defined by

$$E_{ij} v_k = \delta_{jk} v_i,$$

where δ_{jk} is the Kronecker delta. We view each of the following Lie algebras as a Lie-subalgebra of the Lie algebra $\mathrm{End}(V)$ of all $(n+1)\times(n+1)$ matrices, with commutator product $[x, y] = xy - yx$.

The *general linear algebra* is $\mathfrak{gl}_{n+1} \cong \mathfrak{gl}(V) = \mathrm{End}(V)$. It has basis

$$\{b_{ij}, h_i \mid 1 \leq i \neq j \leq n+1\}, \qquad \text{where} \qquad b_{ij} = E_{ij} \quad \text{and} \quad h_i = E_{ii}.$$

The dual basis with respect to $\langle , \rangle$ is given by

$$b_{ij}^* = b_{ji} \quad \text{and} \quad h_i^* = h_i.$$

The triangular decomposition is given by $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{h_i \mid 1 \leq i \leq n\}$, $\mathfrak{n}^+ = \mathrm{span}_{\mathbb{C}}\{b_{ij} \mid 1 \leq i < j \leq n\}$, and $\mathfrak{n}^- = \mathrm{span}_{\mathbb{C}}\{b_{ij}^* \mid b_{ij} \in \mathfrak{n}^+\}$.

Next, A_n is the *special linear algebra* $\mathfrak{sl}_{n+1} \cong \mathfrak{sl}(V) = \{x \in \mathrm{End}(V) \mid \mathrm{tr}(x) = 0\}$. It has basis

$$\{b_{ij}, h_k \mid 1 \leq i \neq j \leq n+1, 1 \leq k \leq n\}, \qquad \text{where} \qquad b_{ij} = E_{ij} \text{ and } h_k = E_{kk} - E_{k+1\ k+1},$$

with dual basis with respect to $\langle , \rangle$ given by

$$b_{ij}^* = b_{ji} \quad \text{and} \quad h_k^* = E_{11} + \cdots + E_{kk} - \frac{k}{n+1}\left(E_{11} + \cdots + E_{n+1\ n+1}\right).$$

The triangular decomposition is given by $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{h_i \mid 1 \leq i \leq n\}$, $\mathfrak{n}^+ = \mathrm{span}_{\mathbb{C}}\{b_{ij} \mid 1 \leq i < j \leq n+1\}$, and $\mathfrak{n}^- = \mathrm{span}_{\mathbb{C}}\{b_{ij}^* \mid b_{ij} \in \mathfrak{n}^+\}$.

The center of $\mathfrak{gl}_{n+1}$ is the span of $\sum_{i=1}^{n+1} h_i$, and therefore $\mathfrak{gl}_{n+1}$ is not semisimple. However,

$$\mathfrak{gl}_{n+1} \cong \mathfrak{sl}_{n+1} \oplus \mathbb{C},$$

so $\mathfrak{gl}_{n+1}$ is reductive.

Now, we calculate the difference between $\mathcal{K}(,)$ and $\langle , \rangle$ on $\mathfrak{sl}_{n+1}$. Since $\mathfrak{sl}_{n+1}$ is simple, we need only calculate the difference in one case; we consider the element $h_1 = E_{11} - E_{22}$. First,

$$\langle h_1, h_1 \rangle = \mathrm{tr}(E_{11}^2 + E_{22}^2) = 2.$$

Next,

$$\mathrm{ad}_{h_1}(E_{ij}) = \begin{cases} 0 & \text{if } i = j \text{ or } i, j \neq 1, 2, \\ 2E_{12} & \text{if } i = 1, j = 2, \\ -2E_{21} & \text{if } i = 2, j = 1, \\ E_{ij} & \text{if } i = 1 \text{ and } j \neq 1, 2 \text{ or } j = 2 \text{ and } i \neq 1, 2, \\ -E_{ij} & \text{if } i = 2 \text{ and } j \neq 1, 2 \text{ or } j = 1 \text{ and } i \neq 1, 2. \end{cases}$$

So

$$\mathcal{K}(h_1, h_1) = 4 + 4 + 2(n-1) + 2(n-1) = 4n + 4 = 4(n+1)\langle h_1, h_1\rangle.$$

Thus $\mathcal{K}(x, y) = 2(n+1)\langle x, y\rangle$.

2.2.1 The weights of $\mathfrak{sl}_n$

Let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal basis of the vector space $\mathbb{R}^n$. Then

$$\mathfrak{h}^* \cong \{\ a_1\varepsilon_1 + \cdots + a_n\varepsilon_n \mid a_i \in \mathbb{R}, a_1 + \cdots + a_n = 0\ \},$$

where ε_i is identified with the weight

$$\varepsilon_i(h_j) = \delta_{i,j} - \delta_{i,j+1}.$$

The roots for the classical Lie algebras can be expressed in terms of the ε_i. For $\mathfrak{sl}_n$,

$$R = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n\}$$

The roots span $\mathfrak{h}^*$, and we choose as a basis the set $\{\alpha_i\}_{i=1}^{n-1}$ of *simple roots*,

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \qquad \text{for } i = 1, 2, \dots, n-1.$$

Then the positive roots (those roots arising from nonnegative integral linear combinations of the simple roots) are

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}.$$

The finite dimensional irreducible $\mathfrak{g}$-modules are indexed by the set of *dominant integral weights*, denoted $P^+ \subseteq \mathfrak{h}^*$. The *fundamental weights* $\omega_i \in \mathfrak{h}^*$ form a $\mathbb{Z}_{\ge 0}$ basis for P^+, i.e.

$$P^+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0}\omega_i = \{\lambda_1\omega_1 + \cdots + \lambda_{n-1}\omega_{n-1} \mid \lambda_i \in \mathbb{Z}_{\ge 0}\}.$$

The fundamental weights have the property that

$$\langle \omega_i, \alpha_j\rangle = \delta_{i,j}, \qquad \text{i.e.} \quad \omega_i(h_{\alpha_j}) = \delta_{i.j},$$

so are given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{n}(\varepsilon_1 + \cdots + \varepsilon_n) \quad 1 \le i \le n-1.$$

Thus the dominant integral weights are given by

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} - \frac{|\lambda|}{n}(\varepsilon_1 + \cdots + \varepsilon_n), \qquad \text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z}, \\ \lambda_1 \ge \cdots \ge \lambda_{n-1} \ge 0, \\ |\lambda| = \lambda_1 + \cdots + \lambda_{n-1}. \end{array}$$

Finally, let

$$\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = \sum_{j=1}^{n-1} \omega_j = \frac{1}{2}\sum_{i=1}^{n}(n + 1 - 2i)\varepsilon_i. \tag{2.7}$$

2.2.2 The weights of $\mathfrak{gl}_n$

Let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal basis of the vector space $\mathbb{R}^n$. Then

$$\mathfrak{h}^* \cong \mathbb{R}^n,$$

where ε_i is identified with the weight

$$\varepsilon_i(h_j) = \delta_{i,j}.$$

We no longer choose a root system which spans $\mathfrak{h}^*$, but instead inherit the structure from the simple part of $\mathfrak{gl}_n$, i.e. just as with $\mathfrak{sl}_n$, write

$$R = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\},$$

We choose as the basis the set of simple roots

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \qquad 1 \leq i \leq n-1.$$

Then the positive roots are

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

The finite dimensional irreducible $\mathfrak{gl}_n$-modules are indexed by elements of

$$P^+ = \mathbb{Z}\omega_n + \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\omega_i, \qquad \text{where} \quad \omega_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \leq i \leq n.$$

So the dominant integral weights for $\mathfrak{gl}_n$ are given by

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, \quad \text{where } \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq \lambda_n.$$

The choice of ρ is no longer unique. To serve as the analogous weight, define

$$\delta = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} = \sum_{i=1}^{n}(n-i)\varepsilon_i. \tag{2.8}$$

This choice matches [Mac, I,1].

2.3 Partitions and highest weight modules

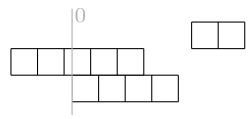
In this section, we explore the combinatorial properties of modules for $\mathfrak{g}$ either a finite dimensional complex simple Lie algebra or $\mathfrak{gl}_n$. Though we concentrate on the specific cases where $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$, many analagous results hold for other simple Lie algebras.

Definition 2.1. *For a positive integer k, a k-*multisegment*, or simply a* multisegment, *λ, is a sequence of pairs*

$$\lambda = \{(\mu_1, \lambda_1), \ldots, (\mu_\ell, \lambda_\ell)\}$$

where $\lambda_i \in \mathbb{Z}_{>0}$, $\mu_i \in \mathbb{C}$, and $\sum_i \lambda_i = k$. We call $\ell = \mathrm{ht}(\lambda)$ the height *of λ. Pictorially, a (real-valued) multisegment can be represented by rows of boxes of length λ_i with displacement μ_i. For example,*

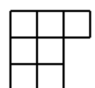
$$\{(4.5, 2), (-2.3, 5), (0, 4)\} \;=$$



is an 11-multisegment. A partition *is a multisegment with $\mu_1 = \cdots = \mu_\ell = 0$. A* skew shape *$\lambda/\mu$ is a multisegment $\{(\mu_1, \nu_1), \ldots, (\mu_\ell, \nu_\ell)\}$ for which $\mu_i \in \mathbb{Z}_{\geq 0}$ and $\mu_i + \nu_i \geq \mu_{i+1} + \nu_{i+1}$ for each $1 \leq i \leq \ell$. Each skew shape is a partition $\lambda = (\mu_1 + \nu_1, \ldots, \mu_\ell + \nu_\ell)$ with the partition $\mu = (\mu_1, \ldots, \mu_{\ell'})$, $\ell' \leq \ell$, removed.*

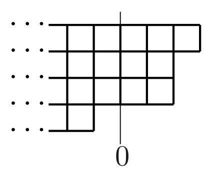
For $\mathfrak{g} = \mathfrak{sl}_n$, identify each dominant integral weight λ with the partition with λ_i boxes in row i. For example, if $\lambda = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 - \frac{7}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$, the associated partition is

$$\lambda =$$



For $\mathfrak{g} = \mathfrak{gl}_n$, associate to weight $\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ an infinite multisegment. This multisegment has n rows extending to the left infinitely and ending in column λ_i. So, for example, if $\lambda = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 - \varepsilon_4$,

$$\lambda =$$



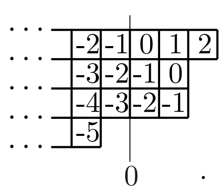
If $\lambda_i \geq 0$ for all $1 \leq i \leq n$, we often represent this multisegment as a partition, leaving off boxes to the left of 0.

Remark 2.2. *When we refer to changing a weight by adding or removing a box in row i, this specifically means changing λ_i by ± 1.*

If B is the box in row i and column j of λ, the *content* of B is

$$c(B) = j - i. \tag{2.9}$$

For example, if we fill in the boxes in the above λ with their respective contents, we get



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Let $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$ and let $L(\lambda)$ be the finite dimensional irreducible highest-weight $\mathfrak{g}$-module of weight λ, where $\lambda \in P^+$, i.e. the irreducible $\mathfrak{g}$-module generated by highest weight vector v_λ^+ of weight λ with action

$$hv_\lambda^+ = \lambda(h)v_\lambda^+ \quad \text{and} \quad xv_\lambda^+ = 0, \qquad \text{for } h \in \mathfrak{h},\ x \in \mathfrak{n}^+.$$

If $\mathfrak{g}$ is reductive, every finite dimensional $\mathfrak{g}$-module M decomposes as the direct sum of simple modules:

$$M = \bigoplus_{\lambda \in P^+} c_\lambda L(\lambda).$$

For example, if $\mathfrak{g} = \mathfrak{sl}_n$ and $\mu \in P^+$,

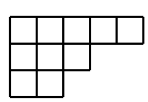
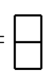
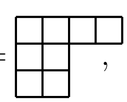
$$L(\mu) \otimes L(\omega_1) = \bigoplus_{\lambda \in \mu^+} L(\lambda), \tag{2.10}$$

where

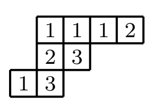
$$\mu^+ = \left\{ \begin{array}{c} \text{partitions of height } \leq n-1 \\ \text{obtained by adding a box to } \mu \end{array} \right\}.$$

In general, the decomposition numbers for the tensor product of two highest weight modules can be calculated using the Littlewood-Richardson rule (Theorem 2.4).

Definition 2.3. *Let μ, ν, λ be partitions such that $\mu, \nu \subset \lambda$, and suppose* $\mathrm{ht}(\nu) = \ell$. *A* tableau T of shape λ/μ of weight ν *is a filling of the boxes of λ which are not in μ with ν_1 1's, ν_2 2's, ..., ν_ℓ ℓ's, such that the row fillings are weakly increasing left to right and the column fillings are strictly increasing top to bottom. We can derive a word $w(T) = a_1a_2\cdots a_N$, where $N = |\lambda| - |\mu|$, by reading off the numbers in T right to left, top to bottom. For example, if*

$\lambda =$ , $\quad \mu =$ , $\quad \nu =$ ,

then one tableau is

$T =$ 

,

and this tableau has word $w(T) = 2\ 1\ 1\ 1\ 3\ 2\ 3\ 1$.

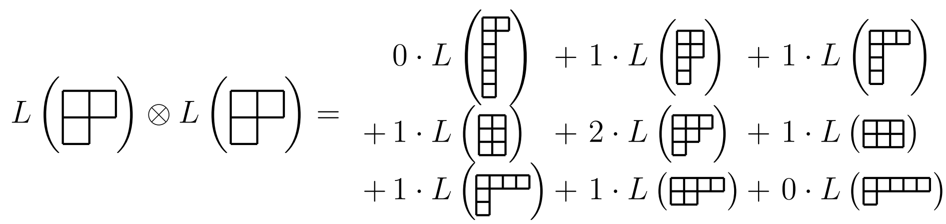
A word $w = a_1a_2\cdots a_N$ is a lattice permutation *if for each $1 \leq r \leq N$ and $2 \leq i \leq \ell$, there are no more occurrences of i than of $i-1$ in $a_1a_2\ldots a_r$. So* 2 1 1 1 3 2 3 1 *is not a lattice permutation, but* 1 2 1 1 3 2 3 1 *is. In the above example, the only tableau of shape λ/μ and weight ν which also generates a lattice permutation is the one generating the word* 1 1 1 1 2 2 3 3.

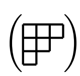
Theorem 2.4 (Littlewood-Richardson rule, [Mac, I,1])**.** *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ or $\mathfrak{gl}_n$.*

$$L(\mu) \otimes L(\nu) = \sum_{\substack{\mu,\nu \subset \lambda \\ |\lambda| = |\mu| + |\nu| \\ ht(\lambda) \leq n}} c^{\lambda}_{\mu,\nu} L(\lambda)$$

where $c^{\lambda}_{\mu\nu}$ is the number of tableaux of shape λ/μ and weight ν which generate lattice permutations.

For example,



The interesting factor here is $2 \cdot L$ , which comes from fillings

If a and b are positive integers, let (a^b) be the partition with b rows of length a.

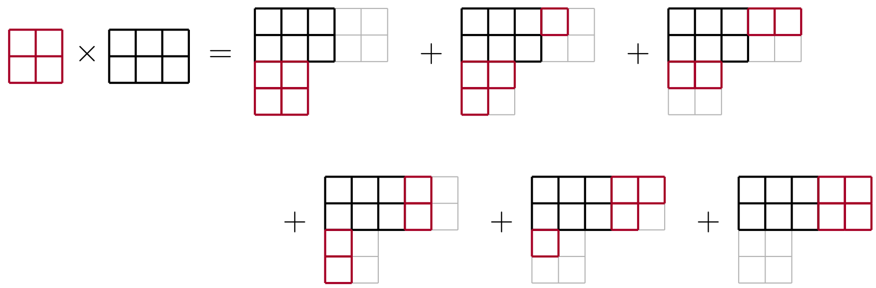
Example 2.5 (Horizontal strips)**.** *If $\nu = (a)$, $a \in \mathbb{Z}_{>0}$, then each $c^{\lambda}_{\mu\nu}$ is at most 1, since there is only one way to fill in all boxes with 1's. Moreover, since columns must be filled with strictly increasing values, λ/μ must be a* horizonal strip *(a skew shape where every column has at most one box).*

Example 2.6 (Vertical strips)**.** *Similarly, if $\nu = (1^a)$, then each $c^{\lambda}_{\mu\nu}$ is at most 1, and is nonzero exactly when λ/μ is a length-a* vertical strip *(a skew shape where every row has at most one box).*

Example 2.7 (Rectangles)**.** *(See [St, Lem. 3.3], [Ok, Thm 2.4]) Let $p \geq q$ and a, b be non-negative integers. Then each $c^{\lambda}_{(a^p)(b^q)}$ is 1 if $\lambda \in \mathcal{P}((a^p),(b^q))$, and is zero otherwise, where $\mathcal{P}((a^p),(b^q))$ is the set of partitions λ with height $\leq p+q$ such that*

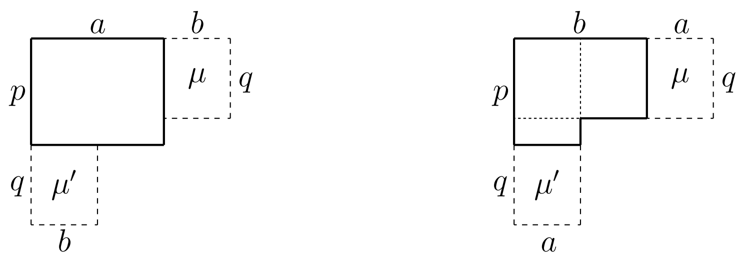
$$\begin{gathered} \lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_p = a, \\ \lambda_q \geq \max(a,b), \\ \lambda_i + \lambda_{p+q-i+1} = a + b, \quad i = 1, \ldots, q. \end{gathered} \tag{2.11}$$

In other words, $\mathcal{P}((a^p),(b^q))$ is the set of partitions made by placing (a^p) to the left of (b^q), carving a corner out of (b^q), rotating it $180°$ and gluing it to the bottom of (a^p). For example,



A useful visualization of these partitions is as follows. The outlined section is filled with boxes, and the dashed regions are filled with complementary partitions.

$a > b$: $a < b$:



μ is a partition in a $b \times q$ box μ is a partition in an $a \times q$ box
μ' is the $180°$ rotation of $(b^q)/\mu$ μ' is the $180°$ rotation of $(a^q)/\mu$

Remark 2.8. *As a consequence of the requirements in (2.17), if a box in $\lambda \in \mathcal{P}((a^b),(p^q))$ is moved from position (i,j) to form another partition in $\mathcal{P}((a^p),(b^q))$, it must be moved to position $(a+b+1-i, p+q+1-j)$.*

2.3.1 Orders on sets of partitions

Finally, we briefly introduce several orders on the set of partitions of n.

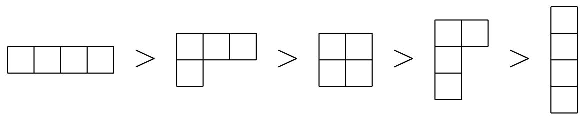
Definition 2.9 ([Mac, §I.1])**.** *Fix $n \in \mathbb{Z}_{>0}$.* Lexicographical order *is a linear ordering on the set of partitions of n given by*

$$\lambda > \mu \qquad \text{if} \qquad \lambda_i \geq \mu_i \qquad \text{for all } i \text{ and } \lambda \neq \mu.$$

Dominance order *(or* natural order*) is a partial ordering on the set of partitions of n given by*

$$\lambda > \mu \qquad \text{if} \qquad \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \qquad \text{for all } j \text{ and } \lambda \neq \mu.$$

For example, if $n = 4$,



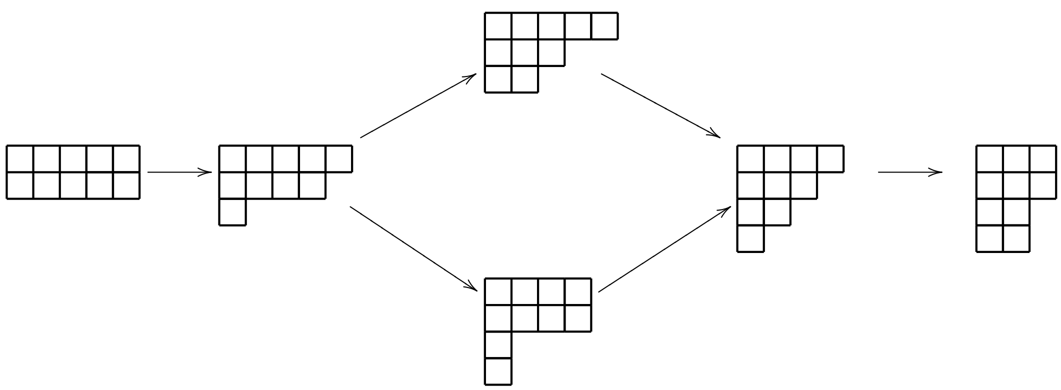
in both lexicographical and dominance order. Dominance ordering of two partitions implies lexicographical ordering, and the two orders are equivalent if and only if $n \leq 5$.

Finally, we return to the set of partitions described in Example 2.7, and provide a third partial order, this time only on the set of partitions $\mathcal{P}((a^p),(b^q))$. Observe that any partition in $\mathcal{P}((a^p),(b^q))$ can be built iteratively by beginning with the partition

$$(a^p) + (b^q) = (\underbrace{a+b,\ldots,a+b}_{q},\underbrace{a,\ldots,a}_{p-q})$$

and moving successive boxes down (by Remark 2.8, this process is well-defined). For example, see Figure 2.10.

Figure 2.10.



Nodes in this lattice represent partitions in $\mathcal{P}((3^2),(2^2))$, and arrows represent moving one box to build the rightmost partition from the leftmost.

This process produces a partial order on partitions in $\mathcal{P}((a^p),(b^q))$, with the highest element in $\mathcal{P}((a^p),(b^q))$ being $(a^p)+(b^q)$, and $\lambda > \mu$ when μ can be built from λ by moving successive boxes down. If two partitions are comparable with respect to this partial order, then their ordering will agree with dominance ordering, and therefore lexicographical ordering. However, this partial order is not the same as dominance order restricted to $\mathcal{P}((a^p),(b^q))$. For example, the two incomparable partitions in Figure 2.10 are comparable in dominance ordering since

$$5 \geq 4, \quad 5+3 \geq 4+4, \quad 5+3+2 \geq 4+4+1, \quad \text{and} \quad 5+3+2+0 \geq 4+4+1+1.$$

2.4 The Casimir element and the operator γ

If $\mathfrak{g}$ is semisimple or $\mathfrak{g} = \mathfrak{gl}_n$, the *Casimir element* of $\mathfrak{g}$ is

$$\kappa = \sum_i b_i b_i^*,$$

where if $\{b_i\}$ is a basis for $\mathfrak{g}$, then $\{b_i^*\}$ is the dual basis to $\{b_i\}$ with respect to the trace form, i.e. $\langle b_i^*, b_j \rangle = \delta_{ij}$. If M and N are $\mathfrak{g}$-modules, κ acts on $M \otimes N$ by

$$\kappa \otimes 1 + 1 \otimes \kappa + 2\gamma, \qquad \text{where} \quad \gamma = \sum_i b_i \otimes b_i^*. \tag{2.12}$$

The Casimir is central in the enveloping algebra of $\mathfrak{g}$, so it acts on each irreducible component of a $\mathfrak{g}$-module M as a scalar.

Theorem 2.11.

(a) *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, and let λ, μ, and ν be dominant integral weights for $\mathfrak{g}$. Recall $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$.*

(i) *The Casimir element κ acts on a $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ by the constant*

$$\langle \lambda, \lambda + 2\rho \rangle.$$

(ii) *If $L(\lambda)$ is a submodule of $L(\mu) \otimes L(\nu)$, then γ acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ by the constant*

$$\gamma_{\mu\nu}^{\lambda} = \tfrac{1}{2}(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle). \tag{2.13}$$

(b) *Let $\mathfrak{g} = \mathfrak{gl}_n$, and let λ, μ, and ν be dominant integral weights for $\mathfrak{g}$. As in equation* (2.8), *let*

$$\delta = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} = \sum_{i=1}^{n} (n-i)\varepsilon_i.$$

(i) *The Casimir element κ acts on a $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ by the constant*

$$\langle \lambda, \lambda + 2\delta \rangle - (n-1)|\lambda|.$$

(ii) *If $L(\lambda)$ is a submodule of $L(\mu) \otimes L(\nu)$, then γ acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ by the constant*

$$\gamma_{\mu\nu}^{\lambda} = \tfrac{1}{2}\big(\langle \lambda, \lambda + 2\delta \rangle - \langle \mu, \mu + 2\delta \rangle - \langle \nu, \nu + 2\delta \rangle\big) - \tfrac{n-1}{2}.$$

Proof.

(a) First, take $\mathfrak{g}$ to be semisimple.

(i) We will first need to choose a very specific basis for $\mathfrak{g}$, one which corresponds to the triangular decomposition in (2.2). To this end, first notice that if $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, then

$$\alpha(h)\langle x, y\rangle = \langle [h,x], y\rangle = -\langle x, [h,y]\rangle = -\beta(h)\langle x, y\rangle.$$

So

$$\text{if } \alpha \neq -\beta, \text{ then } \langle x, y\rangle = 0, \tag{2.14}$$

i.e. since the form $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is nondegenerate, the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal precisely when $\alpha + \beta \neq 0$. Also, by (2.4),

$$\langle h, [x,y]\rangle = \langle [h,x], y\rangle = \alpha(h)\langle x, y\rangle = \langle h, h_\alpha\rangle\langle x, y\rangle = \langle h, \langle x, y\rangle h_\alpha\rangle.$$

So, again since $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is nondegenerate,

$$[x, y] = \langle x, y\rangle h_\alpha, \qquad \text{for } x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_\beta. \tag{2.15}$$

Now let $\{h_1, \ldots, h_n\}$ be a basis for $\mathfrak{h}$ and let $\{h_1^*, \ldots, h_n^*\}$ be the dual basis with respect to $\langle , \rangle : \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{C}$ (i.e. $\langle h_i, h_j^*\rangle = \delta_{ij}$). For each $\alpha \in R^+$, choose $b_\alpha \in \mathfrak{g}_\alpha$ and the corresponding $b_\alpha^* \in \mathfrak{g}_{-\alpha}$ satisfying $\langle b_\alpha, b_\alpha^*\rangle = 1$. Then

$$\{h_1, \ldots, h_n\} \cup \{b_\alpha, b_\alpha^* \mid \alpha \in R^+\} \tag{2.16}$$

is a basis of $\mathfrak{g}$, and has dual basis

$$\{h_1^*, \ldots, h_n^*\} \cup \{b_\alpha^*, b_\alpha \mid \alpha \in R^+\}.$$

Rewriting in terms of the basis (2.16), we have

$$\kappa = \sum_{i=1}^n h_i h_i^* + \sum_{\alpha \in R^+} b_\alpha b_\alpha^* + b_\alpha^* b_\alpha.$$

The module $L(\lambda)$ is the finite-dimensional $\mathfrak{g}$-module generated by the highest weight vector v_λ^+ of weight λ, i.e.

$$hv_\lambda^+ = \lambda(h)v_\lambda^+ \quad \text{and} \quad b_\alpha v_\lambda^+ = 0 \quad \text{for } \alpha \in R^+, h \in \mathfrak{h}.$$

So

$$
\begin{aligned}
\kappa v_\lambda^+ &= \left(\sum_{i=1}^n h_i h_i^* + \sum_{\alpha\in R^+}(b_\alpha b_\alpha^* + b_\alpha^* b_\alpha)\right) v_\lambda^+ \\
&= \left(\sum_{i=1}^n h_i \lambda(h_i^*) + \sum_{\alpha\in R^+}([b_\alpha, b_\alpha^*] + 2b_\alpha^* b_\alpha)\right) v_\lambda^+ \\
&= \left(\sum_{i=1}^n h_i \langle h_\lambda, h_i^*\rangle + \sum_{\alpha\in R^+}(\langle b_\alpha, b_\alpha^*\rangle h_\alpha + 2b_\alpha^* b_\alpha)\right) v_\lambda^+ \qquad \text{by (2.4), (2.15)} \\
&= \left(h_\lambda + \sum_{\alpha\in R^+}(h_\alpha + 0)\right) v_\lambda^+ = \left(\lambda(h_\lambda) + \sum_{\alpha\in R^+}\lambda(h_\alpha)\right) v_\lambda^+ \\
&= \left(\langle\lambda,\lambda\rangle + \left\langle \lambda, \sum_{\alpha\in R^+}\alpha\right\rangle\right) v_\lambda^+ \quad = \quad \langle \lambda, \lambda+2\rho\rangle v_\lambda^+,
\end{aligned}
$$

since $\lambda(h_\mu) = \langle\lambda,\mu\rangle$ by the definition of $\langle,\rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ in(2.5).

(ii) Since κ acts on $L(\mu)\otimes L(\nu)$ by $(\kappa\otimes 1_{L(\nu)}) + (1_{L(\mu)}\otimes\kappa) + 2\gamma$, it follows directly from part (i) that γ acts on the $L(\lambda)$ isotypic component of $L(\mu)\otimes L(\nu)$ by

$$\tfrac{1}{2}(\langle\lambda,\lambda+2\rho\rangle - \langle\mu,\mu+2\rho\rangle - \langle\nu,\nu+2\rho\rangle)$$

as desired.

(b) (i) In the case where $\mathfrak{g} = \mathfrak{gl}_n$, recall $\{E_{ij} \mid 1\le i,j\le n\}$ forms a basis with dual basis $\{E_{ji} \mid 1\le i,j\le n\}$ with respect to the trace form. So

$$
\begin{aligned}
\kappa = \sum_{1\le i,j\le n} E_{ij}E_{ji} &= \sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n} E_{ij}E_{ji} + E_{ji}E_{ij} \\
&= \sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n}([E_{ij}, E_{ji}] + 2E_{ji}E_{ij}) \\
&= \sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n}(E_{ii} - E_{jj} + 2E_{ji}E_{ij})
\end{aligned}
$$

The module $L(\lambda)$ is the $\mathfrak{g}$-module generated by highest weight vector v_λ^+ of weight λ, i.e.

$$E_{ii}v_\lambda^+ = \lambda_i v_\lambda^+ \quad \text{and} \quad E_{ij}v_\lambda^+ = 0 \quad \text{for } i<j.$$

So

$$\begin{aligned}
\kappa v_\lambda^+ &= \left(\sum_{i=1}^{n} E_{ii}E_{ii} + \sum_{1\le i<j\le n} (E_{ii} - E_{jj} + 2E_{ji}E_{ij}) \right) v_\lambda^+ \\
&= \left(\sum_{i=1}^{n} \lambda_i^2 + \sum_{1\le i<j\le n} \lambda_i - \lambda_j + 0 \right) v_\lambda^+ \\
&= \left(\langle \lambda, \lambda \rangle + \sum_{i=1}^{n} \big((n-i) - (i-1)\big)\lambda_i \right) v_\lambda^+ \\
&= \left(\langle \lambda, \lambda \rangle + \sum_{i=1}^{n} \big(2n-2i\big)\lambda_i - (n-1)\lambda_i \right) v_\lambda^+ \\
&= \left(\langle \lambda, \lambda \rangle + \langle \lambda, 2\delta \rangle - (n-1)|\lambda| \right) v_\lambda^+
\end{aligned}$$

(ii) Again, since κ acts on $L(\mu) \otimes L(\nu)$ by $(\kappa \otimes 1_{L(\nu)}) + (1_{L(\mu)} \otimes \kappa) + 2\gamma$, it follows directly from part (i) that γ acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ by

$$\tfrac{1}{2}\Big(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle - (n-1)\big(|\lambda| - |\mu| - |\nu|\big) \Big).$$

But by Theorem 2.4, $|\lambda| = |\mu| + |\nu|$, so the desired result follows.

□

From equation (2.10), we know exactly how $L(\mu) \otimes L(\omega_1)$ decomposes. In fact, we can express the constants in Theorem 2.11 in terms of contents of boxes added or removed in the associated partitions. Recall, from (2.9), if B is the box in row i and column j of λ, the *content* of B is

$$c(B) = j - i.$$

See Section 2.3 for a discussion of this construction.

Theorem 2.12. *Let $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$, let μ be a dominant integral weight for $\mathfrak{g}$, and let $V = L(\omega_1)$. If $L(\lambda)$ has a nontrivial isotypic component in $L(\mu) \otimes L(\omega_1)$, write $\gamma^\lambda_{\mu\omega_1}$ to mean the constant by which γ acts on this component.*

1. *If $\mathfrak{g} = \mathfrak{gl}_n$, then*

$$\gamma^\lambda_{\mu\omega_1} = c(B),$$

where B is the box added to obtain λ from μ.

2. If $\mathfrak{g} = \mathfrak{sl}_n$*, then*

$$\gamma^\lambda_{\mu\omega_1} = c(B) - \frac{|\mu|}{n},$$

where B *is the box added to obtain* λ *from* μ*.*

Proof.

Case $\mathfrak{g} = \mathfrak{gl}_n$**.** Write $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n$. Adding a box to μ in the i^{th} row is equivalent to adding ε_i to μ. Recall $\omega_1 = \varepsilon_1$ and $\delta = \sum_{i=1}^{n}(n-i)\varepsilon_i$.

Thus

$$\begin{aligned} 2\gamma^\lambda_{\mu\omega_1} &= \big(\langle \mu + \varepsilon_i, \mu + \varepsilon_i + 2\delta\rangle - \langle \mu, \mu + 2\delta\rangle - \langle \omega_1, \omega_1 + 2\delta\rangle\big) \\ &= 2\langle \mu, \varepsilon_1\rangle + 2\langle \varepsilon_i - \varepsilon_1, \mu\rangle + 2\langle \varepsilon_i - \varepsilon_1, \varepsilon_1\rangle + \langle \varepsilon_i - \varepsilon_1, \varepsilon_i - \varepsilon_1 + 2\delta\rangle \\ &= 2\big(\mu_1 + \mu_i - \mu_1 - 1 + 1 + (n-i) - (n-1)\big) \\ &= 2(\mu_i + 1 - i) \end{aligned}$$

A box added to row i of μ is in position $(i, \mu_i + 1)$ and has content $(\mu_i + 1) - i$, so

$$\gamma^\lambda_{\mu\omega_1} = c(B).$$

Case $\mathfrak{g} = \mathfrak{sl}_n$**.** Write $\mu = \mu_1\varepsilon_1 + \cdots + \mu_{n-1}\varepsilon_{n-1} - \frac{|\mu|}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$. Recall that $\omega_1 = \varepsilon_1 - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$ and $2\rho = \sum_{i=1}^{n-1}\omega_i = \sum_{j=1}^{n}(n+1-2j)\varepsilon_j$. Adding a box to μ in the i^{th} row is equivalent to adding $\varepsilon_i - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$ to μ, and so

$$\lambda = \mu + \varepsilon_i - \varepsilon_1 + \omega_1.$$

Thus

$$\begin{aligned} 2\gamma^\lambda_{\mu\omega_1} =& \big(\langle \mu + \varepsilon_i - \varepsilon_1 + \omega_1, \mu + \varepsilon_i - \varepsilon_1 + \omega_1 + 2\rho\rangle - \langle \mu, \mu + 2\rho\rangle - \langle \omega_1, \omega_1 + 2\rho\rangle\big) \\ =& \Big(2\langle \mu, \omega_1\rangle + 2\langle \varepsilon_i - \varepsilon_1, \mu\rangle + 2\langle \varepsilon_i - \varepsilon_1, \omega_1\rangle + \langle \varepsilon_i - \varepsilon_1, \varepsilon_i - \varepsilon_1 + 2\rho\rangle\Big) \\ =& 2\left(\mu_1 - \frac{|\mu|}{n} + \mu_i - \mu_1 - 1 + 1 + \tfrac{1}{2}(n + 1 - 2i - (n + 1 - 2))\right) \\ =& 2\left(\mu_i + 1 - i - \frac{|\mu|}{n}\right) \end{aligned}$$

The content of a box added from row i of μ is $(\mu_i + 1) - i$, and so

$$\gamma^\lambda_{\mu\omega_1} = c(B) - \frac{|\mu|}{n}.$$

□

2.4.1 Rectangles

Finally, we will need to understand the the action of γ on $M \otimes N$, where M and N are indexed by rectangles.

Let $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$. Fix a, b, p, q be positive integers with $p \geq q$ and

$$p + q \leq \begin{cases} n & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ n-1 & \text{if } \mathfrak{g} = \mathfrak{sl}_n. \end{cases}$$

Let $M = L((a^p))$, $N = L((b^q))$. Recall from Example 2.7 that the nontrivial irreducible submodules of $M \otimes N$ each have multiplicity one and are indexed by the partitions $\lambda \in \mathcal{P}((a^p), (b^q))$ where $\mathcal{P}((a^p), (b^q))$ is the set of partitions λ with height $\leq p + q$ such that

$$\begin{gathered} \lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_p = a, \\ \lambda_q \geq \max(a, b), \\ \lambda_i + \lambda_{p+q-i+1} = a + b, \quad i = 1, \ldots, q. \end{gathered} \tag{2.17}$$

In Section 2.3.1 we observed that that any partition in $\mathcal{P}((a^p), (b^q))$ can be built iteratively by beginning with the partition

$$(a^p) + (b^q) = \begin{cases} a(\varepsilon_1 + \cdots \varepsilon_p) + b(\varepsilon_1 + \cdots + \varepsilon_q) & \text{when } \mathfrak{g} = \mathfrak{gl}_n, \\ a(\varepsilon_1 + \cdots \varepsilon_p) + b(\varepsilon_1 + \cdots + \varepsilon_q) - \frac{ap+bq}{n}(\varepsilon_1 + \cdots + \varepsilon_n) & \text{when } \mathfrak{g} = \mathfrak{sl}_n, \end{cases}$$

and moving successive boxes down. An example of this process is given in Figure 2.10.

Lemma 2.13. *Let μ and λ index distinct non-trivial components of $M \otimes N$, assume λ differs from μ by moving one box from position (μ_i, i). Denote the constant by which κ acts on an irreducible component $L(\nu)$ as $\kappa_{L(\nu)}$. Then*

$$\kappa_{L(\lambda)} = \kappa_{L(\mu)} - 4((\mu_i - i) - \tfrac{1}{2}(a - p + b - q)).$$

Proof. If $\mathfrak{g} = \mathfrak{gl}_n$ and $\lambda = \mu - \varepsilon_i + \varepsilon_j$ is obtained from μ by moving a box from row i into row j, then

$$\begin{aligned} \kappa_{L(\lambda)} &= \langle \lambda, \lambda + 2\delta \rangle - (n-1)|\lambda| \\ &= \langle \mu, \mu + 2\delta \rangle - (n-1)|\mu| + 2\langle \mu, \varepsilon_j - \varepsilon_i \rangle + \langle \varepsilon_j - \varepsilon_i, \varepsilon_j - \varepsilon_i + 2\delta \rangle \\ &= \kappa_{L(\mu)} + 2(\mu_j - \mu_i) + 2 + (n - 2j) - (n - 2i) \\ &= \kappa_{L(\mu)} - 2((\mu_i - i) - (\mu_j + 1 - j)) \\ &= \kappa_{L(\mu)} - 2\Big((\mu_i - i) - (\lambda_j - j)\Big) \\ &= \kappa_{L(\mu)} - 2(\text{content of old box} - \text{content of new box}) \end{aligned}$$

If $\mathfrak{g} = \mathfrak{sl}_n$ and $\lambda = \mu - \varepsilon_i + \varepsilon_j$ is obtained from μ by moving a box from row i into row j, then

$$\begin{aligned}
\kappa_{L(\lambda)} &= \langle \lambda, \lambda + 2\rho \rangle \\
&= \langle \mu, \mu + 2\rho \rangle + 2\langle \mu, \varepsilon_j - \varepsilon_i \rangle + \langle \varepsilon_j - \varepsilon_i, \varepsilon_j - \varepsilon_i + 2\rho \rangle \\
&= \kappa_{L(\mu)} + 2(\mu_j - \mu_i) + 2 + (n + 1 - 2j) - (n + 1 - 2i) \\
&= \kappa_{L(\mu)} - 2((\mu_i - i) - (\mu_j + 1 - j)) \\
&= \kappa_{L(\mu)} - 2\Big((\mu_i - i) - (\lambda_j - j)\Big) \\
&= \kappa_{L(\mu)} - 2(\text{content of old box} - \text{content of new box})
\end{aligned}$$

Now, if λ and μ are both elements of $\mathcal{P}((a^p), (b^q))$, then we have already seen

$$j = p + q + 1 - i \text{ and } \lambda_j = a + b + 1 - \mu_i.$$

So

$$\begin{aligned}
\kappa_{L(\lambda)} &= \kappa_{L(\mu)} - 2\Big((\mu_i - i) - \big((a - p) + (b - q) - (\mu_i - i)\big)\Big) \\
&= \kappa_{L(\mu)} - 4((\mu_i - i) - \tfrac{1}{2}(a - p + b - q))
\end{aligned}$$

□

Remark 2.14. *If λ and μ satisfy the criteria in Lemma 2.13, then*

$$\kappa_{L(\mu)} - \kappa_{L(\lambda)} = 4((\mu_i - i) - \tfrac{1}{2}(a - p + b - q)) \neq 0.$$

Proof. If a box in position (i, j) in $\mu \in \mathcal{P}((a^p), (b^q))$ can be moved to get another partition in $\mathcal{P}((a^p), (b^q))$, then that box must satisfy either

(1) $\max(a, b) < i \leq a + b$ and $0 < j \leq q$, or

(2) $0 < i \leq \min(a, b)$ and $p < j \leq p + q$

The first case is when position (i, j) is to the upper right, and the second case is when (i, j) is to the lower left. If (i, j) satisfies (1), then

$$\max(a, b) - q < i - j < a + b - q.$$

If (i, j) satisfies (2), then

$$-p - q < i - j < \min(a, b) - p.$$

So since

$$\tfrac{1}{2}(a + b) + \tfrac{1}{2}(p + q) \leq \max(a, b) - q \qquad \text{and} \qquad \min(a, b) - p \leq \tfrac{1}{2}(a + b) + \tfrac{1}{2}(p + q),$$

we have

$$i - j \neq \tfrac{1}{2}(a - p + b - q). \tag{2.18}$$

In other words, any movable box is off of the diagonal $\frac{1}{2}(a - p + b - q)$, and the difference between the contents in μ and λ above is nonzero. □

Lemma 2.15. *Let $\lambda \in \mathcal{P}((a^p),(b^q))$ and define $\mathcal{B}_\lambda$ to be the set of boxes in λ in rows $p+1$ and below. Recall that we denote the constant by which γ acts on an irreducible component $L(\lambda)$ of $L((a^p)) \otimes L((b^q))$ as $\gamma^{\lambda}_{(a^p)(b^q)}$.*

1. *If $\mathfrak{g} = \mathfrak{gl}_n$,*

$$\gamma^{\lambda}_{(a^p)(b^q)} = abq + 2 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a-p+b-q) \right).$$

2. *If $\mathfrak{g} = \mathfrak{sl}_n$,*

$$\gamma^{\lambda}_{(a^p)(b^q)} = abq - \frac{abpq}{n} + 2 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a-p+b-q) \right).$$

Proof.

1. Let $\mathfrak{g} = \mathfrak{gl}_n$. Again, denote the constant by which κ acts on an irreducible component $L(\nu)$ as $\kappa_{L(\nu)}$). By Theorem 2.11

$$\kappa_{L(\lambda)} = \langle \lambda, \lambda + 2\delta \rangle - (n-1)|\lambda|,$$

where $\delta = \sum_{i=1}^{n} (n-i)\varepsilon_i$. So

$$\begin{aligned} \kappa_{L((a^p)+(b^q))} &= \langle (a^p)+(b^q), (a^p)+(b^q)+2\delta \rangle - (n-1)(ap+bq) \\ &= \langle (a^p), (a^p)+2\delta \rangle - (n-1)ap \\ &\quad + \langle (b^q), (b^q)+2\delta \rangle - (n-1)bq + 2\langle (a^p), (b^q) \rangle \\ &= \kappa_M + \kappa_N + 2\langle a(\varepsilon_1 + \cdots + \varepsilon_p), b(\varepsilon_1 + \cdots + \varepsilon_q) \rangle \\ &= \kappa_M + \kappa_N + 2abq. \end{aligned}$$

Therefore γ acts on the $L((a^p)+(b^q))$ component of $M \otimes N$ by

$$\begin{aligned} \gamma^{(a^p)+(b^q)}_{(a^p),(b^q)} &= \tfrac{1}{2}\left(\kappa_{L((a^p)+(b^q))} - \kappa_M - \kappa_N \right) \\ &= abq. \end{aligned} \tag{2.19}$$

Let λ be a partition indexing a component of $M \otimes N$. If we build another component λ' by moving a box (i,j) to the place $(a+b+1-i, p+q+1-j)$, Lemma 2.13 implies

$$\kappa_{L(\lambda')} = \kappa_{L(\lambda)} - 4(i - j - \tfrac{1}{2}(a-p+b-q))$$

Since any partition indexing a component of $M \otimes N$ can be arrived at recursively through this process by beginning with $\lambda_0 = (a^p) + (b^q)$, this tells us

$$\begin{aligned}\kappa_{L(\lambda)} &= \kappa_{L(\lambda_0)} + 4 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a - p + b - q)\right) \\ &= \kappa_M + \kappa_N + 2abq + 4 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a - p + b - q)\right).\end{aligned}$$

So γ acts on the $L(\lambda)$ component of $M \otimes N$ by

$$\gamma^{\lambda}_{(a^p),(b^q)} = abq + 2 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a - p + b - q)\right).$$

2. Let $\mathfrak{g} = \mathfrak{sl}_n$. By Theorem 2.11

$$\kappa_{L(\lambda)} = \langle \lambda, \lambda + 2\rho \rangle,$$

where $\rho = \sum_{i=1}^{n} (n + 1 - 2i)\varepsilon_i$. So

$$\begin{aligned}\kappa_{L((a^p)+(b^q))} &= \langle (a^p) + (b^q), (a^p) + (b^q) + 2\rho \rangle \\ &= \langle (a^p), (a^p) + 2\rho \rangle + \langle (b^q), (b^q) + 2\rho \rangle + 2\langle (a^p), (b^q) \rangle \\ &= \kappa_M + \kappa_N \\ &\qquad + 2\langle a(\varepsilon_1 + \cdots + \varepsilon_p) - \tfrac{ap}{n}(\varepsilon_1 + \cdots + \varepsilon_n), \\ &\qquad\qquad b(\varepsilon_1 + \cdots + \varepsilon_q) - \tfrac{bq}{n}(\varepsilon_1 + \cdots + \varepsilon_n) \rangle \\ &= \kappa_M + \kappa_N + 2\left(abq - bq\frac{ap}{n} - ap\frac{bq}{n} + n\frac{apbq}{n^2}\right) \\ &= \kappa_M + \kappa_N + 2abq - 2\frac{apbq}{n}.\end{aligned}$$

Therefore γ acts on the $L((a^p) + (b^q))$ component of $M \otimes N$ by

$$\begin{aligned}\gamma^{(a^p)+(b^q)}_{(a^p),(b^q)} &= \tfrac{1}{2}\left(\kappa_{L((a^p)+(b^q))} - \kappa_M - \kappa_N\right) \\ &= abq - \frac{apbq}{n}.\end{aligned} \tag{2.20}$$

Let λ be a partition indexing a component of $M \otimes N$. If we build another component λ' by moving a box (i, j) to the place $(a + b + 1 - i, p + q + 1 - j)$, Lemma 2.13 implies

$$\kappa_{L(\lambda')} = \kappa_{L(\lambda)} - 4(i - j - \tfrac{1}{2}(a - p + b - q))$$

Since any partition indexing a component of $M \otimes N$ can be arrived at recursively through this process by beginning with $\lambda_0 = (a^p) + (b^q)$, this tells us

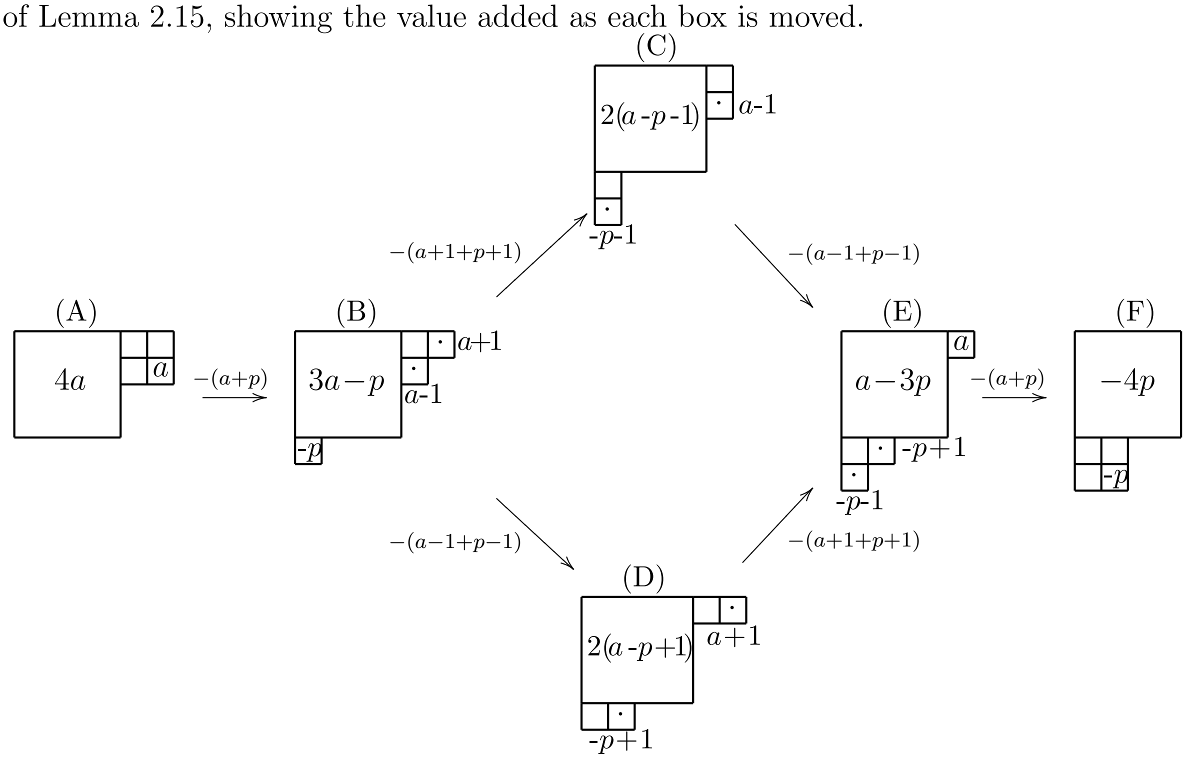
$$\begin{aligned}\kappa_{L(\lambda)} &= \kappa_{L(\lambda_0)} + 4 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a - p + b - q)\right) \\ &= \kappa_M + \kappa_N + 2abq - 2\frac{abpq}{n} + 4 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a - p + b - q)\right).\end{aligned}$$

So γ acts on the $L(\lambda)$ component of $M \otimes N$ by

$$\gamma^{\lambda}_{(a^p),(b^q)} = abq - \frac{abpq}{n} + 2 \sum_{B \in \mathcal{B}_\lambda} \left(c(B) - \tfrac{1}{2}(a - p + b - q)\right)$$

□

Let's examine this lemma a little closer with an example. The diagram below shows the six partitions indexing irreducible submodules of $M \otimes N$ when $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^p))$ with $2 \leq a, p$, and $N = L((2^2))$, as given by the Littlewood Richardson rule (see Example 2.7). The values by which γ acts are shown in the middle of each partition, and can be calculated in two ways. First, the diagram begins with the value on partition (A) being $a(2)(2) = 4a$ as given by (2.19), and follows the recursive process outlined in the proof of Lemma 2.15, showing the value added as each box is moved.



We could also take any of these partitions individually, and compute the value of γ using the result of Lemma 2.15. For example, partition (E) has three boxes in the set $\mathcal{B}_{(\mathrm{E})}$, with contents $-p$, $-p-1$, and $-p+1$ respectively, so the value of γ on the irreducible module indexed by this partition is

$$a(2)(2) - 3\big((a-p) + (2-2)\big) + 2\big(-p + (-p-1) + (-p+1)\big) = a - 3p.$$

2.5 Centralizer algebras

Let V be a finite-dimensional vector space and $A \subseteq \mathrm{End}(V)$ a semisimple algebra. Define the *centralizer* of A by

$$\mathrm{End}_A(V) = \{\, x \in \mathrm{End}(V) \mid xa = ax \text{ for all } a \in A\}.$$

Note that $\mathrm{End}_A(V)$ is an associative algebra with unit id_V.

Theorem 2.16 (Double centralizer theorem)**.** *[GW, Thm 3.3.7]*
The algebra $B = \mathrm{End}_A(V)$ is semisimple, one has $\mathrm{End}_B(V) = A$, and V has the multiplicity-free decomposition

$$V \cong \bigoplus_i V_i \otimes U_i \tag{2.21}$$

as an (A, B)-bimodule, where the V_i are mutually distinct irreducible A-modules and the U_i are mutually distinct irreducible B-modules.

This theorem allows us to simultaneously decompose V as an A-module and as a B-module. As an A-module, V_i occurs with multiplicity $\dim(U_i)$, and vice versa. This duality also provides a correspondence between irreducible A-modules and irreducible B-modules.

In this work, we will be concentrating predominantly on modules of the form $M \otimes N \otimes V^{\otimes k}$, $\mathfrak{g}$ is a finite-dimensional complex reductive Lie algebra and M, N, and V are finite-dimensional irreducible $\mathfrak{g}$-modules, with a goal of building $\mathrm{End}_{\mathfrak{g}}\left(M \otimes N \otimes V^{\otimes k}\right)$. It will be useful to understand how the centralizers for successive tensor products of modules are related. To this end, let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra, and let M and N be $\mathfrak{g}$-modules.

Lemma 2.17. *The map given by*

$$\begin{aligned} \mathrm{End}_{\mathfrak{g}}(M) &\to \mathrm{End}_{\mathfrak{g}}(M \otimes N) \\ \phi &\mapsto \phi \otimes \mathrm{id}_N \end{aligned}$$

is an injective algebra homomorphism.

Proof. Let $x \in \mathfrak{g}$, $u \in M$, $v \in N$, and suppose $\phi \in \mathrm{End}_{\mathfrak{g}}(M)$. Then

$$(\phi \otimes \mathrm{id}_N)x(u \otimes v) = (\phi \otimes \mathrm{id}_N)(x \otimes 1 + 1 \otimes x)(u \otimes v) = (\phi x \otimes \mathrm{id}_N + \phi \otimes x)(u \otimes v)$$

$$= (x\phi \otimes \mathrm{id}_N + \phi \otimes x)(u \otimes v) = (x \otimes 1 + 1 \otimes x)(\phi \otimes \mathrm{id}_N)(u \otimes v) = x(\phi \otimes \mathrm{id}_N)(u \otimes v).$$

So $\phi \otimes \mathrm{id}_N \in \mathrm{End}_{\mathfrak{g}}(M \otimes N)$. If $\psi \in \mathrm{End}_{\mathfrak{g}}(M)$, then

$$(\phi \otimes \mathrm{id}_N - \psi \otimes \mathrm{id}_N)(u \otimes v) = (\phi(u) - \psi(u)) \otimes v$$

which is zero only when $v = 0$ or $\phi(u) - \psi(u) = 0$, so the map

$$\phi \mapsto \phi \otimes \mathrm{id}_N$$

is injective. □

Denote the operator γ acting on the j and j' factors of $V^{\otimes k}$ by $\gamma_{j,j'}$, i.e.

$$\gamma_{j,j'} \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_b (v_{i_1} \otimes \cdots \otimes bv_{i_j} \otimes \cdots \otimes b^* v_{i_{j'}} \otimes \cdots \otimes v_{i_k}). \tag{2.22}$$

If $\{b\}$ is a basis of $\mathfrak{g}$, then $\{b^*\}$ presents an alternate basis for $\mathfrak{g}$, with dual basis $\{b\}$. Therefore,

$$\gamma_{i,j} = \gamma_{j,i}. \tag{2.23}$$

Denote by

$\gamma_{X,Y}$	γ acting on factors X and Y in a tensor space,
κ_X	κ acting on the factor X in a tensor space.

Since κ is central in the enveloping algebra of $\mathfrak{g}$, it follows from lemma 2.17 that

$$\begin{aligned} 0 &= \kappa_{M\otimes N\otimes V}\kappa_{M\otimes N} - \kappa_{M\otimes N}\kappa_{M\otimes N\otimes V} \\ &= (\kappa_M + \kappa_N + \kappa_V + 2(\gamma_{M,N} + \gamma_{M,V} + \gamma_{N,V}))\,(\kappa_M + \kappa_N + 2\gamma_{M,N}) \\ &\quad - (\kappa_M + \kappa_N + 2\gamma_{M,N})\,(\kappa_M + \kappa_N + \kappa_V + 2(\gamma_{M,N} + \gamma_{M,V} + \gamma_{N,V})) \\ &= 4(\gamma_{M,V} + \gamma_{N,V})\gamma_{M,N} - 4\gamma_{M,N}(\gamma_{M,V} + \gamma_{N,V}). \end{aligned}$$

Therefore, $\gamma_{N,V}\gamma_{M,N} - \gamma_{M,N}\gamma_{N,V} = \gamma_{M,N}\gamma_{M,V} - \gamma_{M,V}\gamma_{M,N}$. By similarly comparing $\kappa_{M\otimes N\otimes V}$ to $\kappa_{M\otimes V}$ and $\kappa_{N\otimes V}$, we find

$$\gamma_{M,V}\gamma_{N,V} - \gamma_{N,V}\gamma_{M,V} = \gamma_{N,V}\gamma_{M,N} - \gamma_{M,N}\gamma_{N,V} = \gamma_{M,N}\gamma_{M,V} - \gamma_{M,V}\gamma_{M,N}. \tag{2.24}$$

Chapter 3

The degenerate two-boundary braid group

The following definition is motivated by a desire to study the centralizer of the action of a reductive Lie algebra on the tensor space $M \otimes N \otimes V^{\otimes k}$. The the goal is to build an algebra which will have many centralizers as quotients. In the case where $\mathfrak{g}$ is reductive, we will provide one homomorphism of this universal algebra into $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.

Fix $k \in \mathbb{Z}_{>0}$.

Definition 3.1. *The* group algebra of the symmetric group $\mathbb{C}S_k$ *is generated by*

$$t_{s_1}, \ldots, t_{s_{k-1}}$$

with relations

$$t_{s_i}^2 = 1, \qquad i = 1, \ldots, k-1 \tag{3.1}$$

$$t_{s_i} t_{s_j} = t_{s_j} t_{s_i} \qquad j \neq i \pm 1 \tag{3.2}$$

$$t_{s_i} t_{s_{i+1}} t_{s_i} = t_{s_{i+1}} t_{s_i} t_{s_{i+1}}, \qquad i = 1, \ldots, k-2 \tag{3.3}$$

We write

$$t_{(i\ i+1)} = t_{s_i} \quad \text{and} \quad t_{(i\ j)} = t_{(i+1\ j)} t_{s_i} t_{(i+1\ j)},$$

so $t_{(i\ j)}$ acts as the transposition between i and j.

Throughout, we identify scalars $c \in \mathbb{C}$ with the elements $c\,\mathrm{id} \in \mathbb{C}S_k$.

Definition 3.2. *The* degenerate two-boundary braid group *is the algebra $\mathcal{G}_k$ generated over $\mathbb{C}$ by subalgebras*

$$\mathbb{C}[x_1, \ldots, x_k], \quad \mathbb{C}[y_1, \ldots, y_k], \quad \mathbb{C}[z_0, \ldots, z_k], \quad \text{and} \quad \mathbb{C}S_k$$

with relations

$$t_{s_i}x_j = x_jt_{s_i}, \quad t_{s_i}y_j = y_jt_{s_i}, \quad t_{s_i}z_j = z_jt_{s_i}, \qquad \text{for } j \neq i, i+1 \tag{3.4}$$

$$\begin{aligned}(z_0 + \cdots + z_i)\, x_j = x_j\, (z_0 + \cdots + z_i)\,, \\ (z_0 + \cdots + z_i)\, y_j = y_j\, (z_0 + \cdots + z_i)\,,\end{aligned} \qquad \text{for } i \geq j \tag{3.5}$$

$$t_{s_i}(x_i + x_{i+1}) = (x_i + x_{i+1})t_{s_i}, \quad t_{s_i}(y_i + y_{i+1}) = (y_i + y_{i+1})t_{s_i}, \qquad \text{for } 1 \leq i \leq k-1 \tag{3.6}$$

$$\begin{aligned}(t_{s_i}t_{s_{i+1}})\,(x_{i+1} - t_{s_i}x_it_{s_i})\,(t_{s_{i+i}}t_{s_i}) = x_{i+2} - t_{s_{i+1}}x_{i+1}t_{s_{i+1}} \\ (t_{s_i}t_{s_{i+1}})\,(y_{i+1} - t_{s_i}y_it_{s_i})\,(t_{s_{i+i}}t_{s_i}) = y_{i+2} - t_{s_{i+1}}y_{i+1}t_{s_{i+1}}\end{aligned} \qquad \text{for } 1 \leq i \leq k-2, \tag{3.7}$$

$$x_{i+1} - t_{s_i}x_it_{s_i} = y_{i+1} - t_{s_i}y_it_{s_i} \qquad \text{for } 1 \leq i \leq k-1, \tag{3.8}$$

and

$$z_i = x_i + y_i - m_i, \qquad 1 \leq i \leq k, \tag{3.9}$$

where, if we define

$$m_{i,j} = \begin{cases} x_{i+1} - t_{s_i}x_it_{s_i} & \text{if } j = i+1, \\ t_{(j-1\ \ i)}m_{j-1,j}t_{(j-1\ \ i)} & \text{if } j \neq i+1, \end{cases} \tag{3.10}$$

for $i = 1, 2, \ldots, k-1$ *(where* $(i+1\ \ j)$ *is the transposition acting on the* $i+1$ *and* j *terms), then* m_i *is the element*

$$m_1 = 0, \quad m_i = \sum_{1<j<i} m_{j,i}. \tag{3.11}$$

As we will see, $m_{i,j}$ has been constructed to mimic the operator $t_{i,j}$ (defined in Section 2.4). In fact, notice that from (3.6) and (3.7), we have

$$\begin{aligned} m_{i+1,i} &= (t_{s_i}t_{s_{i-1}}t_{s_i})\ m_{i-1,i}\ (t_{s_i}t_{s_{i-1}}t_{s_i}) \\ &= t_{s_i}(t_{s_{i-1}}t_{s_i}(x_i - t_{s_{i-1}}x_{i-1}t_{s_{i-1}})t_{s_i}t_{s_{i-1}})t_{s_i} \\ &= t_{s_i}(x_{i+1} - t_{s_i}x_it_{s_i})t_{s_i} \\ &= x_{i+1} - t_{s_i}x_it_{s_i} \\ &= m_{i,i+1}. \end{aligned}$$

Similarly, we can show $m_{i,j} = m_{j,i}$.

The *degenerate one-boundary braid group* $\mathcal{G}_k^{(1)}$ subalgebra of $\mathcal{G}_k$ generated by $z_1, \ldots, z_k$ and $t_{s_1}, \ldots t_{s_{k-1}}$.

3.1 Action on tensor space

Let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra and let M, N, and V be finite dimensional simple $\mathfrak{g}$-modules. Recall from Section 2.4 that the Casimir invariant is

$$\kappa = \sum_i b_i b_i^*,$$

where $\{b_i\}$ is a basis of $\mathfrak{g}$ and $\{b_i^*\}$ is its dual basis with respect to the trace form $\langle , \rangle$. The Casimir invariant is central in the enveloping algebra $\mathcal{U}\mathfrak{g}$ and acts on $M \otimes N$ as

$$\kappa \otimes \mathrm{id}_N + \mathrm{id}_M \otimes \kappa + 2\gamma, \qquad \text{where} \quad \gamma = \sum_i b_i \otimes b_i^*.$$

Consider the action of $\mathfrak{g}$ on the tensor space $M \otimes N \otimes V^{\otimes k}$. We denote the operator γ acting on the j and j' factors of $V^{\otimes k}$ by $\gamma_{j,j'}$, i.e.

$$\begin{aligned} \gamma_{j,j+1} &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(j-1)} \otimes \gamma \otimes \mathrm{id}_V^{\otimes(k-j-1)} \qquad \text{and} \\ \gamma_{j,j'} &= (\mathrm{id}_M \otimes \mathrm{id}_N \otimes (j' \; j+1))\gamma_{j,j+1}(\mathrm{id}_M \otimes \mathrm{id}_N \otimes (j' \; j+1)) \end{aligned}$$

as an operator on $M \otimes N \otimes V^{\otimes k}$, where $(j' \; j+1)$ transposes the j' and $j+1$ factors of $V^{\otimes k}$ and fixes all others. Similarly denote by

$\gamma_{X,Y}$	γ acting on factors X and Y in a tensor space,
$\gamma_{X,i}$	γ acting on factor X and the i^{th} copy of V in a tensor space,
κ_X	κ acting on the factor X in a tensor space.
$\kappa_{X,j}$	κ acting on the factor X and the first j factors of V, where $\kappa_{X,0} = \kappa_X$.

Notice that since M, N, and V are simple, κ_M, κ_N, and κ_V act as constants.

Applying $\kappa = \sum_b bb^*$ iteratively to $M \otimes V^{\otimes k}$, $N \otimes V^{\otimes k}$, and $M \otimes N \otimes V^{\otimes k}$, we find that as operators on $M \otimes N \otimes V^{\otimes k}$,

$$\kappa_{X,j} = \kappa_X + j\kappa_V + 2\left(\sum_{1\le i\le j} \gamma_{X,i} + \sum_{1\le r<s\le j} \gamma_{r,s}\right), \tag{3.12}$$

where $X = M, N$ or $M \otimes N$, and so

$$\kappa_{M\otimes N,j} = \kappa_M + \kappa_N + j\kappa_V + 2\left(\gamma_{M,N} + \sum_{1\le i\le j} (\gamma_{M,i} + \gamma_{N,i}) + \sum_{1\le r<s\le j} \gamma_{r,s}\right).$$

Theorem 3.3. *There is a well-defined algebra homomorphism*

$$\Phi\colon \mathcal{G}_k \to \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$$

defined by

$$\begin{aligned}
\Phi(t_{s_j}) &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(j-1)} \otimes s_1 \otimes \mathrm{id}_V^{\otimes(k-j-1)}, \\
\Phi(y_j) &= \tfrac{1}{2}(\kappa_{N,j} - \kappa_{N,j-1}), \\
\Phi(x_j) &= \tfrac{1}{2}(\kappa_{M,j} - \kappa_{M,j-1}), \\
\Phi(z_j) &= \tfrac{1}{2}(\kappa_{M\otimes N,j} - \kappa_{M\otimes N,j-1} + \kappa_V), \\
\Phi(z_0) &= \tfrac{1}{2}(\kappa_{M\otimes N} - \kappa_M - \kappa_N) = \gamma_{M,N},
\end{aligned} \tag{3.13}$$

where $s_1 \cdot (v_{i_1} \otimes v_{i_2}) = v_{i_2} \otimes v_{i_1}$.

Proof. The t_{s_i} act by simple transpositions, so they generate an action of $\mathbb{C}S_k$ on $V^{\otimes k}$. Since the coproduct structure (the action of $\mathfrak{g}$ on tensor space) is symmetric with respect to place permutation, this action of $\mathbb{C}S_k$ in turn commutes with the $\mathfrak{g}$-action.

The Casimir invariant κ is central in the enveloping algebra of $\mathfrak{g}$, so $\kappa_{M,i} \in \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes i})$. Therefore $\kappa_{M,i} \otimes \mathrm{id}_V^{j-i}$ is an element of $\mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes j})$ for $i < j$. Thus, by Lemma 2.17, $\kappa_{M,j}$ commutes with $\kappa_{M,i}$ as operators on $M \otimes V^{\otimes j}$ (and therefore as operators on $M \otimes N \otimes V^{\otimes k}$). So the actions of $\kappa_{M,i}$, $i = 1, 2, \ldots, k$, pairwise commute and are elements of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$. Similarly, the actions of $\kappa_{N,i}$, $i = 1, 2, \ldots, k$, pairwise commute and are elements of $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$. This implies that the action of $x_1, \ldots, x_k$ (respectively $y_1, \ldots, y_k$ and $z_0, \ldots, z_k$) on $M \otimes N \otimes V^{\otimes k}$ pairwise commute. Moreover, since κ_M, κ_N, and κ_V each act on $M \otimes N \otimes V^{\otimes k}$ as constants,

$$\Phi(z_0 + \cdots + z_i) = \frac{1}{2}(\kappa_{M\otimes N,i} + i\kappa_V - \kappa_M - \kappa_N)$$

commutes with $\kappa_{M,j}$ and $\kappa_{N,j}$ for $j \le i$ by analogous reasoning, verifying (3.5).

The relations in (3.6) follow from

$$\begin{aligned}
\Phi(t_{s_i}(x_i + x_{i+1})) &= \tfrac{1}{2} t_{s_i}(\kappa_{M,i+1} - \kappa_{M,i-1}) \\
&= \tfrac{1}{2} t_{s_i} \left(\kappa_M + (i+1)\kappa_V + 2 \left(\sum_{1 \le j \le i+1} \gamma_{M,j} + \sum_{1 \le r < s \le i+1} \gamma_{r,s} \right) \right. \\
&\qquad \left. - \left(\kappa_M + (i-1)\kappa_V + 2 \left(\sum_{1 \le j \le i-1} \gamma_{M,j} + \sum_{1 \le r < s \le i-1} \gamma_{r,s} \right) \right) \right) \\
&= \tfrac{1}{2} t_{s_i} \left(2\kappa_V + 2 \sum_{\ell=1}^{i-1} (\gamma_{\ell,i} + \gamma_{\ell,i+1}) + 2\gamma_{i,i+1} \right) \\
&= \frac{1}{2} \left(2\kappa_V + 2 \sum_{\ell=1}^{i-1} (\gamma_{\ell,i+1} + \gamma_{\ell,i}) + 2\gamma_{i+1,i} \right) t_{s_i} \\
&= \Phi((x_i + x_{i+1}) t_{s_i}).
\end{aligned}$$

(a similar computation will confirm $\Phi(t_{s_i}(y_i + y_{i+1})) = \Phi((y_i + y_{i+1})t_{s_i}))$. Notice that in combination with (3.6), relation (3.4) is equivalent to

$$t_{s_i} \kappa_{X,j} = \kappa_{X,j} t_{s_i}, \text{ if } j \neq i, \text{ and } X = M, N, \text{ or } M \otimes N. \tag{3.14}$$

Since the action of the symmetric group commutes with the action of $\mathfrak{g}$, (3.14) is satisfied for $i < j$. If $j < i$, $\kappa_{X,j}$ acts by the identity on the i and $i+1$ factors of $V^{\otimes k}$, (3.14) is satisfied. Thus (3.14) (and therefore (3.4)) is satisfied for all $i \neq j$.

Finally,

$$\begin{aligned}
x_j &= \tfrac{1}{2} (\kappa_{M,j} - \kappa_{M,j-1}) \\
&= \frac{1}{2} \left(\kappa_M + j\kappa_V + 2 \left(\sum_{1 \le i \le j} \gamma_{M,i} + \sum_{1 \le r < s \le j} \gamma_{r,s} \right) \right. \\
&\qquad \left. - \left(\kappa_M + (j-1)\kappa_V + 2 \left(\sum_{1 \le i \le j-1} \gamma_{M,i} + \sum_{1 \le r < s \le j-1} \gamma_{r,s} \right) \right) \right) \\
&= \tfrac{1}{2}\kappa_V + \gamma_{M,j} + \sum_{1 \le \ell < j} \gamma_{\ell,j},
\end{aligned}$$

and similarly

$$\begin{aligned}
y_j &= \tfrac{1}{2} (\kappa_{N,j} - \kappa_{N,j-1}) \\
&= \tfrac{1}{2}\kappa_j + \gamma_{N,j} + \sum_{1 \le \ell < j} \gamma_{\ell,j}, \\
z_j &= \tfrac{1}{2} (\kappa_{M \otimes N,j} - \kappa_{M \otimes N,j-1} + \kappa_V) \\
&= \kappa_V + \gamma_{N,j} + \gamma_{M,j} + \sum_{1 \le \ell < j} \gamma_{\ell,j}.
\end{aligned}$$

So

$$x_{j+1} - t_{s_j} x_j t_{s_j} = y_{j+1} - t_{s_j} y_j t_{s_j} = z_{j+1} - t_{s_j} z_j t_{s_j} = \gamma_{j,j+1}. \tag{3.15}$$

So (3.8) and (3.9) are satisfied. Since $t_{s_j} t_{s_{j+1}} \gamma_{j,j+1} t_{s_{j+1}} t_{s_j} = t_{s_j} \gamma_{j,j+2} t_{s_j} = \gamma_{j+1,j+2}$, relation (3.7) follows from (3.15).

□

Lemma 3.4. *Fix c_i^x, c_i^y, c_i^z, $c_0^z \in \mathbb{C}$, $1 \leq i \leq k$, satisfying*

$$c_{i+1}^x - c_i^x = c_{i+1}^y - c_i^y = c \quad \text{and} \quad c_i^z = c_i^x + c_i^y - (i-1)c$$

for $i \geq 1$, where $c = c_2^x - c_1^x$. The map $\phi : \mathcal{G}_k \to \mathcal{G}_k$ given by

$$t_{s_i} \mapsto t_{s_i}, \quad x_i \mapsto x_i - c_i^x, \quad y_i \mapsto y_i - c_i^y, \quad z_i \mapsto z_i - c_i^z, \quad z_0 \mapsto z_0 - c_0^z,$$

is an algebra automorphism.

Proof. We need only check that relations (3.7), (3.8), and (3.9) are satisfied:

Relation (3.7): If $c_{i+2}^x - c_{i+1}^x = c_{i+1}^x - c_i^x$ and $c_{i+2}^y - c_{i+1}^y = c_{i+1}^y - c_i^y$, then

$$\begin{aligned}
&\phi((t_{s_i} t_{s_{i+1}})\,(x_{i+1} - t_{s_i} x_i t_{s_i})\,(t_{s_{i+i}} t_{s_i})) \\
&\quad = (t_{s_i} t_{s_{i+1}})\,(x_{i+1} - t_{s_i} x_i t_{s_i})\,(t_{s_{i+i}} t_{s_i})) - (c_{i+1}^x - c_i^x) \\
&\quad = x_{i+2} - t_{s_{i+1}} x_{i+1} t_{s_{i+1}} - (c_{i+2}^x - c_{i+1}^x) \\
&\quad = \phi(x_{i+2} - t_{s_{i+1}} x_{i+1} t_{s_{i+1}}), \text{ and similarly} \\
&\phi((t_{s_i} t_{s_{i+1}})\,(y_{i+1} - t_{s_i} y_i t_{s_i})\,(t_{s_{i+i}} t_{s_i})) \\
&\quad = \phi(y_{i+2} - t_{s_{i+1}} y_{i+1} t_{s_{i+1}})
\end{aligned}$$

Relation (3.8): If $c_{i+1}^x - c_i^x = c_{i+1}^y - c_i^y$, then

$$\begin{aligned}
&\phi(x_{i+1} - t_{s_i} x_i t_{s_i}) = x_{i+1} - t_{s_i} x_i t_{s_i} - (c_{i+1}^x - c_i^x) \\
&\quad = y_{i+1} - t_{s_i} y_i t_{s_i} - (c_{i+1}^y - c_i^y) = \phi(y_{i+1} - t_{s_i} y_i t_{s_i}).
\end{aligned}$$

Relation (3.9): If $c_1^z = c_1^x + c_1^y$ then

$$\phi(x_1 + y_1) = x_1 + y_1 - (c_1^x + c_1^y) = \phi(z_1).$$

If $c_i^z = c_i^x + c_i^y - (i-1)(c_i^x - c_{i-1}^x)$ for $i > 1$, then

$$\begin{aligned}
&\phi(m_i) = \phi\left(\sum_{j=1}^{i-1} t_{(j\ i-1)}(x_i - t_{s_{i-1}} x_{i-1} t_{s_{i-1}}) t_{(j\ i-1)}\right) \\
&\quad = \sum_{j=1}^{i-1} \left(t_{(j\ i-1)}(x_i - t_{s_{i-1}} x_{i-1} t_{s_{i-1}}) t_{(j\ i-1)} - (c_i^x - c_{i-1}^x)\right) \\
&\quad = m_i + (i-1)(c_i^x - c_{i-1}^x), \text{and so} \\
&\phi(z_i) = \phi(x_i + y_i - m_i) \\
&\quad = x_i + y_i - m_i - (c_i^x + c_i^y - (i-1)(c_i^x - c_{i-1}^x)) \\
&\quad = z_i - c_i^z.
\end{aligned}$$

□

This automorphism gives rise to the following corollary to Theorem 3.3.

Corollary 3.5. *The map* $\Phi'\colon \mathcal{G}_k \to \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ *defined by*

$$\begin{aligned}
\Phi'(t_{s_i}) &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes \mathrm{id}_V^{\otimes(i-1)} \otimes s_1 \otimes \mathrm{id}_V^{\otimes(k-i-1)},\\
\Phi'(x_i) &= \tfrac{1}{2}(\kappa_{M,i} - \kappa_{M,i-1}) + c_i^x,\\
\Phi'(y_i) &= \tfrac{1}{2}(\kappa_{N,i} - \kappa_{N,i-1}) + c_i^y,\\
\Phi'(z_i) &= \tfrac{1}{2}(\kappa_{M\otimes N,i} - \kappa_{M\otimes N,i-1} + \kappa_V) + c_i^z,\\
\Phi'(z_0) &= \tfrac{1}{2}(\kappa_{M\otimes N} - \kappa_M - \kappa_N) + c_0^z = \gamma_{M,N} + c_0^z,
\end{aligned} \tag{3.16}$$

is a representation of $\mathcal{G}_k$ *which commutes with the action of* $\mathfrak{g}$ *whenever* c_i^x, c_i^y, c_i^z, $c_0^z \in \mathbb{C}$, $1 \leq i \leq k$ *satisfy*

$$c_{i+1}^x - c_i^x = c_{i+1}^y - c_i^y = c \quad \text{and} \quad c_i^z = c_i^x + c_i^y - (i-1)c \tag{3.17}$$

for $i \geq 1$, *where* $c = c_2^x - c_1^x$.

Chapter 4

The degenerate two-boundary Hecke algebra

We consider the case where $\mathfrak{g}$ is of type $\mathfrak{gl}_n$ or $\mathfrak{sl}_n$, and fix three specific $\mathfrak{g}$-modules (M and N indexed by rectangular partitions, and V being the first fundamental representation). We use the representations of $\mathcal{G}_k$ in Corollary 3.5 to motivate the construction of a new algebra, the degenerate extended two-boundary Hecke algebra. The goal is to find the centralizer of the action of $\mathfrak{g}$ in $M \otimes N \otimes V^{\otimes k}$.

Definition 4.1. *Fix $a, b, p, q \in \mathbb{Z}_{>0}$. The* degenerate extended two-boundary Hecke algebra *$\mathcal{H}_k^{ext}$ is the quotient of the two-boundary graded braid group by the relations*

$$t_{s_i} x_i = x_{i+1} t_{s_i} - 1, \quad t_{s_i} y_i = y_{i+1} t_{s_i} - 1, \quad i = 1, \dots, k-1. \tag{4.1}$$

$$(x_1 - a)(x_1 + p) = 0 \qquad (y_1 - b)(y_1 + q) = 0. \tag{4.2}$$

The degenerate two-boundary Hecke algebra *$\mathcal{H}_k$ is the subalgebra of $\mathcal{H}_k^{ext}$ generated by $x_1, \dots, x_k,\ y_1, \dots, y_k,\ z_1, \dots, z_k,\ t_{s_1}, \dots, t_{s_{k-1}}$.*

The degenerate one-boundary Hecke algebra *$\mathcal{H}_k^{(1)}$ is the quotient of $\mathcal{G}_k^{(1)}$ by the relations*

$$t_{s_i} z_i = z_{i+1} t_{s_i} - 1, \qquad i = 1, \dots k-1. \tag{4.3}$$

Note that though it is not indicated by the notation, each of these Hecke algebras are dependent on the choice of constants a, b, p, q.

The following proposition provides a presentation of $\mathcal{H}_k^{\text{ext}}$ which is a straightforward consolidation of the presentation in Section 3. We will follow this up with Theorem 4.3, which provides a much more efficient presentation that we will make use of in Section 4.3. Finally, we will extract from Theorem 4.3 a preferred presentation of $\mathcal{H}_k$.

Proposition 4.2. *$\mathcal{H}_k^{ext}$ is presented as an algebra over $\mathbb{C}$ by generators*

$$x_1, z_0, z_1, t_{s_1}, \dots, t_{s_{k-1}}$$

and relations

Braid relations:

$$t_{s_i}^2 = 1, \quad t_{s_i} t_{s_j} = t_{s_j} t_{s_i} \quad j \neq i \pm 1, \qquad t_{s_i} t_{s_{i+1}} t_{s_i} = t_{s_{i+1}} t_{s_i} t_{s_{i+1}}, \quad i = 1, \dots, k-2$$

Quadratic relations:

$$(x_1-a)(x_1+p)=0 \qquad (y_1-b)(y_1+q)=0, \qquad a,b,p,q\in\mathbb{Z}_{>0},$$

Commutation relations:

$$t_{s_i}x_j=x_jt_{s_i},\ t_{s_i}z_j=z_jt_{s_i}, \quad \textit{for } j\neq i,i+1,$$
$$x_ix_j=x_jx_i,\ y_iy_j=y_jy_i,\ z_iz_j=z_jz_i,\ z_0z_i=z_iz_0, \quad \textit{for } 1\leq i,j\leq k,$$
$$x_jz_i=z_ix_j \quad \textit{for } i>j$$

Twisting relations:

$$\begin{matrix} x_i(z_0+\cdots+z_i)=(z_0+\cdots+z_i)x_i, \\ y_i(z_0+\cdots+z_i)=(z_0+\cdots+z_i)y_i, \end{matrix} \quad \textit{for } i=1\dots k$$

where

$$x_i=t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}}, \qquad z_i=t_{s_{i-1}}z_{i-1}t_{s_{i-1}}+t_{s_{i-1}}, \quad \textit{for } i=2,\dots,k,$$

and if

$$m_1=0, \qquad m_i=\sum_{j=1}^{i-1}t_{(j\ i)}$$

then

$$y_i=z_i-x_i+m_i \quad \textit{for } i=1,\dots,k.$$

Proof. We begin by simplifying relations (3.1) - (3.11) by introducing (4.1):

1. Equation (3.6) can be rewritten as

   $$x_it_{s_i}-t_{s_i}x_{i+1}=t_{s_i}x_i-x_{i+1}t_{s_i} \quad \text{and} \quad y_it_{s_i}-t_{s_i}y_{i+1}=t_{s_i}y_i-y_{i+1}t_{s_i}$$

   for $1\leq i\leq k-1$. But

   $$x_it_{s_i}-t_{s_i}x_{i+1}=t_{s_i}(t_{s_i}x_i-x_{i+1}t_{s_i})t_{s_i}=t_{s_i}(-1)t_{s_i}=-1.$$

   Similarly, $y_it_{s_i}-t_{s_i}y_{i+1}=-1$. So together with (4.1), equation (3.6) is equivalent to $-1=-1$, and can be discarded.

2. Equation (3.7), given by

   $$\begin{matrix} (t_{s_i}t_{s_{i+1}})(x_{i+1}-t_{s_i}x_it_{s_i})(t_{s_{i+i}}t_{s_i})=x_{i+2}-t_{s_{i+1}}x_{i+1}t_{s_{i+1}} \\ (t_{s_i}t_{s_{i+1}})(y_{i+1}-t_{s_i}y_it_{s_i})(t_{s_{i+i}}t_{s_i})=y_{i+2}-t_{s_{i+1}}y_{i+1}t_{s_{i+1}} \end{matrix} \quad \text{for } 1\leq i\leq k-2,$$

   is equivalent to

   $$(t_{s_i}t_{s_{i+1}})(t_{s_i})(t_{s_{i+i}}t_{s_i})=t_{s_{i+1}} \quad \text{for } 1\leq i\leq k-2.$$

   But $t_{s_i}t_{s_{i+1}}t_{s_i}t_{s_{i+i}}t_{s_i}=t_{s_i}t_{s_{i+i}}t_{s_i}=t_{s_{i+1}}$ by equation (3.3). So (3.7) can also be discarded.

3. Equation (3.8), given by

$$x_{i+1} - t_{s_i} x_i t_{s_i} = y_{i+1} - t_{s_i} y_i t_{s_i} \quad \text{for } 1 \leq i \leq k-1$$

is equivalent to $t_{s_i} = t_{s_i}$ and can be discarded.

4. Equations (3.10) can be rewritten as

$$m_{i,i+1} = x_{i+1} - t_{s_i} x_i t_{s_i} = t_{s_i}$$

$$\text{and} \qquad m_{i,j} = t_{(j-1\ i)} m_{j-1,j} t_{(j-1\ i)} = t_{(i\ j)} \quad \text{if } j \neq i+1.$$

So

$$m_1 = 0, \quad m_i = \sum_{1<j<i} t_{(i\ j)}.$$

5. By the previous item, (3.9) implies

$$\begin{aligned} t_{s_i} z_i t_{s_i} &= t_{s_i}(x_i + y_i - m_i) t_{s_i} \\ &= x_{i+1} - t_{s_i} + y_{i+1} - t_{s_i} - t_{s_i} \left(\sum_{1<j<i} t_{(i\ j)} \right) t_{s_i} \\ &= x_{i+i} + y_{i+1} - t_{s_i} - t_{s_i} - \sum_{1<j<i} t_{(i+1\ j)} \\ &= x_{i+i} + y_{i+1} - m_i - t_{s_i} \\ &= z_{i+1} - t_{s_i}. \end{aligned}$$

Similarly, together with (3.9), any two of

$$x_{i+1} - t_{s_i} x_i t_{s_i}, \quad y_{i+1} - t_{s_i} y_i t_{s_i}, \quad \text{and } z_{i+1} - t_{s_i} z_i t_{s_i}, \qquad i = 1, \ldots, k-1,$$

imply the third.

6. Furthermore, we see that the quotient by (4.1) can be viewed as discarding the generators $x_2, \ldots, x_k$, $y_1, \ldots, y_k$, and $z_2, \ldots, z_k$, by defining

$$x_i = t_{s_{i-1}} x_{i-1} t_{s_{i-1}} + t_{s_{i-1}}, \quad z_i = t_{s_{i-1}} z_{i-1} t_{s_{i-1}} + t_{s_{i-1}}, \quad i = 2, \ldots, k,$$

$$\text{and} \quad y_i = z_i - x_i + m_i, \quad i = 1, \ldots, k$$

Finally, we choose two rearrangements from the original presentation which do not depend on (4.1):

1. The second relation in (3.4) can be discarded since for $j \neq i, i+1$,

$$\begin{aligned} t_{s_i} y_j &= t_{s_i} \left(z_j - x_j + \sum_{\ell=1}^{j-1} t_{(\ell\ j)} \right) \\ &= \left(z_j - x_j + \sum_{\ell=1}^{j-1} t_{(\ell\ j)} \right) t_{s_i} \\ &= y_j t_{s_i}. \end{aligned}$$

2. We can separate (3.5), given by

$$\begin{aligned} (z_0 + \cdots + z_i)\, x_j &= x_j\, (z_0 + \cdots + z_i)\,, \\ (z_0 + \cdots + z_i)\, y_j &= y_j\, (z_0 + \cdots + z_i)\,, \end{aligned} \qquad \text{for } i \geq j$$

into two relations. First, when we set $i = j$, these two equations can be rewritten as

$$\begin{aligned} x_i z_0 &= z_0 x_i + \big((z_1 + \cdots + z_i)x_i - x_i(z_1 + \cdots + z_i)\big), \\ y_i z_0 &= z_0 y_i + \big((z_1 + \cdots + z_i)y_i - y_i(z_1 + \cdots + z_i)\big), \end{aligned} \qquad \text{for } i = 1, \ldots, k.$$

Second, for $i > j$,

$$\begin{aligned} x_j z_i &= x_j(z_0 + \cdots + z_i) - x_j(z_0 + \cdots + z_{i-1}) \\ &= (z_0 + \cdots + z_i)x_j - (z_0 + \cdots + z_{i-1})x_j \\ &= z_i x_j. \end{aligned}$$

Similarly, $y_j z_i = z_i y_j$. In combination, we can also recover (3.5) from these two sets of relations.

So we have found that $\mathcal{H}_k^{\text{ext}}$ is presented as an algebra over $\mathbb{C}$ by generators

$$x_1, z_0, z_1, t_{s_1}, \ldots, t_{s_{k-1}}$$

and relations

(3.1) - (3.3):

$$t_{s_i}^2 = 1, \quad t_{s_i} t_{s_j} = t_{s_j} t_{s_i} \quad j \neq i \pm 1, \qquad t_{s_i} t_{s_{i+1}} t_{s_i} = t_{s_{i+1}} t_{s_i} t_{s_{i+1}}, \quad i = 1, \ldots, k-2$$

(x_i, y_i, and z_j generate polynomial rings):

$$x_i x_j = x_j x_i,\ y_i y_j = y_j y_i,\ z_i z_j = z_j z_i,\ z_0 z_i = z_i z_0, \qquad \text{for } 1 \leq i, j \leq k,$$

(3.4)′:

$$t_{s_i}x_j = x_j t_{s_i},\ t_{s_i}z_j = z_j t_{s_i}, \quad \text{for } j \neq i, i+1,$$

(3.5)′

$$\begin{aligned} x_i z_0 &= z_0 x_i + \big((z_1 + \cdots + z_i)x_i - x_i(z_1 + \cdots + z_i)\big), \\ y_i z_0 &= z_0 y_i + \big((z_1 + \cdots + z_i)y_i - y_i(z_1 + \cdots + z_i)\big), \end{aligned} \quad \text{for } i = 1 \ldots k$$

$$x_j z_i = z_i x_j \quad y_j z_i = z_i y_j \qquad \text{for } i > j$$

(4.2)

$$(x_1 - a)(x_1 + p) = 0 \qquad (y_1 - b)(y_1 + q) = 0$$

where

$$m_1 = 0, \qquad m_i = \sum_{j=1}^{i-1} t_{(j\ i)}$$

$$x_i = t_{s_{i-1}} x_{i-1} t_{s_{i-1}} + t_{s_{i-1}}, \qquad z_i = t_{s_{i-1}} z_{i-1} t_{s_{i-1}} + t_{s_{i-1}},$$

$$\text{and} \qquad y_j = z_j - x_j + m_j$$

for $i = 2, \ldots, k$, $j = 1, \ldots, k$ ((3.9)′, (3.10)′ , (3.11)′, and (4.1)′).

□

While the presentation provided just now in Proposition 4.2 is somehow the most recognizable from the braid group $\mathcal{G}_k$, we can distill further. The following presentation will be our favorite choice in calculating representations in Section 4.3.

Theorem 4.3. *Let $w_i = z_i - \frac{1}{2}(a - p + b - q)$. $\mathcal{H}_k^{\text{ext}}$ is presented as an algebra over $\mathbb{C}$ by generators w_0, w_1, …, w_k, x_1, t_{s_1}, …, $t_{s_{k-1}}$, and relations*

Braid relations:

$$t_{s_i}^2 = 1, \qquad i = 1, \ldots, k-1 \tag{4.4}$$

$$t_{s_i} t_{s_j} = t_{s_j} t_{s_i}, \qquad j \neq i \pm 1 \tag{4.5}$$

$$t_{s_i} t_{s_{i+1}} t_{s_i} = t_{s_{i+1}} t_{s_i} t_{s_{i+1}}, \qquad i = 1, \ldots, k-2 \tag{4.6}$$

$$x_1(t_{s_1} x_1 t_{s_1} + t_{s_1}) = (t_{s_1} x_1 t_{s_1} + t_{s_1})x_1 \tag{4.7}$$

Quadratic relation:

$$(x_1 - a)(x_1 + p) = 0 \tag{4.8}$$

Commutation relations:

$$t_{s_i} w_j = w_j t_{s_i}, \qquad j \neq i, i+1 \tag{4.9}$$

$$x_1 w_i = w_i x_1 \qquad i = 2, \ldots, k \tag{4.10}$$

$$x_1 t_{s_i} = t_{s_i} x_1 \qquad i = 2, \ldots, k-1 \tag{4.11}$$

$$w_i w_j = w_j w_i \qquad i, j = 0, \ldots, k \tag{4.12}$$

Twisting relations:

$$t_{s_i} w_i = w_{i+1} t_{s_i} - 1, \qquad i = 1, \ldots, k-1, \tag{4.13}$$

$$x_1 w_0 = w_0 x_1 - (x_1 w_1 - w_1 x_1), \tag{4.14}$$

and

$$x_1 w_1 = -w_1 x_1 + (a-p) w_1 + w_1^2 + \left(\frac{a+p+b+q}{2}\right)\left(\frac{a+p-(b+q)}{2}\right). \tag{4.15}$$

Proof. Since $\frac{1}{2}(a-p+b-q)$ is central in $\mathcal{H}_k^{\text{ext}}$, each of the commutation and twisting relations concerning z_i in the presentation given in Proposition 4.2 are true with w_i replacing z_i, i.e.

$$t_{s_i} w_j = w_j t_{s_i}, \quad \text{for } j \neq i, i+1,$$
$$w_i w_j = w_j w_i, \; w_0 w_i = w_i w_0, \quad \text{for } 1 \leq i, j \leq k,$$
$$x_j w_i = w_i x_j \quad \text{for } i > j$$

and

$$x_i(w_0 + \cdots + w_i) = (w_0 + \cdots + w_i) x_i,$$
$$y_i(w_0 + \cdots + w_i) = (w_0 + \cdots + w_i) y_i.$$

Next, we address the new twisting relation (4.15) by proving the following claim:

Claim 0: The set of relations
(A): $(x_1 - a)(x_1 + p) = 0$, $(y_1 - b)(y_1 + q) = 0$, and $w_1 = x_1 + y_1 - \frac{1}{2}(a-p+b-q)$

are equivalent to the set of relations

(B):
$$(x_1 - a)(x_1 + p) = 0 \quad \text{and}$$
$$x_1 w_1 = -w_1 x_1 + (a-p) w_1 + w_1^2 + \left(\tfrac{a+p+b+q}{2}\right)\left(\tfrac{a+p-(b+q)}{2}\right)$$

Proof:

(A) $\Longrightarrow$ (B): First notice that

$$x_1^2 = (a-p) x_1 + ap, \quad y_1^2 = (b-q) y_1 + bq,$$

and $\quad z_1^2 = (x_1 + y_1)^2 = x_1 y_1 + y_1 x_1 + (a-p) x_1 + (b-q) y_1 + ap + bq.$

So

$$\begin{aligned} x_1 w_1 + w_1 x_1 &= x_1(x_1 + y_1 - (a-p+b-q)/2) \\ &\quad + (x_1 + y_1 - (a-p+b-q)/2) x_1 \\ &= 2x_1^2 + (x_1 y_1 + y_1 x_1) - (a-p+b-q) x_1 \\ &= (a-p-(b-q)) x_1 + 2ap + (x_1 y_1 + y_1 x_1). \end{aligned}$$

Since

$$\begin{aligned}
w_1^2 &= z_1^2 - (a-p+b-q)z_1 + \frac{1}{4}(a-p+b-q)^2 \\
&= x_1y_1 + y_1x_1 + (a-p)x_1 + (b-q)y_1 + ap + bq \\
&\qquad - (a-p+b-q)(x_1+y_1) + \frac{1}{4}(a-p+b-q)^2 \\
&= (x_1y_1 + y_1x_1) - (b-q)x_1 - (a-p)(w_1 - x_1 + (a-p+b-q)/2) \\
&\qquad + ap + bq + \frac{1}{4}(a-p+b-q)^2 \\
&= (x_1y_1 + y_1x_1) + (a-p-(b-q))x_1 - (a-p)w_1 \\
&\qquad + ap + bq - (a-p)^2/4 + (b-q)^2/4
\end{aligned}$$

we have

$$\begin{aligned}
x_1w_1 + w_1x_1 &= (a-p-(b-q))x_1 + 2ap \\
&\qquad + (w_1^2 - \big((a-p-(b-q))x_1 - (a-p)w_1 \\
&\qquad + ap + bq - (a-p)^2/4 + (b-q)^2/4)\big) \\
&= w_1^2 + (a-p)w_1 + \big(ap + (a-p)^2/4\big) - \big(bq + (b-q)^2/4\big)\,. \\
&= w_1^2 + (a-p)w_1 + \left(\frac{a+p+b+q}{2}\right)\left(\frac{a+p-(b+q)}{2}\right).
\end{aligned}$$

(B) $\Longrightarrow$ (A): Let $y_1 = w_1 - x_1 + \frac{1}{2}(a-p+b-q)$. Then

$$\begin{aligned}
(y_1-b)(y_1+q) &= (w_1 - x_1 + \tfrac{1}{2}(a-p+b-q) - b) \\
&\qquad (w_1 - x_1 + \tfrac{1}{2}(a-p+b-q) + q) \\
&= (w_1-x_1)^2 + (w_1-x_1)(a-p+b-q-b+q) \\
&\qquad + (\tfrac{1}{2}(a-p+b-q) - b)(\tfrac{1}{2}(a-p+b-q) + q) \\
&= w_1^2 + x_1^2 - (w_1x_1 + x_1w_1) + (a-p)w_1 - (a-p)x_1 \\
&\qquad + (\tfrac{1}{2}(a+p-(b+q)) - p)(\tfrac{1}{2}(a+p+b+q) - p) \\
&= \big(w_1^2 + (a-p)w_1 + (\tfrac{1}{2})^2(a+p+b+q)(a+p-(b+q))\big) \\
&\qquad\quad - (\tfrac{1}{2})^2(a+p+b+q)(a+p-(b+q)) \\
&\qquad - (w_1x_1 + x_1w_1) \\
&\qquad + (x_1-a)(x_1+p) + ap \\
&\qquad + (\tfrac{1}{2})^2(a+p-(b+q))(a+p+b+q) \\
&\qquad - p(\tfrac{1}{2}(a+p-(b+q)) + \tfrac{1}{2}(a+p+b+q)) + p^2 \\
&= ap - p(a+p) + p^2 = 0
\end{aligned}$$

Since the remainder of the relations listed in this presentation are a subset of the relations in Proposition 4.2, we will proceed by showing that relations in Proposition 4.2 follow from this shortened list. As in Proposition 4.2, define $x_{i+1} = t_{s_i} x_i t_{s_i} + t_{s_i}$.

Claim 1:

$$\begin{aligned} x_{j+1} = {} & t_{s_j} \cdots t_{s_{i+1}} (t_{s_i} x_i t_{s_i} + t_{s_i}) t_{s_{i+1}} \cdots t_{s_j} \\ & + \sum_{\ell=i+1}^{j} t_{s_j} \cdots t_{s_{\ell+1}} t_{s_\ell} t_{s_{\ell+1}} \cdots t_{s_j} \end{aligned} \tag{4.16}$$

Proof: This follows by induction on j.

Claim 2: $t_{s_i} x_j = x_j t_{s_i}$ for $i > j$.
Proof: If $i > j$, then t_{s_i} commutes with t_{s_ℓ} for all $\ell < j$, so by Claim 1 and 4.4

$$\begin{aligned} t_{s_i} x_j &= t_{s_i} (t_{s_{j-1}} \cdots t_{s_2})(t_{s_1} x_1 t_{s_1} + t_{s_1})(t_{s_2} \cdots t_{s_{j-1}}) \\ &\quad + t_{s_i} \sum_{\ell=2}^{j-1} t_{s_{j-1}} \cdots t_{s_{\ell+1}} t_{s_\ell} t_{s_{\ell+1}} \cdots t_{s_{j-1}} \\ &= (t_{s_{j-1}} \cdots t_{s_2})(t_{s_1} x_1 t_{s_1} + t_{s_1})(t_{s_2} \cdots t_{s_{j-1}}) t_{s_i} \\ &\quad + \left(\sum_{\ell=2}^{j-1} t_{s_{j-1}} \cdots t_{s_{\ell+1}} t_{s_\ell} t_{s_{\ell+1}} \cdots t_{s_{j-1}} \right) t_{s_i} \\ &= x_j t_{s_i}. \end{aligned}$$

Claim 3: $t_{s_i} x_j = x_j t_{s_i}$ for $i < j - 1$.
Proof: By Claim 1,

$$\begin{aligned} t_{s_i} x_j &= t_{s_i} (t_{s_{j-1}} \cdots t_{s_{i+2}} t_{s_{i+1}})(t_{s_i} x_i t_{s_i} + t_{s_i})(t_{s_{i+1}} t_{s_{i+2}} \cdots t_{s_{j-1}}) \\ &\quad + t_{s_i} t_{s_{j-1}} \cdots t_{s_{i+2}} t_{s_{i+1}} t_{s_{i+2}} \cdots t_{s_{j-1}} \\ &\quad + t_{s_i} \sum_{\ell=i+2}^{j-1} t_{s_{j-1}} \cdots t_{s_{\ell+1}} t_{s_\ell} t_{s_{\ell+1}} \cdots t_{s_{j-1}} \\ &= (t_{s_{j-1}} \cdots t_{s_{i+2}})(t_{s_i}) \left(t_{s_{i+1}} (t_{s_i} x_i t_{s_i} + t_{s_i}) t_{s_{i+1}} + t_{s_{i+1}} \right) (t_{s_{i+2}} \cdots t_{s_{j-1}}) \\ &\quad + \left(\sum_{\ell=i+2}^{j-1} t_{s_{j-1}} \cdots t_{s_{\ell+1}} t_{s_\ell} t_{s_{\ell+1}} \cdots t_{s_{j-1}} \right) t_{s_i} \end{aligned}$$

But, by Claim 2, since $i+1>i$,

$$\begin{aligned} t_{s_i}(t_{s_{i+1}}(t_{s_i}x_it_{s_i}+t_{s_i})t_{s_{i+1}}+t_{s_{i+1}}) &= t_{s_i}t_{s_{i+1}}t_{s_i}x_it_{s_i}t_{s_{i+1}}+t_{s_i}t_{s_{i+1}}t_{s_i}t_{s_{i+1}}+t_{s_i}t_{s_{i+1}} \\ &= t_{s_{i+1}}t_{s_i}t_{s_{i+1}}x_it_{s_i}t_{s_{i+1}}+t_{s_{i+1}}t_{s_i}t_{s_{i+1}}^2+t_{s_i}t_{s_{i+1}}t_{s_i}^2 \\ &= t_{s_{i+1}}t_{s_i}x_it_{s_{i+1}}t_{s_i}t_{s_{i+1}}+t_{s_{i+1}}t_{s_i}+t_{s_{i+1}}t_{s_i}t_{s_{i+1}}t_{s_i} \\ &= t_{s_{i+1}}t_{s_i}x_it_{s_i}t_{s_{i+1}}t_{s_i}+t_{s_{i+1}}t_{s_i}+t_{s_{i+1}}t_{s_i}t_{s_{i+1}}t_{s_i} \\ &= (t_{s_{i+1}}(t_{s_i}x_it_{s_i}+t_{s_i})t_{s_{i+1}}+t_{s_{i+1}})t_{s_i}. \end{aligned}$$

So $t_{s_i}x_j=x_jt_{s_i}$.

Claim 4: $x_ix_{i+1}=x_{i+1}x_i$ for $i=1,\ldots,k-1$.
Proof: This follows by induction on i. First, it is satisfied for $i=1$ by 4.4. Next,

$$\begin{aligned} &x_ix_{i+1} \\ &\quad= x_it_{s_i}x_it_{s_i}+x_it_{s_i} \\ &\quad= \big((t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}})t_{s_i}(t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}})+(t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}})\big)t_{s_i} \\ &\quad= (t_{s_{i-1}}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}}x_{i-1}t_{s_{i-1}})t_{s_i} \\ &\qquad\quad+(t_{s_{i-1}}t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}})t_{s_i} \\ &\qquad\quad+(t_{s_{i-1}}t_{s_i}t_{s_{i-1}}+t_{s_{i-1}})t_{s_i}. \end{aligned}$$

But

$$\begin{aligned} (t_{s_{i-1}}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}}x_{i-1}t_{s_{i-1}})t_{s_i} &= t_{s_{i-1}}x_{i-1}t_{s_i}t_{s_{i-1}}t_{s_i}x_{i-1}t_{s_{i-1}}t_{s_i}+t_{s_{i-1}}x_{i-1}t_{s_i}^2t_{s_{i-1}}t_{s_i} \\ &= t_{s_{i-1}}t_{s_i}x_{i-1}t_{s_{i-1}}x_{i-1}t_{s_i}t_{s_{i-1}}t_{s_i}+t_{s_{i-1}}t_{s_i}x_{i-1}t_{s_i}t_{s_{i-1}}t_{s_i} \\ &= t_{s_{i-1}}t_{s_i}x_{i-1}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}}+t_{s_{i-1}}t_{s_i}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}} \\ &= t_{s_{i-1}}t_{s_i}(x_{i-1}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+x_{i-1}t_{s_{i-1}})t_{s_i}t_{s_{i-1}} \\ &= t_{s_{i-1}}t_{s_i}(t_{s_{i-1}}x_{i-1}t_{s_{i-1}}x_{i-1}+t_{s_{i-1}}x_{i-1})t_{s_i}t_{s_{i-1}} \\ &= t_{s_i}(t_{s_{i-1}}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}}x_{i-1}t_{s_{i-1}}), \end{aligned}$$

$$\begin{aligned} (t_{s_{i-1}}t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}+t_{s_{i-1}}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}})t_{s_i} &= t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_i}t_{s_{i-1}}+t_{s_{i-1}}x_{i-1}t_{s_i}t_{s_{i-1}} \\ &= t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_i}t_{s_{i-1}}+t_{s_i}^2t_{s_{i-1}}t_{s_i}x_{i-1}t_{s_{i-1}} \\ &= t_{s_i}(t_{s_{i-1}}x_{i-1}t_{s_{i-1}}t_{s_i}t_{s_{i-1}}t_{s_{i-1}}t_{s_i}t_{s_{i-1}}x_{i-1}t_{s_{i-1}}), \end{aligned}$$

and

$$(t_{s_{i-1}}t_{s_i}t_{s_{i-1}}+t_{s_{i-1}})t_{s_i}=t_{s_i}t_{s_{i-1}}t_{s_i}^2+t_{s_i}^2t_{s_{i-1}}t_{s_i}=t_{s_i}(t_{s_{i-1}}+t_{s_{i-1}}t_{s_i}t_{s_{i-1}}).$$

So

$$
\begin{aligned}
x_i x_{i+1} &= t_{s_i}\big((t_{s_{i-1}} x_{i-1} t_{s_{i-1}} + t_{s_{i-1}}) t_{s_i} (t_{s_{i-1}} x_{i-1} t_{s_{i-1}} + t_{s_{i-1}}) + (t_{s_{i-1}} x_{i-1} t_{s_{i-1}} + t_{s_{i-1}})\big) \\
&= t_{s_i} x_i t_{s_i} x_i + t_{s_i} x_i \\
&= x_{i+1} x_i .
\end{aligned}
$$

Claim 5: $x_i x_j = x_j x_i$ for $i, j = 1, \ldots, k$.
Proof: Assume, without loss of generality, that $i < j$. By Claim 1,

$$x_j = t_{s_{j-1}} \cdots t_{s_{i+1}} x_{i+1} t_{s_{i+1}} \cdots t_{s_{j-1}} + \sum_{\ell=i+1}^{j} t_{s_j} \cdots t_{s_{\ell+1}} t_{s_\ell} t_{s_{\ell+1}} \cdots t_{s_j} .$$

Thus, by Claims 2 and 4, $x_i x_j = x_j x_i$.

Claim 6: $w_j x_i = x_i w_j$ for $j > i$.
Proof: By Claim 1,

$$x_i = t_{s_{i-1}} \cdots t_{s_1} x_1 t_{s_1} \cdots t_{s_{i-1}} + \sum_{\ell=1}^{i-1} t_{s_{i-1}} \cdots t_{s_{\ell+1}} t_{s_\ell} t_{s_{\ell+1}} \cdots t_{s_{i-1}} .$$

Since $w_j x_1 = x_1 w_j$ for $j > 1$ and $w_j t_{s_\ell} = t_{s_\ell} w_j$ for $\ell < j - 1$, this implies that $w_j x_i = x_i w_j$ for $j > i$.

Claim 7:

$$x_i(w_0 + \cdots + w_i) = (w_0 + \cdots + w_i) x_i$$

Proof: This follows by induction on i. Since $x_1(w_0 + w_1) = (w_1 + w_0) x_1$, this relation is satisfied for $i = 1$. Now assume $x_{i-1}(w_0 + \cdots + w_{i-1}) = (w_0 + \cdots + w_{i-1}) x_{i-1}$. Recall that $t_{s_i} w_i = w_{i+1} t_{s_i} - 1$ implies $t_{s_i}(w_i + w_{i+1}) = (w_i + w_{i+1}) t_{s_i}$. So

$$
\begin{aligned}
x_i(w_0 + \cdots + w_i) &= (t_{s_{i-1}} x_{i-1} t_{s_{i-1}} + t_{s_{i-1}})(w_0 + \cdots + w_i) \\
&= (w_0 + \cdots + w_i)(t_{s_{i-1}} x_{i-1} t_{s_{i-1}} + t_{s_{i-1}}) \\
&= (w_0 + \cdots + w_i) x_i
\end{aligned}
$$

since $x_{i-1} w_i = w_i x_{i-1}$ by Claim 6, and $t_{s_\ell} w_j = w_j t_{s_\ell}$ for $\ell < j$.

Claim 8: If $y_1 = w_1 - x_1 + \frac{1}{2}(a - p + b - q)$ and $y_1 = w_2 - x_2 + t_{s_1} + \frac{1}{2}(a - p + b - q)$, then $y_1 y_2 = y_2 y_1$.

Proof: Let $K = \frac{1}{2}(a - p + b - q)$. So

$$\begin{aligned}
y_1 y_2 &= (w_1 - x_1 + K)(w_2 - (t_{s_1} x_1 t_{s_1} + t_{s_1}) + t_{s_1} + K) \\
&= (w_2 + K)(w_1 - x_1 + K) - (w_1 - x_1 + K) t_{s_1} x_1 t_{s_1} \\
&= (w_2 + K)(w_1 - x_1 + K) - (t_{s_1} x_1 t_{s_1}) K - w_1 t_{s_1} x_1 t_{s_1} + x_1 t_{s_1} x_1 t_{s_1} \\
&= (w_2 + K)(w_1 - x_1 + K) - (t_{s_1} x_1 t_{s_1}) K + x_1 t_{s_1} x_1 t_{s_1} - (t_{s_1} w_2 - 1) x_1 t_{s_1} \\
&= (w_2 + K)(w_1 - x_1 + K) - (t_{s_1} x_1 t_{s_1}) K + x_1 t_{s_1} x_1 t_{s_1} + x_1 t_{s_1} - t_{s_1} x_1 w_2 t_{s_1} \\
&= (w_2 + K)(w_1 - x_1 + K) - (t_{s_1} x_1 t_{s_1}) K + t_{s_1} x_1 t_{s_1} x_1 + t_{s_1} x_1 - t_{s_1} x_1 (t_{s_1} w_1 + 1) \\
&= (w_2 + K)(w_1 - x_1 + K) - (t_{s_1} x_1 t_{s_1}) K + t_{s_1} x_1 t_{s_1} x_1 - t_{s_1} x_1 t_{s_1} w_1 \\
&= (w_2 - t_{s_1} x_1 t_{s_1} + K)(w_1 - x_1 + K) \\
&= y_2 y_1.
\end{aligned}$$

Claim 9: If $y_1 = w_1 - x_1 + \frac{1}{2}(a - p + b - q)$, then $y_1 t_{s_i} = t_{s_i} y_1$ for $i > 1$.
Proof: This follows as t_{s_i} commutes with w_1, x_1, and $\frac{1}{2}(a - p + b - q)$ for $i > 1$.

Claim 10: Let $m_i = \sum_{j=1}^{i-1} t_{(j\ i)}$ and $K = \frac{1}{2}(a - p + b - q)$. If $y_1 = w_1 - x_1 + K$, then

$$y_i = w_i - x_i + m_i + K \qquad \text{and} \qquad y_i = t_{s_{i-1}} y_{i-1} t_{s_{i-1}} + t_{s_{i-1}}$$

for $i = 2, \ldots, k$ are equivalent definitions of y_i.
Proof: First, if $i < j$,

$$t_{(i\ j)} = t_{s_{j-1}} \cdots t_{s_{i+1}} t_{s_i} t_{s_{i+1}} \cdots t_{s_{j-1}},.$$

Thus

$$t_{s_j} t_{(i\ j)} t_{s_j} = t_{(i\ j+1)}, \qquad \text{and so} \qquad t_{s_j} m_j t_{s_j} = m_{j+1} - t_{s_j}$$

$$\begin{aligned}
t_{s_i} y_i t_{s_i} + t_{s_i} &= t_{s_i}(w_i - x_i + m_i + K) t_{s_i} + t_{s_i} \\
&= (w_{i+1} - t_{s_i}) - (x_{i+1} - t_{s_i}) + (m_{i+1} - t_{s_i}) + K + t_{s_i} \\
&= w_{i+1} - x_{i+1} + m_{i+1} + K \\
&= y_{i+1}.
\end{aligned}$$

Claim: If y_i is as in Claim 9, then

$$y_i y_j = y_j y_i \text{ for } i, j = 1, \ldots, k, \quad t_{s_i} y_j = y_j t_{s_i} \text{ for } j \neq i, i+1,$$

$$\text{and } y_i w_0 = w_0 y_i + \big((w_1 + \cdots + w_i) y_i - y_i (w_1 + \cdots + w_i)\big) \text{ for } i = 1, \ldots k.$$

Proof: These follow from Claims 8 - 10, just as the corresponding x_i-valued relations follow above.

□

One advantage to this presentation of Theorem 4.3 is that we can pluck from it a clean presentation of $\mathcal{H}_k$.

Corollary 4.4. *$\mathcal{H}_k$ is presented as an algebra over $\mathbb{C}$ by generators*

$$w_1, \dots, w_k, x_1, t_{s_1}, \dots, t_{s_{k-1}},$$

and relations

Braid relations:

$$t_{s_i}^2 = 1, \quad t_{s_i}t_{s_j} = t_{s_j}t_{s_i} \quad j \neq i \pm 1, \qquad t_{s_i}t_{s_{i+1}}t_{s_i} = t_{s_{i+1}}t_{s_i}t_{s_{i+1}}, \quad i = 1, \dots, k-2$$

$$x_1(t_{s_1}x_1t_{s_1} + t_{s_1}) = (t_{s_1}x_1t_{s_1} + t_{s_1})x_1$$

Quadratic relation:

$$(x_1 - a)(x_1 + p) = 0$$

Commutation relations:

$$t_{s_i}w_j = w_jt_{s_i}, \qquad \text{for } j \neq i, i+1,$$

$$x_1w_i = w_ix_1 \qquad \text{and} \qquad x_1t_{s_i} = t_{s_i}x_1, \quad \text{for } i \geq 2,$$

$$w_iw_j = w_jw_i, \qquad \text{for } i, j = 0, \dots, k,$$

Twisting relations:

$$t_{s_i}w_i = w_{i+1}t_{s_i} - 1, \quad i = 1, \dots, k-1,$$

and

$$x_1w_1 = -w_1x_1 + (a-p)w_1 + w_1^2 + \left(\frac{a+p+b+q}{2}\right)\left(\frac{a+p-(b+q)}{2}\right),$$

Proof. From Theorem 4.3, $\mathcal{H}_k^{\text{ext}} \cong \mathbb{C}[z_0] \otimes \mathcal{H}_k$ as vector spaces. Therefore, we can extract a presentation of $\mathcal{H}_k$ by dropping w_0 from Theorem 4.3. □

Remark 4.5. *Some useful relations include:*

$$w_{i+1}^n t_{s_i} = t_{s_i}w_i^n + \frac{w_{i+1}^n - w_i^n}{w_{i+1} - w_i} = t_{s_i}w_i^n + \sum_{j=1}^{n} w_i^{j-1}w_{i+1}^{n-j} \tag{4.17}$$

$$w_i^n t_{s_i} = t_{s_i}w_{i+1}^n - \frac{w_i^n - w_{i+1}^n}{w_i - w_{i+1}} = t_{s_i}w_{i+1}^n + \sum_{j=1}^{n} w_i^{n-j}w_{i+1}^{j-1} \tag{4.18}$$

So

$$\begin{aligned} w_i^n w_{i+1}^m t_{s_i} &= w_i^n \left(t_{s_i} w_i^m + \frac{w_{i+1}^m - w_i^m}{w_{i+1} - w_i} \right) \\ &= w_i^n t_{s_i} w_i^m + w_i^n \left(\frac{w_{i+1}^m - w_i^m}{w_{i+1} - w_i} \right) \\ &= \left(t_{s_i} w_{i+1}^n - \frac{w_i^n - w_{i+1}^n}{w_i - w_{i+1}} \right) w_i^m + w_i^n \left(\frac{w_{i+1}^m - w_i^m}{w_{i+1} - w_i} \right) \\ &= t_{s_i} w_i^m w_{i+1}^n - \left(\frac{w_i^n - w_{i+1}^n}{w_i - w_{i+1}} \right) w_i^m + w_i^n \left(\frac{w_{i+1}^m - w_i^m}{w_{i+1} - w_i} \right) \end{aligned}$$

Let $\lambda \in \mathbb{Z}_{\geq 0}^k$ *and denote*

$$w^\lambda = w_1^{\lambda_1} \cdots w_k^{\lambda_k}.$$

S_k *acts on* λ *by*

$$s_i \lambda = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_k).$$

So we have

$$w^\lambda t_{s_i} = t_{s_i} w^{s_i \lambda} + \frac{w^\lambda - w^{s_i \lambda}}{w_{i+1} - w_i}. \tag{4.19}$$

Now, renormalizing x_1*, let* $t_0 = \frac{1}{a+p}(2x_1 - (a-p))$ *so that*

$$\begin{aligned} t_0^2 &= \frac{1}{(a+p)^2}(4x_1^2 - 4(a-p)x_1 + (a-p)^2) \\ &= \frac{1}{(a+p)^2}(4(a-p)x_1 + 4ap - 4(a-p)x_1 + (a-p)^2) \\ &= \frac{1}{(a+p)^2}(a+p)^2 = 1 \end{aligned}$$

Then

$$w_1 t_0 = -t_0 w_1 + \frac{2}{a+p} \left(w_1^2 + \left(\frac{a+p+b+q}{2} \right) \left(\frac{a+p-(b+q)}{2} \right) \right). \tag{4.20}$$

Let

$$A = \frac{2}{a+p}, \qquad B = \frac{2}{a+p} \left(\frac{a+p+b+q}{2} \right) \left(\frac{a+p-(b+q)}{2} \right).$$

So

$$w_1 t_0 = -t_0 w_1 + A w_1^2 + B$$

implies

$$w_1^n t_0 = (-1)^n t_0 w_1^n + \begin{cases} w_1^{n-1}(Aw_1^2 + B) & n \text{ is odd,} \\ 0 & n \text{ is even.} \end{cases} \tag{4.21}$$

$$= t_0(-w_1)^n + \frac{w_1^n - (-w_1)^n}{2w_1}\left(Aw_1^2 + B\right) \tag{4.22}$$

Finally,

$$t_0 t_{s_1} t_0 t_{s_1} - t_{s_1} t_0 t_{s_1} t_0 = \frac{2}{(a+p)}\left(t_{s_1} t_0 - t_0 t_{s_1}\right).$$

4.1 Representations

Let $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$. Let $M = L\left((a^p)\right)$ (the finite dimensional $\mathfrak{g}$-module indexed by the rectangular partition with p parts of length a) and $N = ((b^q))$, and let $V = L((1^1))$ (the first fundamental representation, isomorphic to $\mathbb{C}^n$). We return to the representation defined in Corollary 3.5, and see that for special choices of constants, Φ' factors through as a representation of $\mathcal{H}_k^{\text{ext}}$.

Theorem 4.6.

(a) *When* $\mathfrak{g} = \mathfrak{gl}_n$*, fix* $c_i^x = c_i^y = \frac{1}{2}c_i^z = -\frac{1}{2}n$.

(b) *When* $\mathfrak{g} = \mathfrak{sl}_n$*, fix*

$$c_i^x = \frac{ap+i-1}{n} - \frac{1}{2}\left(n + \frac{1}{n}\right), \quad c_i^y = \frac{bq+i-1}{n} - \frac{1}{2}\left(n + \frac{1}{n}\right),$$

$$\text{and} \quad c_i^z = \frac{ap+bq+i}{n} - n.$$

These values satisfy the criteria in Lemma 3.4, so yield a representation

$$\Phi' = \Phi \circ \phi : \mathcal{G}_k \to \operatorname{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

Furthermore, for this choice of constants, Φ' *factors through the quotient defined in* (4.1) *and* (4.2)*, so defines an action of* $\mathcal{H}_k^{\text{ext}}$ *which commutes with the action of* $\mathfrak{g}$.

Proof. First,

1. when $\mathfrak{g} = \mathfrak{gl}_n$,

$$c_{i+1}^x - c_i^x = c_{i+1}^y - c_i^y = 0, \quad \text{and} \quad c_i^x + c_i^y - (i-1)(0) = -n = c_i^z, \quad \text{and}$$

2. when $\mathfrak{g} = \mathfrak{sl}_n$,

$$c^x_{i+1} - c^x_i = c^y_{i+1} - c^y_i = \frac{1}{n}, \quad \text{and}$$
$$c^x_i + c^y_i - (i-1)\frac{1}{n} = \frac{ap+bq+i}{n} - n = c^z_i,$$

so c^x_i, c^y_i, c^z_i satisfy the requirements of Lemma 3.4. Therefore Φ' is a representation of $\mathcal{G}_k$.

The relations in (4.1) can be rewritten as

$$x_{i+1} - t_{s_i} x_i t_{s_i} = t_{s_i}, \quad y_{i+1} - t_{s_i} y_i t_{s_i} = t_{s_i}, \quad i = 1, \dots, k-1.$$

Recall from (3.12) that

$$\kappa_{X,j} = \kappa_X + j\kappa_V + 2\left(\sum_{1\le i\le j} \gamma_{X,i} + \sum_{1\le r<s\le j} \gamma_{r,s}\right)$$

and so

$$\kappa_{X,i} - \kappa_{X,i-1} = \kappa_V + 2\gamma_{X,i} + 2\sum_{1\le \ell<i} \gamma_{\ell,i} \tag{4.23}$$

as an operator on $X \otimes V^{\otimes k}$. Therefore

$$\begin{aligned}
(\kappa_{X,i+1} - &\kappa_{X,i}) - s_i(\kappa_{X,i} - \kappa_{X,i-1})s_i \\
&= \kappa_V + 2\gamma_{X,i+1} + 2\sum_{1\le\ell<i+1} \gamma_{\ell,i+1} - s_i\left(\kappa_V + 2\gamma_{X,i} + 2\sum_{1\le\ell<i}\gamma_{\ell,i}\right)s_i \\
&= \kappa_V + 2\gamma_{X,i+1} + 2\sum_{1\le\ell<i+1} \gamma_{\ell,i+1} - \left(\kappa_V + 2\gamma_{X,i+1} + 2\sum_{1\le\ell<i}\gamma_{\ell,i+1}\right) \\
&= 2\gamma_{i,i+1}.
\end{aligned}$$

This means that to show (4.1), it only remains to be checked that

$$\begin{aligned}
s_i = \Phi'(t_{s_i}) &= \Phi'(x_{i+1} - t_{s_i} x_i t_{s_i}) \\
&= \tfrac{1}{2}\left((\kappa_{M,i+1} - \kappa_{M,i}) + 2c^x_{i+1} - s_i(\kappa_{M,i} - \kappa_{M,i-1} + 2c^x_i)s_i\right) \\
&= \gamma_{i,i+1} + c^x_{i+1} - c^x_i \\
&= \begin{cases} \gamma_{i,i+1} & \text{when } \mathfrak{g} = \mathfrak{gl}_n, \\ \gamma_{i,i+1} + \frac{1}{n} & \text{when } \mathfrak{g} = \mathfrak{sl}_n, \end{cases}
\end{aligned}$$

as operators on $M \otimes N \otimes V^{\otimes k}$ (the check for $\Phi'(t_{s_i}) = \Phi'(y_{i+1} - t_{s_i} y_i t_{s_i})$ is the same).

The decomposition of $V \otimes V$ is

$$V \otimes V = L\left(\square\!\square\right) \oplus L\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right),$$

where if $v_1, \dots, v_n$ is a basis for V, then

$$L\left(\square\!\square\right) = \operatorname{span}_{\mathbb{C}}\{v_i \otimes v_j + v_j \otimes v_i \mid 1 \le i, j \le n\}, \quad \text{and}$$

$$L\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) = \operatorname{span}_{\mathbb{C}}\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \le i, j \le n\}.$$

It follows from this decomposition and Lemma 2.12 that the actions of s_1 and γ are given by

$\mathfrak{g} = \mathfrak{gl}_n$

	$L\left(\square\!\square\right)$	$L\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$
s_1	1	-1
γ	1	-1

$\mathfrak{g} = \mathfrak{sl}_n$

	$L\left(\square\!\square\right)$	$L\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$
s_1	1	-1
γ	$1 - \frac{1}{n}$	$-1 - \frac{1}{n}$

so (4.1) is satisfied.

Next we check $(x_1 - a)(x_1 + p) = 0$. By (4.23), we have

$$\Phi'(x_1) = \tfrac{1}{2}\kappa_V + \gamma_{M,1} + c_1^x.$$

The module $M \otimes V$ decomposes as

$$M \otimes V = L\left(\overset{a}{\boxed{p\quad}}\!\square\right) \oplus L\left(\begin{smallmatrix}\overset{a}{\boxed{p\quad}}\\ \square\end{smallmatrix}\right). \tag{4.24}$$

Case 1: $\mathfrak{g} = \mathfrak{gl}_n$
By Theorem 2.11,

$$\kappa_V = \langle \omega_1, \omega_1 + 2\delta \rangle - (n-1)|\omega_1| = 1 + (n-1) - (n-1) = n, \tag{4.25}$$

so $\frac{1}{2}\kappa_V + c_1^x = 0$. By Lemma 2.12 and the decomposition in (4.24), $\gamma_{M,1} = a$ or $-p$, so $\Phi'(x_1 - a)(x_1 + p) = 0$ as desired.

Case 2: $\mathfrak{g} = \mathfrak{sl}_n$
By Theorem 2.11,

$$\kappa_V = \langle \omega_1, \omega_1 + 2\rho \rangle = n - \frac{1}{n}, \tag{4.26}$$

so $\frac{1}{2}\kappa_V + c_1^x = \frac{ap}{n}$. By Lemma 2.12 and the decomposition in (4.24), $\gamma_{M,1} = \left(a - \frac{ap}{n}\right)$ or $\left(-p - \frac{ap}{n}\right)$ so $\Phi'(x_1 - a)(x_1 + p) = 0$ as desired.

The relation $(y_1 - b)(y_1 + q) = 0$ follows analogously, and therefore (4.2) is satisfied.

$\square$

4.2 Bratteli diagram and seminormal bases

Recall from Example 2.7, if (a^p) and (b^q) are rectangular partitions then $\mathcal{P}((a^p),(b^q))$ is the set of partitions μ for which $L(\mu)$ appears as a submodule of $L((a^p)) \otimes L((b^q))$. Define $\mathcal{P}_1((a^p),(b^q))$ to be the set of partitions which are obtained by adding a box to an element of $\mathcal{P}((a^p),(b^q))$, and define $\mathcal{P}_i((a^p),(b^q))$ to be the set of partitions which are obtained by adding a box to an element of $\mathcal{P}_{i-1}((a^p),(b^q))$.

Definition 4.7. *Define the* Bratteli diagram *for $M \otimes N \otimes V^{\otimes k}$ as a ranked graph, with ranks $-1, 0, 1, \ldots, k$, constructed as follows:*

Vertices: *The vertices are labeled by partitions.*

- *level* -1*:* *On level* -1*, place one vertex, labeled by (a^p).*
- *level* 0*:* *On level 0, place one vertex for each partition in $\mathcal{P}((a^p),(b^q))$.*
- *level i:* *On level i, $i = 1, \ldots k$, place one vertex for each partition in $\mathcal{P}_i((a^p),(b^q))$.*

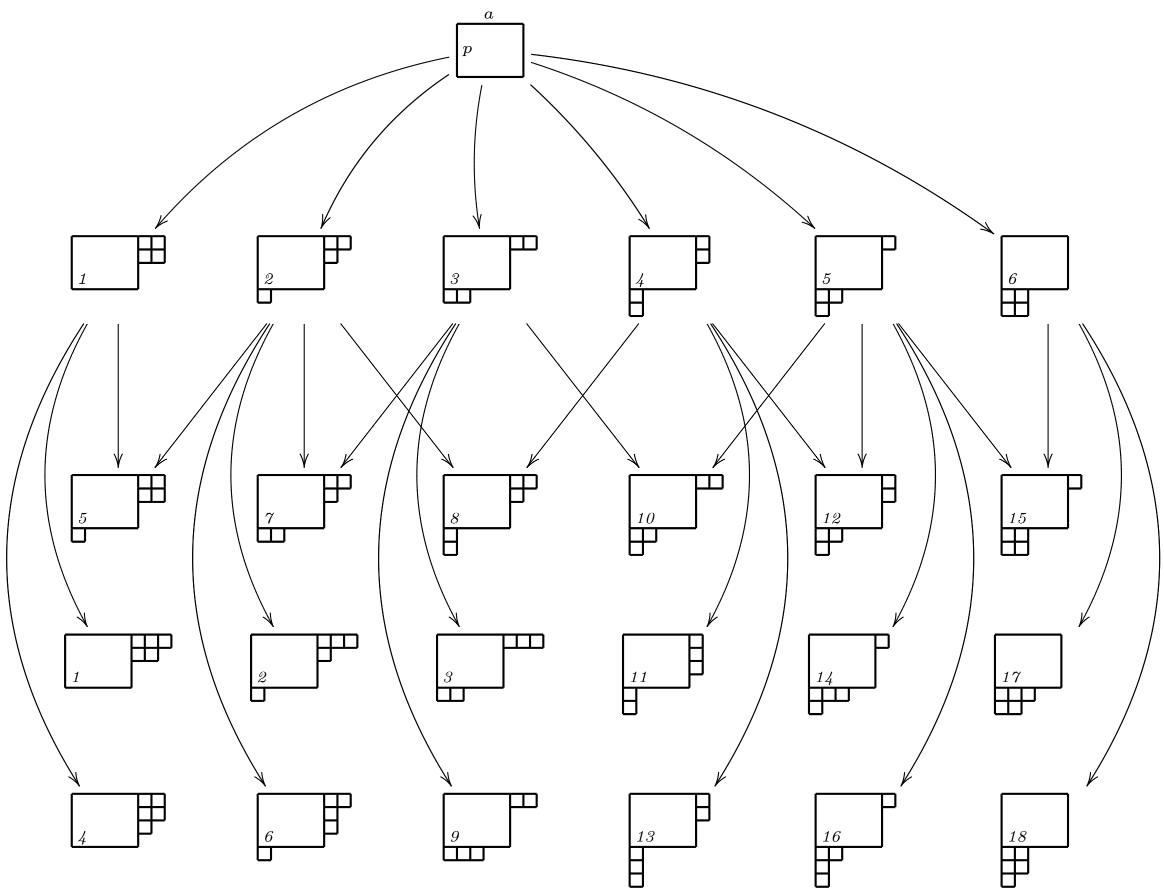
Edges: *The edges will connect two vertices only if the vertices are in adjacent levels.*

Connect the vertex on level -1 to each of the vertices on level 0 with one edge.

Connect each vertex on level i to a vertex on level $i-1$ if the vertex on level i can be obtained by adding a box to the corresponding vertex on level $i-1$.

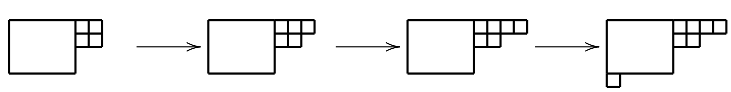
For example, let $k = 1$, $a, p > 2$, and $b = q = 2$. The corresponding Bratteli diagram in given in Figure 4.8.

Figure 4.8 (Bratteli diagram).

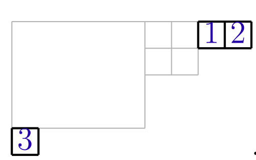


The partitions on levels 0 and 1 are numbered in lexicographical order.

Consider sequences of partitions $T = (\lambda = T^{(0)}, T^{(1)}, \ldots, T^{(k)} = \mu)$ where $T^{(i)}$ is obtained from $T^{(i-1)}$ by adding a box. Identify each such T with the the μ/λ-tableau, as defined in Definition 2.3, built by placing the integer i in the box added at the i^{th} step. For example,



is identified with the filling



.

We call a tableau *standard* if each label appears exactly once. Therefore, we have a bijection

$$\left\{\begin{array}{c}\text{sequences of partitions}\\ T=(\lambda=T^{(0)},T^{(1)},\dots T^{(k)}=\mu)\\ \text{where } T^{(i)}/T^{(i-1)} \text{ is a box}\end{array}\right\} \leftrightarrow \left\{\text{ standard } \mu/\lambda\text{-tableaux}\right\}. \qquad (4.27)$$

Throughout the remainder of this exposition, unless otherwise stated, we assume tableaux to be standard and often think of them as sequences $T = (\lambda = T^{(0)}, T^{(1)}, \dots T^{(k)} = \mu)$ of partitions where $T^{(i)}$ is obtained from $T^{(i-1)}$ by adding a box.

Identify downward-moving paths in the Bratteli diagram from level 0 to level i with μ/λ-tableaux where $\mu \in \mathcal{P}_i((a^p),(b^q))$, $\lambda \in \mathcal{P}((a^p),(b^q))$, and $T^{(j)}$ is the node at level j on the path. We already know from Section 2.3 that the irreducible $\mathfrak{g}$-modules appearing in $M \otimes N \otimes V^{\otimes k}$ are indexed by nodes on level k of the Bratteli diagram. Futhermore, we know from Theorem 2.16 that his implies that the irreducible $\text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$-modules appearing are also indexed by nodes on level k of the Bratteli diagram. For each $\mu \in \mathcal{P}_k((a^p),(b^q))$, let $\mathcal{L}^\mu$ be the irreducible $\text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$-module corresponding to μ. Our next goal is to show that the basis of $\mathcal{L}^\mu$ is indexed by μ/λ-tableaux, and that one such basis consists of highest weight vectors of weight μ.

Let $\mu \in \mathfrak{h}^*$, and recall from (2.1) that the μ-weight space of a finite-dimensional $\mathfrak{g}$-module U is

$$U_\mu = \{u \in U \mid hu = \mu(h)u \text{ for all } h \in \mathfrak{h}\}.$$

If $u \in U_\mu$ and $\phi \in \text{End}_{\mathfrak{g}}(U)$, then

$$h(\phi u) = \phi h u = \phi \mu(h) u = \mu(h)(\phi u),$$

so $\phi u \in U_\mu$, and thus U_μ is a $\text{End}_{\mathfrak{g}}(U)$-module. By Theorem 2.16, U decomposes as

$$U = \bigoplus_{\mu} L(\mu) \otimes \mathcal{L}^\mu \qquad \text{as a } (\mathfrak{g}\text{-}\text{End}_{\mathfrak{g}}(U))\text{-bimodule},$$

where $L(\mu)$ is the irreducible highest weight $\mathfrak{g}$-module of weight μ and $\mathcal{L}^\mu$ is the corresponding irreducible $\text{End}_{\mathfrak{g}}(U)$-module. Since $(L(\mu) \otimes \mathcal{L}^\mu)_\mu$ is the span of all highest weight vectors of weight μ in U, it is a $\text{End}_{\mathfrak{g}}(U)$-module. It also has dimension the multiplicity of $L(\mu)$, i.e. the dimension of $\mathcal{L}^\mu$. Therefore,

$$(L(\mu) \otimes \mathcal{L}^\mu)_\mu \cong \mathcal{L}^\mu.$$

Recall from Lemma 2.17 that there is a natural chain of inclusions

$$\text{End}_{\mathfrak{g}}(M) \hookrightarrow \text{End}_{\mathfrak{g}}(M \otimes N) \hookrightarrow \text{End}_{\mathfrak{g}}(M \otimes N \otimes V) \hookrightarrow \dots \hookrightarrow \text{End}(M \otimes N \otimes V^{\otimes k}).$$

As a $\mathfrak{g}$-module,

$$M \otimes N = \bigoplus_{\lambda \in \mathcal{P}((a^p),(b^q))} L(\lambda).$$

Therefore for each $\lambda \in \mathcal{P}((a^p),(b^q))$, $\mathcal{L}^\lambda$ is one-dimensional and isomorphic to the $\mathrm{End}_{\mathfrak{g}}(M \otimes N)$-module generated by the (unique up to scaling) highest weight vector v_λ of $L(\lambda)$. Next,

$$\begin{aligned} M \otimes N \otimes V &= \left(\bigoplus_{\lambda \in \mathcal{P}((a^p),(b^q))} L(\lambda) \right) \otimes V \\ &= \bigoplus_{\lambda \in \mathcal{P}((a^p),(b^q))} (L(\lambda) \otimes V) \\ &= \bigoplus_{\lambda \in \mathcal{P}((a^p),(b^q))} \left(\bigoplus_{\mu \in \lambda^+} L(\mu) \right). \end{aligned}$$

So, for each $\lambda \in \mathcal{P}((a^p),(b^q))$, there is exactly one copy of $L(\mu)$ in $M \otimes N \otimes V$ for every μ/λ-tableau (a path of length one). Since $\mathcal{L}^\mu$ is isomorphic to the μ-weight space of $L(\mu) \otimes \mathcal{L}^\mu$, $\mathcal{L}^\mu$ then has a basis indexed by μ/λ-tableaux (with $\lambda \in \mathcal{P}((a^p),(b^q))$ and $\mu \in \mathcal{P}_i((a^p),(b^q)))$. Specifically, $v_{(\lambda,\mu)}$ is the (unique up to scaling) highest weight vector of the copy of $L(\mu)$ coming from $L(\lambda) \otimes V$.

Inductively, if $L(\mu) \otimes \mathcal{L}^\mu$ is a nontrivial isotypic component of $M \otimes N \otimes V^{\otimes k}$, then $\mathcal{L}^\mu$ has basis

$$v_T^{(z)} \text{ indexed by } \mu/\lambda\text{-tableaux } T = (\lambda = T^{(0)}, T^{(1)}, \dots T^{(k)} = \mu), \tag{4.28}$$

with $\lambda \in \mathcal{P}((a^p),(b^q))$ (the notation will be made more transparent when we show how $v_T^{(z)}$ corresponds to the set of generators $z_1, \dots, z_k$ in $\mathcal{H}_k^{\text{ext}}$). Specifically, $v^{(z)}$ is the (unique up to scaling) vector for which

$$v_T^{(z)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \subseteq M \otimes N \otimes V^{\otimes k}, \quad i = 0, \dots, k,$$

where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq M \otimes N \otimes V^{\otimes i}$. This basis is exactly indexed by the paths in the Bratteli diagram from level 0 (and therefore from level -1) to μ.

Another basis can be constructed by considering the chain

$$\mathrm{End}_{\mathfrak{g}}(M) \hookrightarrow \mathrm{End}_{\mathfrak{g}}(M \otimes V) \hookrightarrow \dots \hookrightarrow \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes k}) \hookrightarrow \mathrm{End}(M \otimes N \otimes V^{\otimes k}),$$

where the last containment is given by the maps

$$\mathrm{End}(M \otimes V^{\otimes k}) \xrightarrow{\phi \mapsto \phi \otimes \mathrm{id}_N} \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes k} \otimes N) \xrightarrow{\sim} \mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}).$$

By similar induction, this chain admits a basis of each $\mathcal{L}^\mu$

$$v_T^{(x)} \text{ labeled by sequences } T = ((a^p) = T^{(0)}, T^{(1)}, \dots, T^{(k)}, \mu), \tag{4.29}$$

where $((a^p) = T^{(0)}, T^{(1)}, \ldots, T^{(k)})$ is a tableau, and $\mathcal{L}^{T^{(k)}}$ is a nontrivial component of

$$\mathrm{Res}^{\mathrm{End}_{\mathfrak{g}}(M\otimes N\otimes V^{\otimes k})}_{\mathrm{End}_{\mathfrak{g}}(M\otimes V^{\otimes k})}(\mathcal{L}^{\mu}).$$

Specifically, $v_T^{(x)}$ is a vector for which

$$v_T^{(x)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \otimes N \subseteq M \otimes V^{\otimes k} \otimes N \cong M \otimes N \otimes V^{\otimes k}, \quad i = 0, \ldots, k,$$

where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq M \otimes V^{\otimes i}$.

Similarly, the chain

$$\mathrm{End}_{\mathfrak{g}}(N) \hookrightarrow \mathrm{End}_{\mathfrak{g}}(N \otimes V) \hookrightarrow \ldots \hookrightarrow \mathrm{End}_{\mathfrak{g}}(N \otimes V^{\otimes k}) \hookrightarrow \mathrm{End}(M \otimes N \otimes V^{\otimes k}),$$

admits a basis

$$v_T^{(y)} \text{ labeled by sequences } T = ((b^q) = T^{(0)}, T^{(1)}, \ldots, T^{(k)}, \mu), \tag{4.30}$$

where $((b^q) = T^{(0)}, T^{(1)}, \ldots, T^{(k)})$ is a tableau, and $\mathcal{L}^{T^{(k)}}$ is a nontrivial component of

$$\mathrm{Res}^{\mathrm{End}_{\mathfrak{g}}(M\otimes N\otimes V^{\otimes k})}_{\mathrm{End}_{\mathfrak{g}}(N\otimes V^{\otimes k})}(\mathcal{L}^{\mu}).$$

Specifically, $v_T^{(y)}$ is a vector for which

$$v_T^{(y)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \otimes M \subseteq N \otimes V^{\otimes k} \otimes M \cong M \otimes N \otimes V^{\otimes k}, \quad i = 0, \ldots, k,$$

where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq N \otimes V^{\otimes i}$.

Theorem 4.9. *If Φ' is the representation in Theorem 4.6, then for each $\mathcal{L}^{\mu} \subseteq M \otimes N \otimes V^{\otimes k}$,*

1. *if $v_T^{(x)}$ is as in (4.29), $\Phi'(x_i)$ acts by*

$$\Phi'(x_i)v_T^{(x)} = c(T^{(i)}/T^{(i-1)})v_T^{(x)}, \qquad i = 1, \ldots k,$$

2. *if $v_T^{(y)}$ is as in (4.30), $\Phi'(y_i)$ acts by*

$$\Phi'(y_i)v_T^{(y)} = c(T^{(i)}/T^{(i-1)})v_T^{(y)}, \qquad i = 1, \ldots k,$$

and

3. *if $v_T^{(z)}$ is as in (4.28), $\Phi'(z_i)$ acts by*

$$\Phi'(z_i)v_T^{(z)} = c(T^{(i)}/T^{(i-1)})v_T^{(x)}, \qquad i = 1, \ldots k,$$

where if $T^{(i)}$ and $T^{(i-1)}$ differ by a box in column j and row ℓ, then

$$c(T^{(i)}/T^{(i-1)}) = j - \ell$$

is the content of the box added.

Proof. We can rewrite

$$\Phi(x_i) = \tfrac{1}{2}(\kappa_{M,i} - \kappa_{M,i-1}) = \gamma_{M\otimes V^{\otimes i-1},V} + \tfrac{1}{2}\kappa_V, \tag{4.31}$$
$$\Phi(y_i) = \tfrac{1}{2}(\kappa_{N,i} - \kappa_{N,i-1}) = \gamma_{N\otimes V^{\otimes i-1},V} + \tfrac{1}{2}\kappa_V, \tag{4.32}$$
$$\Phi(z_i) = \tfrac{1}{2}(\kappa_{M\otimes N,i} - \kappa_{M\otimes N,i-1} + \kappa_V) = \gamma_{M\otimes N\otimes V^{\otimes i-1},V} + \tfrac{1}{2}\kappa_V. \tag{4.33}$$

Case 1: $\mathfrak{g} = \mathfrak{gl}_n$. By Theorem 2.11,

$$\kappa_V = \langle \omega_1, \omega_1 + 2\delta\rangle - (n-1)|\omega_1| = 1 + (n-1) - (n-1) = n. \tag{4.34}$$

1. If $v_T^{(x)}$ is as in (4.29), then

$$v_T^{(x)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \otimes N \subseteq M \otimes V^{\otimes k} \otimes N, \quad \text{i=0, \dots, k},$$

   where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq M \otimes V^{\otimes i}$. Therefore

$$\begin{aligned} \Phi'(x_i)v_T^{(x)} &= \left(\tfrac{1}{2}(\kappa_{M,i} - \kappa_{M,i-1}) - \tfrac{1}{2}n\right)\cdot v_T^{(x)} \\ &= \left(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V - \tfrac{1}{2}n\right)\cdot v_T^{(x)}, && \text{by (4.31),} \\ &= (c(T^{(i)}/T^{(i-1)}) + \tfrac{1}{2}n - \tfrac{1}{2}n)v_T^{(x)}, && \text{by (4.34) and Theorem 2.12,} \\ &= c(T^{(i)}/T^{(i-1)})v_T^{(x)}. \end{aligned}$$

2. If $v_T^{(y)}$ is as in (4.30), then

$$v_T^{(y)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \otimes M \subseteq N \otimes V^{\otimes k} \otimes M, \quad \text{i=0, \dots, k},$$

   where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq N \otimes V^{\otimes i}$. Therefore

$$\begin{aligned} \Phi'(y_i)v_T^{(y)} &= \left(\tfrac{1}{2}(\kappa_{N,i} - \kappa_{N,i-1}) - \tfrac{1}{2}n\right)\cdot v_T^{(y)} \\ &= \left(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V - \tfrac{1}{2}n\right)\cdot v_T^{(y)}, && \text{by (4.32),} \\ &= (c(T^{(i)}/T^{(i-1)}) + \tfrac{1}{2}n - \tfrac{1}{2}n)v_T^{(y)}, && \text{by (4.34) and Theorem 2.12,} \\ &= c(T^{(i)}/T^{(i-1)})v_T^{(y)}. \end{aligned}$$

3. If $v_T^{(z)}$ is as in (4.28), then

$$v_T^{(z)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \subseteq M \otimes N \otimes V^{\otimes k}, \quad \text{i=0, \ldots, k},$$

where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq M \otimes N \otimes V^{\otimes i}$. Therefore

$$\begin{aligned}
\Phi'(z_i) \cdot v_T^{(z)} &= \big(\tfrac{1}{2}(\kappa_{M\otimes N,i} - \kappa_{M\otimes N,i-1} + \kappa_V) - n\big) \cdot v_T^{(z)} \\
&= \big(\gamma_{L(T^{(i-1)}),V} + \kappa_V - n\big)\, v_T^{(z)}, && \text{by (4.33),} \\
&= \big(c(T^{(i)}/T^{(i-1)}) + n - n\big)\, v_T^{(z)}, && \text{by (4.34) and Theorem 2.12,} \\
&= c(T^{(i)}/T^{(i-1)}) v_T^{(z)}.
\end{aligned}$$

Case 1: $\mathfrak{g} = \mathfrak{sl}_n$. By Theorem 2.11,

$$\kappa_V = \langle \omega_1, \omega_1 + 2\rho \rangle = n - \frac{1}{n}. \tag{4.35}$$

1. If $v_T^{(x)}$ is as in (4.29), then

$$v_T^{(x)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \otimes N \subseteq M \otimes V^{\otimes k} \otimes N, \quad \text{i=0, \ldots, k},$$

where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq M \otimes V^{\otimes i}$. Since

$$|T^{(i-1)}| = ap + i - 1,$$

we have

$$(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V) v_T^{(x)} = \left(c(T^{(i)}/T^{(i-1)}) - \frac{ap + i - 1}{n} + \frac{1}{2}\left(n - \frac{1}{n}\right)\right) v_T^{(x)}$$

by (4.35) and Theorem 2.12. Therefore

$$\begin{aligned}
\Phi'(x_i) v_T^{(x)} &= \left(\tfrac{1}{2}(\kappa_{M,i} - \kappa_{M,i-1}) + \frac{ap+i-1}{n} - \frac{1}{2}\left(n + \frac{1}{n}\right)\right) \cdot v_T^{(x)} \\
&= \left(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V + \frac{ap+i-1}{n} - \frac{1}{2}\left(n + \frac{1}{n}\right)\right) \cdot v_T^{(x)}, \quad \text{by (4.31),} \\
&= c(T^{(i)}/T^{(i-1)}) v_T^{(x)}.
\end{aligned}$$

2. If $v_T^{(y)}$ is as in (4.30), then

$$v_T^{(y)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \otimes M \subseteq N \otimes V^{\otimes k} \otimes M, \quad \text{i=0, \ldots, k},$$

where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq N \otimes V^{\otimes i}$. Since

$$|T^{(i-1)}| = bq + i - 1$$

we have

$$(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V)v_T^{(y)} = \left(c(T^{(i)}/T^{(i-1)}) - \frac{bq+i-1}{n} + \frac{1}{2}\left(n + \frac{1}{n}\right)\right) v_T^{(y)}$$

by (4.35) and Theorem 2.12. Therefore

$$\begin{aligned}
\Phi'(y_i)v_T^{(y)} &= \left(\tfrac{1}{2}(\kappa_{N,i} - \kappa_{N,i-1}) + \frac{bq+i-1}{n} - \frac{1}{2}\left(n + \frac{1}{n}\right)\right) \cdot v_T^{(y)} \\
&= \left(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V + \frac{bq+i-1}{n} - \frac{1}{2}\left(n + \frac{1}{n}\right)\right) \cdot v_T^{(y)}, \quad \text{by (4.32)}, \\
&= c(T^{(i)}/T^{(i-1)})v_T^{(y)}.
\end{aligned}$$

3. If $v_T^{(z)}$ is as in (4.28), then

$$v_T^{(z)} \in v_{T^{(i)}} \otimes V^{\otimes(k-i)} \subseteq M \otimes N \otimes V^{\otimes k}, \quad \text{i=0, \dots, k},$$

where $v_{T^{(i)}}$ is a highest weight vector of weight $T^{(i)}$ in $L(T^{(i)}) \subseteq M \otimes N \otimes V^{\otimes i}$. Since

$$|T^{(i-1)}| = ap + bq + i - 1$$

we have

$$(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V)v_T^{(z)} = \left(c(T^{(i)}/T^{(i-1)}) - \frac{ap+bq+i-1}{n} + n - \frac{1}{n}\right) v_T^{(z)}$$

by (4.35) and Theorem 2.12. Therefore

$$\begin{aligned}
\Phi'(z_i)v_T^{(z)} &= \left(\tfrac{1}{2}(\kappa_{M\otimes N,i} - \kappa_{M\otimes N,i-1} + \kappa_V) + \frac{ap+bq+i}{n} - n\right) \cdot v_T^{(z)} \\
&= \left(\gamma_{L(T^{(i-1)}),V} + \tfrac{1}{2}\kappa_V + \frac{ap+bq+i}{n} - n\right) \cdot v_T^{(z)}, \quad \text{by (4.33)}, \\
&= c(T^{(i)}/T^{(i-1)})v_T^{(z)}.
\end{aligned}$$

□

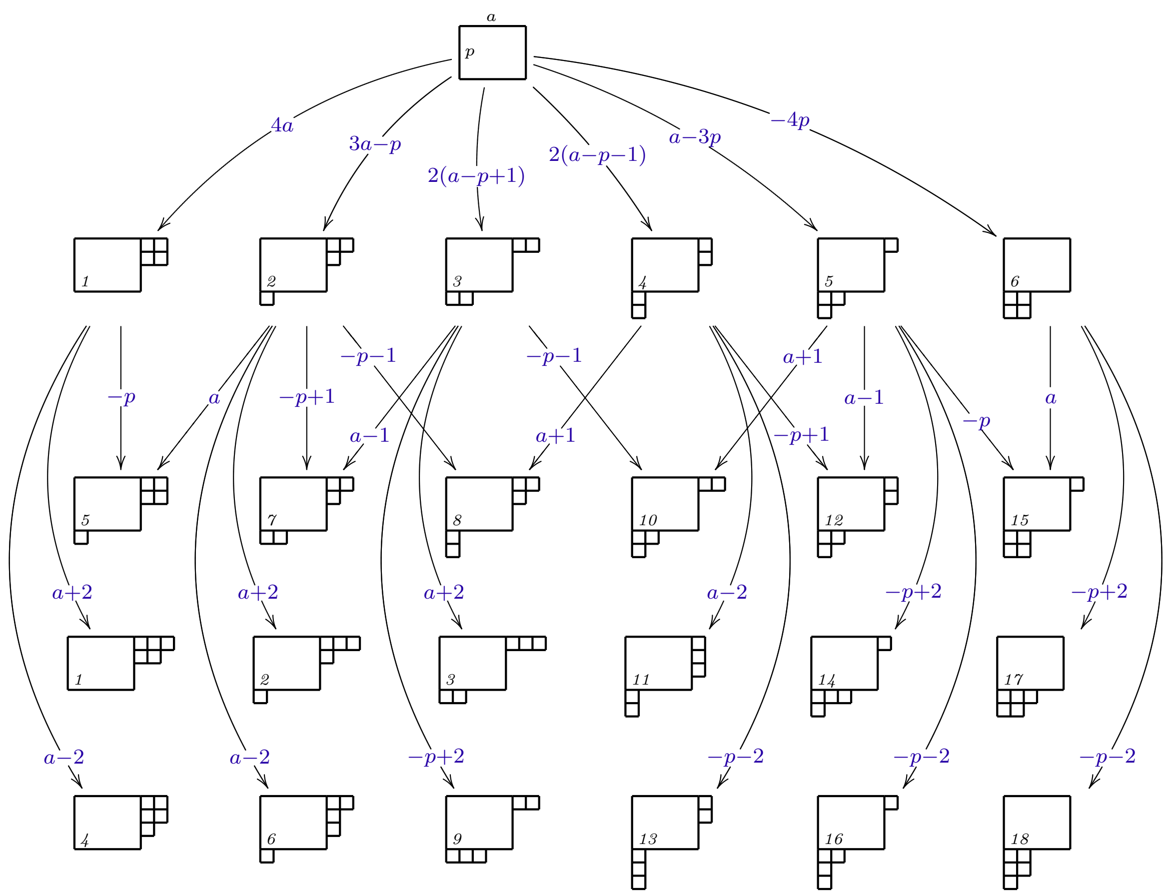
Example 4.10. *To illustrate, we apply Theorem 4.9 to the example where $\mathfrak{g} = \mathfrak{gl}_n$, $a, p > 2$, $b = q = 2$, and $k = 1$. Figure 4.11 is the Bratteli diagram in 4.8 with edges labeled by functions of box contents. The edges connecting level -1 to level 0 are labeled* *by the action of $\gamma_{M,N}$ as given in Lemma 2.15. The edges connecting level 0 to level 1 are labeled by the content of the box added. In general, we would label the edges connecting level i to level $i+1$, $i = 1, 2, \ldots, k-1$ by the content of the box added.*

By Theorem 4.9, if Φ' satisfies

$$c_i^x = c_i^y = \tfrac{1}{2}c_i^z = -\tfrac{1}{2}n,$$

then the descending paths in this diagram from (a^p) to $\lambda \in \mathcal{P}_1((a^p),(b^q))$ index the basis of $\mathcal{L}^\lambda$ in (4.28), and $\Phi'(z_1)$ and $\Phi'(z_0)$ act on those basis elements by the corresponding edge labels. Therefore, there are eighteen distinct isotypic components of $M \otimes N \otimes V$, six of which correspond to 2-dimensional $\mathcal{H}_1^{\text{ext}}$-modules and twelve of which correspond to 1-dimensional $\mathcal{H}_1^{\text{ext}}$-modules.

Figure 4.11 (Isotypic components of $M \otimes N \otimes V$)**.**



Theorem 4.15 will provide explicit formulas for x_1, but we can already ascertain the eigenvalues of x_1 and y_1. Recall the relations $(x_1 - a)(x_1 + p) = 0 = (y_1 - b)(y_1 - q)$ and $x_1 + y_1 = z_1$. So on any given one- or two-dimensional module, x_1 has eigenvalues from the set $\{a, -p\}$ and y_1 has the eigenvalues from the set $\{2, -2\}$. Furthermore, the sum

of the values z_1 is equal to the sum of the eigenvalues of x_1 and y_1. On two-dimensional irreducible modules, x_1 and y_1 must not act diagonally on this choice of basis, so must have two distinct eigenvalues (there is one way to achieve this); in fact, we can see that the sum of the values on each two-dimensional $\mathcal{H}_k^{\text{ext}}$-module is $a-p$, as expected. On the one-dimensional components, we can see, case-by-case that there is one way to chose one value from $\{a,-p\}$ and one value from $\{2,-2\}$ which sum to the value of z_1, determining the action of x_1 and y_1.

Notice that in Figure 4.8, each of the partitions in $\mathcal{P}_1((a^p),(b^q))$ comes from exactly one or two partitions in $\mathcal{P}((a^p),(b^q))$. This will happen in general.

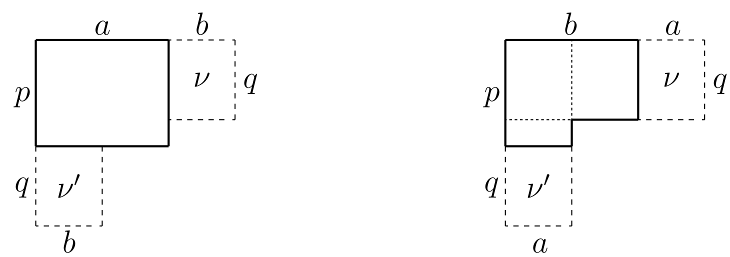
Lemma 4.12. *If $\mu \in \mathcal{P}_1((a^p),(b^q))$, then there are exactly one or two $\lambda \in \mathcal{P}((a^p),(b^q))$ for which $\lambda \subseteq \mu$.*

Proof. Recall that $\mathcal{P}((a^p),(b^q))$ is the set of partitions λ with height $\leq p+q$ such that

$$\begin{gathered} \lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_p = a, \\ \lambda_q \geq \max(a,b), \\ \lambda_i + \lambda_{p+q-i+1} = a+b, \quad i = 1,\ldots,q. \end{gathered} \tag{4.36}$$

Again, a useful visualization of these partitions is as follows.

$a > b$: $a < b$:



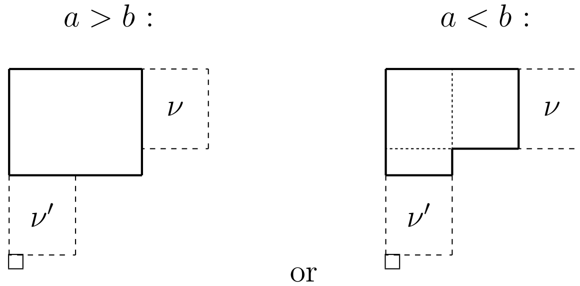
ν is a partition in a $b \times q$ box
ν' is the 180° rotation of $(b^q)/\nu$

ν is a partition in a $a \times q$ box
ν' is the 180° rotation of $(a^q)/\nu$

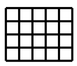
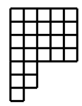
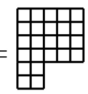
We have discussed that this means if you remove a box from $\lambda \in \mathcal{P}((a^p),(b^q))$ in position (i,j), then a box must be added to position $(a+b+1-i, p+q+1-j)$ to get another partition in $\mathcal{P}((a^p),(b^q))$. So now, consider a partition $\mu \in \mathcal{P}_1((a^p),(b^q))$. To make things just a bit easier, assume, in addition to having $p \geq q$, that if $p = q$, we choose $a \geq b$. By moving through the criteria in (4.36) and considering addable boxes for a partition which meets these criteria, we can see that this partition falls into one of the following categories.

1. μ has height $p+q+1$: In this case, exactly one box can be removed to a partition

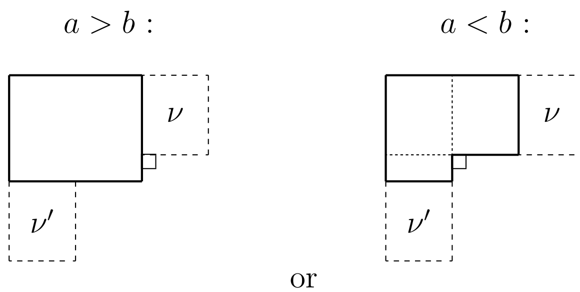
which satisfies (4.36), the box in position $(1, p+q+1)$. This partition μ looks like



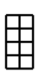
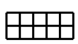
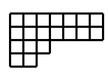
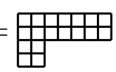
For example

if $(a^p) =$ , $(b^q) =$ , and $\mu =$ , then μ came from $\lambda =$ .

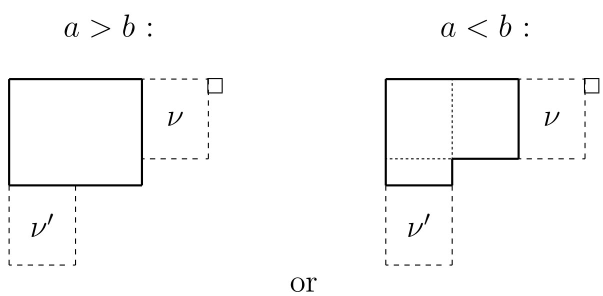
2. $\mu_{q+1} = a+1$: In this case, there is exactly one box which can be removed to obtain a partition which satisfies (4.36), the box in position $(a+1, q+1)$. This partition μ looks like



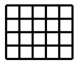
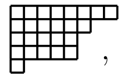
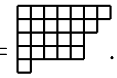
For example,

if $(a^p) =$ , $(b^q) =$ , and $\mu =$ , then μ came from $\lambda =$ .

3. $\mu_1 = a+b+1$: Again, there is exactly one box which can be removed to obtain a partition which satisfies (4.36), the box in position $(a+b+1, 1)$. This partition μ looks like

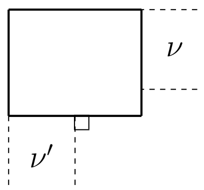


For example,

if $(a^p) =$  , $(b^q) =$  , and $\mu =$  , then μ came from $\lambda =$  .

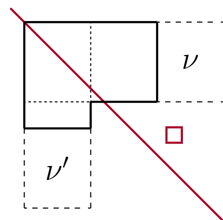
4. $\mu_{p+1} = b+1$: This is similar to the case above, but is a little more complex. We can only see $\mu_{p+1} = b+1$ when $a > b$ and $\mu_q = a$. So the only removable box is the one in position $(b+1, p+1)$. This partition μ looks like

$a > b$:

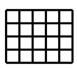
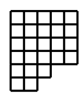
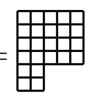


or

$a < b$:

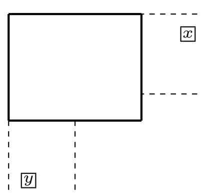


For example,

if $(a^p) =$  , $(b^q) =$  , and $\mu =$  , then μ came from $\lambda =$  .

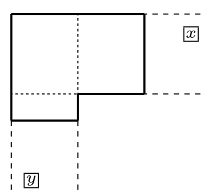
5. $\mu_j + \mu_{p+q-j+1} = a+b+1$ for some $1 \leq j \leq p$, but $\mu_j < a+b+1$ and $\mu_{p+q-j+1} < \min(a,b)+1$: This is the case which will yield two partitions. One is the partition in which we remove the box in position (μ_j, j); the other is the partition in which we remove the box in position $(a+b+1-\mu_j, p+q+1-j)$. This partition μ looks like

$a > b$:

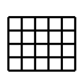
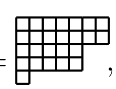


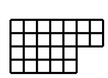
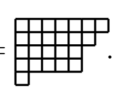
or

$a < b$:



where the boxes marked x and y are corner boxes, one of x or y has position (i, j), and the other has position $(a+b+1-\mu_j, p+q+1-j)$. For example,

if $(a^p) =$  , $(b^q) =$  , and $\mu =$  ,

then μ came from $\lambda =$  or $\lambda =$  .

□

The following lemma specifies the contents of the boxes being removed in cases 1-5 in the previous lemma.

Lemma 4.13. *Suppose* $\mu \in \mathcal{P}_1((a^p),(b^q))$ *and* $\lambda \in \mathcal{P}((a^p),(b^q))$ *differ by a box. Then*

1. *there is exactly one such* λ *if and only if* $c(\mu/\lambda) = -p-q,\ a-q,\ a+b,\ \text{ or }\ b-p$, *and*
2. *if* $c(\mu/\lambda) \neq -p-q,\ a-q,\ a+b,\ \text{ or }\ b-p$, *then there is exactly one* $\lambda' \in \mathcal{P}((a^p),(b^q))$ *distinct from* λ *which differs from* μ *by a box, and*

$$c(\mu/\lambda') = a - p + b - q - c(\mu/\lambda).$$

Proof. If $\mu \in \mathcal{P}_1((a^p),(b^q))$ satisfies cases 1-4 in Lemma 4.12, and B is the unique removable box, then

$$c(B) = i - j = -p - q,\ a - q,\ a + b,\ \text{ or }\ b - p.$$

The final case yielded two partitions which differ by the movement of one box. As we saw in Remark 2.14, if a box in position (i,j) in $\lambda \in \mathcal{P}((a^p),(b^q))$ can be moved to get another partition in $\mathcal{P}((a^p),(b^q))$, then that box must satisfy either

(1) $\max(a,b) < i \leq a+b$ and $0 < j \leq q$, or

(2) $0 < i \leq \min(a,b)$ and $p < j \leq p+q$.

If (i,j) satisfies (1), then

$$\max(a,b) - q < i - j < a + b - q.$$

So since $p \geq q$,

$$-p-q < i-j, \quad a-q < i-j, \quad b-p < i-j, \quad \text{ and } i-j < a+b.$$

If (i,j) satisfies (2), then

$$-p-q < i-j < \min(a,b) - p.$$

So, similarly,

$$-p-q < i-j, \quad i-j < a-q, \quad i-j < b-p \quad \text{ and } i-j < a+b.$$

Thus, if there are two partitions in $\mathcal{P}((a^p),(b^q))$ which can be obtained by removing a box from μ, then the contents of those boxes are distinct from $-p-q,\ a-q,\ a+b,\ \text{ and }\ b-p$. □

4.3 Seminormal Representations

Recall, a, b, p, q are non-negative integers with $q \leq p$. Throughout this section, we consider the sets of tableaux

$$\mathcal{T}_\lambda = \left\{ T = (T^{(0)}, \ldots, T^{(k)} = \lambda) \mid T^{(0)} \in \mathcal{P}((a^p), (b^q)),\ T^{(i)} \in \mathcal{P}_i((a^p), (b^q)) \right\}. \tag{4.37}$$

The box added to $T^{(i)}$ to get $T^{(i-1)}$ is $b_i = T^{(i)}/T^{(i-1)}$. Define

$$c_T(0) = abq + 2 \sum_{B \in \mathcal{B}_\mu} \left(c(B) - \tfrac{1}{2}(a - p + b - q) \right),$$
$$c_T(i) = c(T^{(i)}/T^{(i-1)}) - \tfrac{1}{2}(a - p + b - q),$$

where $\mathcal{B}_\mu$ if the set of boxes in μ in rows $p + 1$ and below as described in Lemma 2.15. We can think of the values $c_T(1), \ldots, c_T(k)$ as *shifted contents*.

Lemma 4.14. *Given the information*

$$c_T(1), \ldots, c_T(k), \quad \textit{and} \quad T^{(k)}$$

the tableau T is determined.

Proof. This can be shown by induction on k. The key observation is that the value $c_T(i)$, $i > 0$, determines the diagonal on which $T^{(i)}/T^{(i-1)}$ lies. In any given partition, there is at most one removable box on any diagonal. So $c_T(k)$ and $T^{(k)}$ determines $T^{(k-1)}$. By iterating, $c_T(i)$ and $T^{(i)}$ determines $T^{(i-1)}$, so we can recover $T^{(k-1)}$, $T^{(k-2)}$, ..., $T^{(0)}$. □

Two consecutive boxes b_i and b_{i+1} are in the same row or column if and only if $c(b_i) = c(b_{i+1}) \pm 1$. So for any i for which $c_T(i) \neq c_T(i+1) \pm 1$, we can define

$$s_i T = (T^{(0)}, T^{(1)}, \ldots, T^{(i+1)}, T^{(i)}, \ldots, T^{(k)}) \tag{4.38}$$

as the tableau constructed from T by switching the order of adding the i^{th} and $(i+1)^{\text{th}}$ boxes. Notice that if $c_T(i) \neq c_T(i+1) \pm 1$, then $s_i T$ is the only tableau which varies from T only at the i^{th} position; otherwise, if $c_T(i) = c_T(i+1) \pm 1$, then there is no such tableau.

Similarly, for any $\mu \in \mathcal{P}_1((a^p), (b^q))$, there are exactly one or two partitions $\nu \in \mathcal{P}((a^p), (b^q))$ which differ from μ by a box (see Lemma 4.12). In other words, there are exactly one or two $\nu \in \mathcal{P}((a^p), (b^q))$ which could be the first step in a tableau with a given shifted content list $c_T(2), \ldots, c_T(k)$. Lemma 4.13 tells us that this difference is determined by $c_T(1)$; i.e. there is one when $c_T(1) = \frac{1}{2}(\pm(a + p) \pm (b + q))$, and there are two otherwise (remember to shift the values given in Lemma 4.13). So if $c_T(1) \neq \frac{1}{2}(\pm(a + p) \pm (b + q))$ define

$$s_0 T = (s_0 T^{(0)}, T^{(1)}, \ldots, T^{(k)}), \tag{4.39}$$

where $s_0T^{(0)}$ is the unique partition built by moving $T^{(1)}/T^{(0)}$ to its complementary position (see Remark 2.8). Recall by 4.13,

$$c(T^{(1)}/s_0T^{(0)}) = a - p + b - q - 2c(T^{(1)}/T^{(0)}).$$

So

$$c_{s_0T}(1) = -c_T(1). \tag{4.40}$$

Theorem 4.15. *Fix some $\lambda \in \mathcal{P}_k((a^p),(b^q))$, and define*

$$\mathcal{L}_\lambda^{(a^p),(b^q)} = \operatorname{span}_{\mathbb{C}}\{\ v_T \mid T \in \mathcal{T}_\lambda\ \},$$

as a vector space with basis indexed by all tableaux from any $\mu \in \mathcal{P}((a^p),(b^q))$ to λ ($\mathcal{T}_\lambda$ is defined in (4.37)). Define an action of $\mathcal{H}_k^{\mathrm{ext}}$ by

$$\begin{aligned} w_i \cdot v_T &= c_T(i) v_T, && \textit{for } 0 \le i \le k \\ t_{s_i} \cdot v_T &= [t_i]_{T,T} v_T + [t_i]_{T,s_iT} v_{s_iT}, && \textit{for } 1 \le i \le k-1 \\ x_1 \cdot v_T &= [x_1]_{T,T} v_T + [x_1]_{T,s_0T} v_{s_0T} \end{aligned}$$

where $[t_i]_{T,s_iT} = 0$ if and only if $c_T(i) = c_T(i+1) \pm 1$, and $[x_1]_{T,s_0T} = 0$ if and only if $c_T(1) = \frac{1}{2}(\pm(a+p) \pm (b+q))$. Then $\mathcal{L}_\lambda^{(a^p),(b^q)}$ is a simple $\mathcal{H}_k^{\mathrm{ext}}$-module with respect to this action if

1. $[t_i]_{T,T} = 1/(c_T(i+1) - c_T(i))$
2. $$[x_1]_{T,T} = \frac{(a-p)c_T(1) + c_T^2(1) + \left(\frac{(a+p)+(b+q)}{2}\right)\left(\frac{(a+p)-(b+q)}{2}\right)}{2c_T(1)}$$
3. *Commutation:*
$$[t_i]_{s_0T,s_is_0T}[x_1]_{T,s_0T} = [t_i]_{T,s_iT}[x_1]_{s_iT,s_0s_iT} \qquad \textit{for } i > 1,$$
4. *Involutions:*
$$[t_i]_{T,s_iT}[t_i]_{s_iT,T} = 1 - ([t_i]_{T,T})^2$$
5. *Quadratic relation:*
$$[x_1]_{T,s_0T}[x_1]_{s_0T,T} = \begin{aligned} -\tfrac{1}{(2c_T(1))^2}&\left(c_T(1) + \tfrac{(a+p)+(b+q)}{2}\right)\left(c_T(1) - \tfrac{(a+p)-(b+q)}{2}\right) \\ &\cdot\left(c_T(1) - \tfrac{(a+p)+(b+q)}{2}\right)\left(c_T(1) + \tfrac{(a+p)-(b+q)}{2}\right) \end{aligned}$$
6. *Braid relations:*
$$[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT} = [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT}$$
$$\begin{aligned} &[x_1]_{s_1T,s_0s_1T}[x_1]_{s_1s_0s_1T,s_0s_1s_0s_1T}[t_1]_{T,s_1T}[t_1]_{s_0s_1T,s_1s_0s_1T} \\ &\qquad = [x_1]_{T,s_0T}[x_1]_{s_1s_0T,s_0s_1s_0T}[t_1]_{s_0T,s_1s_0T}[t_1]_{s_0s_1s_0T,s_1s_0s_1s_0T} \end{aligned}$$

Before we provide a proof of this theorem, we will give a nice example of such a seminormal representation.

Corollary 4.16. *Define an action of* $\mathcal{H}_k^{\text{ext}}$ *on* $\mathcal{L}_\lambda^{(a^p),(b^q)}$ *by*

$$\begin{aligned} w_i \cdot v_T &= c_T(i) v_T, && \text{for } 0 \le i \le k \\ t_{s_i} \cdot v_T &= [t_i]_{T,T} v_T + [t_i]_{T,s_iT} v_{s_iT}, && \text{for } 1 \le i \le k-1 \\ x_1 \cdot v_T &= [x_1]_{T,T} v_T + [x_1]_{T,s_0T} v_{s_0T} \end{aligned}$$

and

$$[t_i]_{T,S} = \begin{cases} \sqrt{1 - [t_i]_{T,T}^2}, & \text{if } S \neq T \\ 1/(c_T(i+1) - c_T(i)), & \text{if } S = T. \end{cases}$$

$$[x_1]_{T,S} = \begin{cases} \sqrt{\begin{array}{c} -\frac{1}{(2c_T(1))^2}\left(c_T(1) + \frac{(a+p)+(b+q)}{2}\right)\left(c_T(1) - \frac{(a+p)-(b+q)}{2}\right) \\ \cdot\left(c_T(1) - \frac{(a+p)+(b+q)}{2}\right)\left(c_T(1) + \frac{(a+p)-(b+q)}{2}\right)\end{array}}, & \text{if } S \neq T \\ \dfrac{(a-p)c_T(1) + c_T^2(1) + \left(\frac{(a+p)+(b+q)}{2}\right)\left(\frac{(a+p)-(b+q)}{2}\right)}{2c_T(1)} & \text{if } S = T. \end{cases}$$

then $\mathcal{L}_\lambda^{(a^p),(b^q)}$ *is a simple* $\mathcal{H}_k^{\text{ext}}$ *-module.*

Proof. The values for $[t_i]_{T,T}$ and $[x_1]_{T,T}$ are pulled directly from Theorem 4.15, so we need only check criteria 3-6: Commutation, Quadratic relation, and Braid relations. We will verify these using the fact that $[x_1]_{T,S}$ and $[t_i]_{T,S}$ for $S \neq T$ are functions of shifted contents $c_T(j)$.

Commutation: For $j \neq i \pm 1$, $c_T(i) = c_{s_jT}(i)$, $c_T(i+1) = c_{s_jT}(i+1)$, $c_T(j) = c_{s_iT}(j)$, and $c_T(j+1) = c_{s_iT}(j+1)$, so

$$[t_i]_{s_jT, s_is_jT} = [t_i]_{T,s_iT} \quad \text{and} \quad [t_j]_{T,s_jT} = [t_j]_{s_iT, s_js_iT}.$$

Similarly, for $i > 1$, $c_T(i) = c_{s_0T}(i)$ and $c_T(i+1) = c_{s_0T}(i+1)$, so

$$[t_i]_{s_0T, s_is_0T} = [t_i]_{T,s_iT},$$

and $c_T(1) = c_{s_iT}(1)$, so

$$[x_1]_{T,s_0T} = [x_1]_{s_iT, s_0s_iT}.$$

Thus criteria 3 is satisfied.

Quadratic Relation: By equation (4.40),

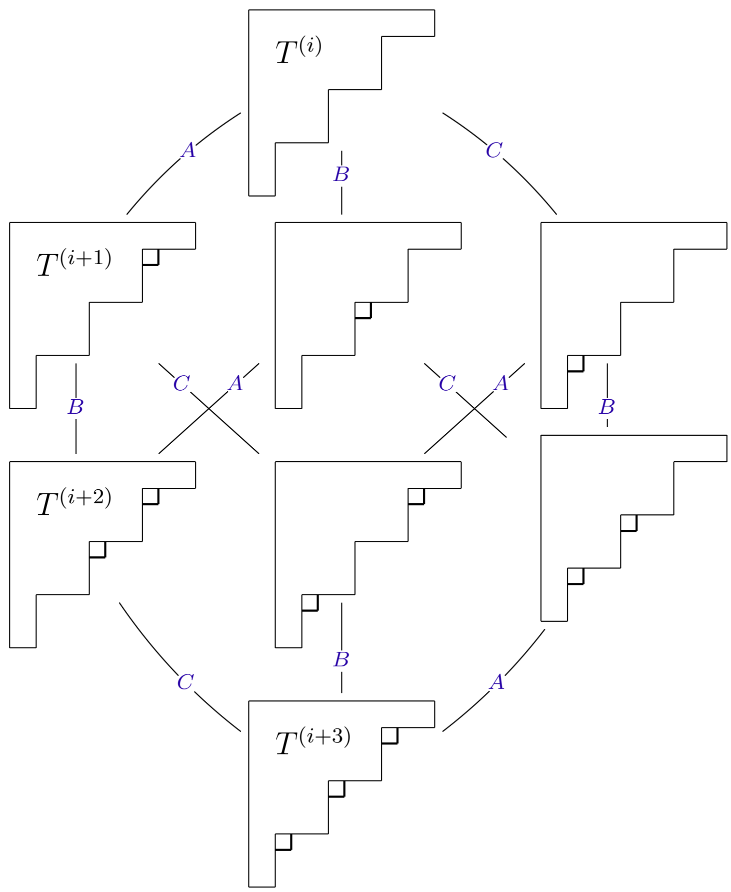
$$[x_1]_{T,s_0T} = [x_1]_{T,s_0T},$$

so criteria 4 is satisfied.

Braid relations: For the first braid relation, let $A = c_T(i)$, $B = c_T(i+1)$, and $C = c_T(i+2)$. Either both sides of the equality

$$[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT} = [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT}$$

are zero, or the six tableaux involved sit in a subgraph of the Bratteli diagram depicted as follows.



This encodes the fact that for whichever of these S exist, their shifted contents are given by the following table:

$S \rightarrow$	T	s_iT	$s_{i+1}T$	$s_is_{i+1}T$	$s_{i+1}s_iT$	$s_is_{i+1}s_iT$
$c_S(i)$	A	B	A	C	B	C
$c_S(i+1)$	B	A	C	A	C	B
$c_S(i+2)$	C	C	B	B	A	A

So

$$[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT} = [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT}$$

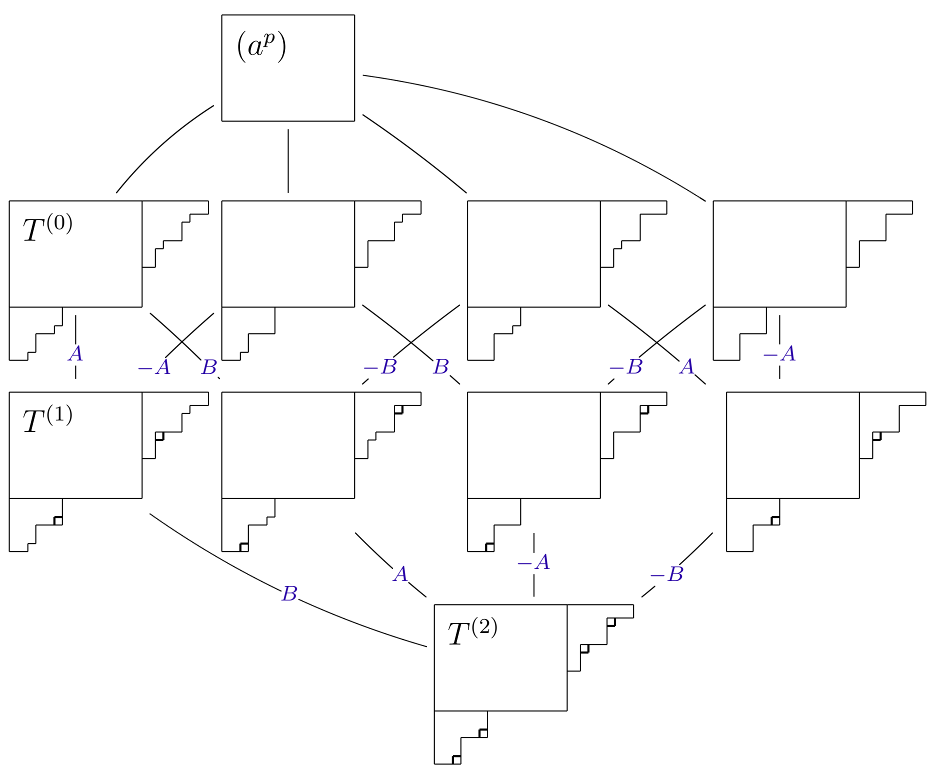
since

$$\begin{aligned}
[t_i]_{T,s_iT} &= \sqrt{1-\left(\tfrac{1}{B-A}\right)^2} \\
[t_{i+1}]_{s_iT,s_{i+1}s_iT} &= \sqrt{1-\left(\tfrac{1}{C-A}\right)^2} \\
[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT} &= \sqrt{1-\left(\tfrac{1}{C-B}\right)^2} \\
[t_{i+1}]_{T,s_{i+1}T} &= \sqrt{1-\left(\tfrac{1}{C-B}\right)^2} \\
[t_i]_{s_{i+1}T,s_is_{i+1}T} &= \sqrt{1-\left(\tfrac{1}{C-A}\right)^2} \\
[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT} &= \sqrt{1-\left(\tfrac{1}{B-A}\right)^2}.
\end{aligned}$$

For the second braid relation, let $A = c_T(1)$, and $B = c_T(2)$. So either both sides of the equality

$$\begin{aligned}
&[x_1]_{s_1T,s_0s_1T}[x_1]_{s_1s_0s_1T,s_0s_1s_0s_1T}[t_1]_{T,s_1T}[t_1]_{s_0s_1T,s_1s_0s_1T} \\
&\qquad = [x_1]_{T,s_0T}[x_1]_{s_1s_0T,s_0s_1s_0T}[t_1]_{s_0T,s_1s_0T}[t_1]_{s_0s_1s_0T,s_1s_0s_1s_0T}
\end{aligned}$$

are zero, or the eight tableaux involved sit in a subgraph of the Bratteli diagram depicted as follows.



This encodes the fact that for whichever of these S exist, their shifted contents are given by the following table:

$S \to$	T	s_0T	s_1T	s_0s_1T	s_1s_0T	$s_0s_1s_0T$	$s_1s_0s_1T$	$s_0s_1s_0s_1T$
$c_S(1)$	A	$-A$	B	$-B$	B	$-B$	A	$-A$
$c_S(2)$	B	B	A	A	$-A$	$-A$	$-B$	$-B$

Therefore,

$$\begin{aligned} [x_1]_{s_1T,s_0s_1T} &= [x_1]_{s_1s_0T,s_0s_1s_0T}, \\ [x_1]_{s_1s_0s_1T,s_0s_1s_0s_1T} &= [x_1]_{T,s_0T}, \\ [t_1]_{T,s_1T} &= [t_1]_{s_0s_1s_0T,s_1s_0s_1s_0T}, \quad \text{and} \\ [t_1]_{s_0s_1T,s_1s_0s_1T} &= [t_1]_{s_0T,s_1s_0T}, \end{aligned}$$

and so

$$\begin{aligned} &[x_1]_{s_1T,s_0s_1T}[x_1]_{s_1s_0s_1T,s_0s_1s_0s_1T}[t_1]_{T,s_1T}[t_1]_{s_0s_1T,s_1s_0s_1T} \\ &\qquad = [x_1]_{T,s_0T}[x_1]_{s_1s_0T,s_0s_1s_0T}[t_1]_{s_0T,s_1s_0T}[t_1]_{s_0s_1s_0T,s_1s_0s_1s_0T}. \end{aligned}$$

Thus criteria 6 is satisfied. □

Proof of Theorem 4.15. We prove Theorem 4.15 in two parts. In Part 1, we will check that the relations in the presentation of $\mathcal{H}_k^{\text{ext}}$ given in Theorem 4.3 hold, showing that $\mathcal{L}_\lambda^{(a^p),(b^q)}$ is a $\mathcal{H}_k^{\text{ext}}$-module. In Part 2, we will verify that $\mathcal{L}_\lambda^{(a^p),(b^q)}$ is simple.

Part 1: $\mathcal{L}_\lambda^{(a^p),(b^q)}$ is a $\mathcal{H}_k^{\text{ext}}$-module. By (4.12), the elements $w_0, w_1, \dots w_k$ generate a commutative subalgebra of $\mathcal{H}_k^{\text{ext}}$, so we can begin by fixing the diagonal action as stated above,

$$\begin{aligned} w_0 \cdot v_T &= \left(abq + 2 \sum_{B \in \mathcal{B}_\lambda} \big(c(B) - \tfrac{1}{2}(a - p + b - q)\big) \right) v_T \\ w_i \cdot v_T &= c_T(i) v_T, \quad \text{for } 1 \le i \le k. \end{aligned}$$

Now write

$$t_{s_i} v_T = \sum_{S \in \mathcal{T}_\lambda} [t_i]_{T,S} v_S \quad \text{and} \quad x_1 v_T = \sum_{S \in \mathcal{T}_\lambda} [x_1]_{T,S} v_S,$$

where $\mathcal{T}_\lambda$ is the set of tableaux (4.37) and $[t_i]_{T,S}, [x_1]_{T,S} \in \mathbb{C}$.

Claim 1: Relations relations (4.4), and (4.9), (4.13) are satisfied if

$$t_{s_i}v_T = [t_i]_{T,T}v_T + [t_i]_{T,s_iT}v_{s_iT}, \quad \text{for } i = 1, \dots, k,$$

$$[t_i]_{T,T} = \frac{1}{c_T(i+1) - c_T(i)} \quad \text{and } [t_i]_{T,S}[t_i]_{S,T} = 1 - ([t_i]_{T,T})^2.$$

Proof: The first commutation relation (4.9), $t_{s_i}w_j = w_j t_{s_i}$ for $j \neq i, i+1$, implies

$$\begin{aligned} t_{s_i}w_j \cdot v_T &= \sum_{S \in \mathcal{T}_\lambda} c_T(j)[t_i]_{T,S}v_S \\ = w_j t_{s_i} \cdot v_T &= \sum_{S \in \mathcal{T}_\lambda} c_S(j)[t_i]_{T,S}v_S. \end{aligned}$$

So for each S, either

$$[t_i]_{T,S} = 0 \qquad \text{or} \qquad c_T(j) = c_S(j) \quad \text{for all } j \neq i, i+1. \tag{4.41}$$

The first twisting relation (4.13), together with relation (4.4), reqire

$$t_{s_i}w_i - w_{i+1}t_{s_i} = -1 = w_i t_{s_i} - t_{s_i}w_{i+1},$$

i.e.,

$$\begin{aligned} (t_{s_i}w_i - w_{i+1}t_{s_i}) \cdot v_T &= \sum_{S \in \mathcal{T}_\lambda} (c_T(i) - c_S(i+1))[t_i]_{T,S}v_S \\ &= -v_T, \\ = (w_i t_{s_i} - t_{s_i}w_{i+1}) \cdot v_T &= \sum_{S \in \mathcal{T}_\lambda} (c_S(i) - c_T(i+1))[t_i]_{T,S}v_S. \end{aligned}$$

So

$$[t_i]_{T,T} = \frac{1}{c_T(i+1) - c_T(i)} \tag{4.42}$$

and for $S \neq T$, either

$$c_S(i+1) = c_T(i) \quad \text{and} \quad c_S(i) = c_T(i+1) \qquad \text{or} \qquad [t_i]_{T,S} = 0. \tag{4.43}$$

By Lemma 4.14, equations (4.41) and (4.43) tell us

$$t_{s_i}v_T = [t_i]_{T,T}v_T + [t_i]_{T,s_iT}v_{s_iT}, \quad \text{for } i = 1, \dots, k,$$

where $[t_i]_{T,s_iT} = 0$ if $c_T(i) = c_T(i+1) \pm 1$. Finally, he involution relation (4.4), $t_{s_i}^2 = 1$, implies

$$[t_i]_{T,T} = -[t_i]_{s_iT,s_iT} \quad \text{and } [t_i]_{T,S}[t_i]_{S,T} = 1 - ([t_i]_{T,T})^2.$$

The first is implied by $[t_i]_{T,T} = \frac{1}{c_T(i+1) - c_T(i)}$, but the second places a new condition on coefficients. This concludes the proof of Claim 1. □

Claim 2: Relation (4.10) is satisfied if

$x_1 v_T = [x_1]_{T,T} v_T + [x_1]_{T,s_0T} v_{s_0T}$, where $[x_1]_{T,s_0T} = 0$ if $c_T(1) = \pm(a+p) \pm (b+q)$.

Furthermore, (4.8), (4.14), and (4.15) are additionally satisfied if

$$[x_1]_{T,T} = \frac{(a-p)c_T(1) + c_T^2(1) + \left(\frac{(a+p)+(b+q)}{2}\right)\left(\frac{(a+p)-(b+q)}{2}\right)}{2c_T(1)}$$

and

$$\begin{aligned}[x_1]_{T,s_0T}[x_1]_{s_0T,T} = \; & -\tfrac{1}{(2c_T(1))^2}\left(c_T(1) + \tfrac{(a+p)+(b+q)}{2}\right)\left(c_T(1) - \tfrac{(a+p)-(b+q)}{2}\right) \\ & \cdot \left(c_T(1) - \tfrac{(a+p)+(b+q)}{2}\right)\left(c_T(1) + \tfrac{(a+p)-(b+q)}{2}\right).\end{aligned}$$

Proof: The relation $x_1 w_i = w_i x_1$ for $i > 1$ implies

$$\begin{aligned} x_1 w_i v_T &= \sum_{S \in T_\lambda} c_T(i)[x_1]_{T,S} v_S \\ = w_i x_1 v_T &= \sum_{S \in T_\lambda} c_S(i)[x_1]_{T,S} v_S \end{aligned}$$

So by Lemmas 4.12 and 4.14,

$$x_1 v_T = [x_1]_{T,T} v_T + [x_1]_{T,s_0T} v_{s_0T}, \tag{4.44}$$

where $[x_1]_{T,s_0T} = 0$ if $c_T(1) = \pm(a+p) \pm (b+q)$.

Now let $K = \left(\frac{a+p+b+q}{2}\right)\left(\frac{a+p-(b+q)}{2}\right)$, so the third twisting relation (4.15),

$$x_1 w_1 = -w_1 x_1 + (a-p)w_1 + w_1^2 + K,$$

says

$$\begin{aligned} (x_1 w_1 + w_1 x_1) v_T &= (c_T(1) + c_T(1))[x_1]_{T,T} v_T + (c_T(1) + c_{s_0T}(1))[x_1]_{T,s_0T} v_{s_0T} \\ &= 2c_T(1)[x_1]_{T,T} v_T \\ = \left((a-p)w_1 + w_1^2 + K\right) v_T &= \left((a-p)c_T(1) + (c_T(1))^2 + K\right) v_T \end{aligned}$$

So

$$[x_1]_{T,T} = \frac{((a-p)c_T(1) + (c_T(1))^2 + K)}{2c_T(1)}. \tag{4.45}$$

If $S = s_0 T$ exists, then the quadratic relation (4.8), $(x_1 - a)(x_1 + p) = 0$, implies

$$\begin{aligned} x_1^2 v_T &= ([x_1]_{T,T}^2 + [x_1]_{T,S}[x_1]_{S,T}) v_T \\ &\quad + ([x_1]_{T,T}[x_1]_{T,S} + [x_1]_{T,S}[x_1]_{S,S}) v_S \\ = (a-p)x_1 + ap &= ((a-p)[x_1]_{T,T} + ap) v_T + (a-p)[x_1]_{T,S} v_S. \end{aligned}$$

We already could conclude $([x_1]_{T,T}[x_1]_{T,S} + [x_1]_{T,S}[x_1]_{S,S}) = (a-p)[x_1]_{T,S}$ from (4.45), so this simply tells us that

$$
\begin{aligned}
[x_1]_{T,S}[x_1]_{S,T} &= -[x_1]^2_{T,T} + (a-p)[x_1]_{T,T} + ap \\
&= -\left(\frac{((a-p)c_T(1) + c_T^2(1) + K)}{2c_T(1)}\right)^2 \\
&\qquad + (a-p)\left(\frac{((a-p)c_T(1) + c_T^2(1) + K)}{2c_T(1)}\right) + ap \\
&= \left(\frac{(a-p)c_T(1) + c_T^2(1) + K}{2c_T(1)}\right)\left(\frac{(a-p)c_T(1) - (c_T^2(1) + K)}{2c_T(1)}\right) + ap \\
&= \frac{(a-p)^2 c_T^2(1) - (c_T^2(1) + K)^2 + 4c_T^2(1)ap}{4c_T^2(1)} \\
&= -\frac{c_T^4(1) - ((a-p)^2 - 2K + 4ap)c_T^2(1) + K^2}{4c_T^2(1)} \\
&= -\frac{c_T^4(1) - \frac{1}{2}((a+p)^2 - (b+q)^2)c_T^2(1) + \left(\frac{1}{4}((a+p)^2 - (b+q)^2)\right)^2}{4c_T^2(1)} \\
&= -\frac{1}{4c_T^2(1)}\left(c_T(1) + \frac{(a+p)+(b+q)}{2}\right)\left(c_T(1) - \frac{(a+p)-(b+q)}{2}\right) \\
&\qquad \left(c_T(1) - \frac{(a+p)+(b+q)}{2}\right)\left(c_T(1) + \frac{(a+p)-(b+q)}{2}\right).
\end{aligned}
$$

Finally, the second twisting relation (4.14), $x_1(w_0+w_1) = (w_0+w_1)x_1$, implies

$$
\begin{aligned}
x_1(w_0+w_1)v_T &= (c_T(0) + c_T(1))[x_1]_{T,T}v_T + (c_T(0) + c_T(1))[x_1]_{T,s_0T}v_{s_0T} \\
&= (w_0+w_1)x_1 v_T = (c_T(0) + c_T(1))[x_1]_{T,T}v_T + (c_{s_0T}(0) + c_{s_0T}(1))[x_1]_{T,s_0T}v_{s_0T}.
\end{aligned}
$$

So we require

$$[x_1]_{T,s_0T} = [x_1]_{T,s_0T} = 0 \quad \text{or} \quad c_T(0) + c_T(1) = c_{s_0T}(0) + c_{s_0T}(1).$$

Recall from (4.40) that if v_{s_0T} exists, then $c_{s_0T}(1) = -c_T(1)$. Therefore we need

$$[x_1]_{T,s_0T} = [x_1]_{T,s_0T} = 0 \quad \text{or} \quad c_T(1) = \tfrac{1}{2}(c_{s_0T}(0) - c_T(0)).$$

But this is just a consequence of the construction in Lemmas 2.13 and 2.15. This concludes the proof of Claim 2.□

Claim 3: Relations (4.5) and (4.11) are satisfied if

$$[t_i]_{s_jT,s_is_jT}[t_j]_{T,s_jT} = [t_i]_{T,s_iT}[t_j]_{s_iT,s_js_iT} \qquad \text{for } j \neq i \pm 1,$$

and

$$[t_i]_{s_0T,s_is_0T}[x_1]_{T,s_0T} = [t_i]_{T,s_iT}[x_1]_{s_iT,s_0s_iT} \qquad \text{for } i > 1,$$

respectively.

Proof: For $j \neq i \pm 1$, relation (4.5) implies

$$\begin{aligned} t_{s_i}t_{s_j} &= [t_i]_{T,T}[t_j]_{T,T}v_T + [t_i]_{T,s_iT}[t_j]_{T,T}v_{s_iT} \\ &\quad + [t_i]_{s_jT,s_jT}[t_j]_{T,s_jT}v_{s_jT} + [t_i]_{s_jT,s_is_jT}[t_j]_{T,s_jT}v_{s_is_jT} \\ = t_{s_j}t_{s_i} &= [t_i]_{T,T}[t_j]_{T,T}v_T + [t_i]_{T,s_iT}[t_j]_{s_iT,s_iT}v_{s_iT} \\ &\quad + [t_i]_{T,T}[t_j]_{T,s_iT}v_{s_jT} + [t_i]_{T,s_iT}[t_j]_{s_iT,s_js_iT}v_{s_js_iT}. \end{aligned}$$

If s_iT and s_jT exist, we already know $[t_j]_{T,T} = [t_j]_{s_iT,s_iT}$ and $[t_i]_{s_jT,s_jT} = [t_i]_{T,T}$ because $c_T(j) = c_{s_iT}(j)$ and $c_T(i) = c_{s_jT}(i)$ for $j \neq i \pm 1$. However, since $s_is_jT = s_js_iT$, we gain the requirement

$$[t_i]_{s_jT,s_is_jT}[t_j]_{T,s_jT} = [t_i]_{T,s_iT}[t_j]_{s_iT,s_js_iT}.$$

Similarly, for $i > 1$, relation (4.11) implies

$$\begin{aligned} t_{s_i}x_1v_T &= [t_i]_{T,T}[x_1]_{T,T}v_T + [t_i]_{s_0T,s_0T}[x_1]_{T,s_0T}v_{s_0T} \\ &\quad + [t_i]_{T,s_iT}[x_1]_{T,T}v_{s_iT} + [t_i]_{s_0T,s_is_0T}[x_1]_{T,s_0T}v_{s_is_0T} \\ = x_1t_{s_i}v_T &= [t_i]_{T,T}[x_1]_{T,T}v_T + [t_i]_{T,T}[x_1]_{T,s_0T}v_{s_0T} \\ &\quad + [t_i]_{T,s_iT}[x_1]_{s_iT,s_iT}v_{s_iT} + [t_i]_{T,s_iT}[x_1]_{s_iT,s_0s_iT}v_{s_is_0T} \end{aligned}$$

since $s_0s_iT = s_is_0T$ for $i > 1$. If s_0T and s_iT exists, we already require that

$$[t_i]_{s_0T,s_0T} = [t_i]_{T,T} \qquad \text{and} [x_1]_{T,T} = [x_1]_{s_iT,s_iT}$$

since $c_T(i) = c_{s_0T}(i)$, $c_T(i+1) = c_{s_0T}(i+1)$, and $c_T(1) = c_{s_iT}(1)$. However, given s_0T, s_iT, and s_0s_iT exist, we gain the requirement

$$[t_i]_{s_0T,s_is_0T}[x_1]_{T,s_0T} = [t_i]_{T,s_iT}[x_1]_{s_iT,s_0s_iT}, \tag{4.46}$$

concluding the proof of Claim 3. □

Claim 4: If

$$\begin{aligned} &[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT} \\ &\quad =[t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT} \end{aligned}$$

then the braid relation (4.6)

$$t_{s_i}t_{s_{i+1}}t_{s_i}v_T = t_{s_{i+1}}t_{s_i}t_{s_{i+1}}v_T$$

is satisfied.

Proof: If v_S exists for $S = s_iT,\ s_{i+1}T,\ s_is_{i+1}T,\ s_{i+1}s_iT,\ s_is_{i+1}s_iT$, then

$$\begin{aligned}
t_{s_i}t_{s_{i+1}}t_{s_i}v_T &= t_{s_i}t_{s_{i+1}}\left([t_i]_{T,T}v_T + [t_i]_{T,s_iT}v_{s_iT}\right)\\
&= t_{s_i}\Big([t_i]_{T,T}\left([t_{i+1}]_{T,T}v_T + [t_{i+1}]_{T,s_{i+1}T}v_{s_{i+1}T}\right)\\
&\qquad + [t_i]_{T,s_iT}\left([t_{i+1}]_{s_iT,s_iT}v_{s_iT} + [t_{i+1}]_{s_iT,s_{i+1}s_iT}v_{s_{i+1}s_iT}\right)\Big)\\
&= [t_i]_{T,T}[t_{i+1}]_{T,T}\left([t_i]_{T,T}v_T + [t_i]_{T,s_iT}v_{s_iT}\right)\\
&\qquad + [t_i]_{T,T}[t_{i+1}]_{T,s_{i+1}T}\left([t_i]_{s_{i+1}T,s_{i+1}T}v_{s_{i+1}T} + [t_i]_{s_{i+1}T,s_is_{i+1}T}v_{s_is_{i+1}T}\right)\\
&\qquad + [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_iT}\left([t_i]_{s_iT,s_iT}v_{s_iT} + [t_i]_{s_iT,s_is_iT}v_{s_is_iT}\right)\\
&\qquad + [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}\\
&\qquad\qquad \left([t_i]_{s_{i+1}s_iT,s_{i+1}s_iT}v_{s_{i+1}s_iT} + [t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT}v_{s_is_{i+1}s_iT}\right)\\
&= \left([t_i]^2_{T,T}[t_{i+1}]_{T,T} + [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_iT}[t_i]_{s_iT,T}\right)v_T\\
&\qquad + \left([t_i]_{T,T}[t_{i+1}]_{T,T}[t_i]_{T,s_iT} + [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_iT}[t_i]_{s_iT,s_iT}\right)v_{s_iT}\\
&\qquad + \left([t_i]_{T,T}[t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_{i+1}T}\right)v_{s_{i+1}T}\\
&\qquad + \left([t_i]_{T,T}[t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}\right)v_{s_is_{i+1}T}\\
&\qquad + \left([t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_{i+1}s_iT}\right)v_{s_{i+1}s_iT}\\
&\qquad + \left([t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT}\right)v_{s_is_{i+1}s_iT}
\end{aligned}$$

because $s_is_iT = T$. Similarly,

$$\begin{aligned}
t_{s_{i+1}}t_{s_i}t_{s_{i+1}}v_T = &\left([t_{i+1}]^2_{T,T}[t_i]_{T,T} + [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_{i+1}T}[t_{i+1}]_{s_{i+1}T,T}\right)v_T\\
&+ \Big([t_{i+1}]_{T,T}[t_i]_{T,T}[t_{i+1}]_{T,s_{i+1}T}\\
&\qquad + [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_{i+1}T}[t_{i+1}]_{s_{i+1}T,s_{i+1}T}\Big)v_{s_{i+1}T}\\
&+ [t_{i+1}]_{T,T}[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_iT}v_{s_iT}\\
&+ [t_{i+1}]_{T,T}[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}v_{s_{i+1}s_iT}\\
&+ [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}T}v_{s_is_{i+1}T}\\
&+ [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT}v_{s_is_{i+1}s_iT}.
\end{aligned}$$

Notice that if one of the above v_S does not exist, the result is the same, as whenever S is undefined the coefficient on v_S is 0. So, to check the identity $t_{s_i}t_{s_{i+1}}t_{s_i}v_T = t_{s_{i+1}}t_{s_i}t_{s_{i+1}}v_T$, we show that each coefficient in $t_{s_i}t_{s_{i+1}}t_{s_i}v_T - t_{s_{i+1}}t_{s_i}t_{s_{i+1}}v_T$ is 0.

Let $A = c_T(i)$, $B = c_T(i+1)$, and $C = c_T(i+2)$. By definition, for whichever of these S exist, their shifted contents are given by the following table.

$S \to$	T	s_iT	$s_{i+1}T$	$s_is_{i+1}T$	$s_{i+1}s_iT$	$s_is_{i+1}s_iT$
$c_S(i)$	A	B	A	C	B	C
$c_S(i+1)$	B	A	C	A	C	B
$c_S(i+2)$	C	C	B	B	A	A

So, in the expansion of $t_{s_i}t_{s_{i+1}}t_{s_i}v_T - t_{s_{i+1}}t_{s_i}t_{s_{i+1}}v_T$, the coefficient on v_S is given by the following.

$S = T$:

$$\begin{aligned}
&\big([t_i]^2_{T,T}[t_{i+1}]_{T,T} + [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_iT}[t_i]_{s_iT,T}\big)\\
&\quad -\big([t_{i+1}]^2_{T,T}[t_i]_{T,T} + [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_{i+1}T}[t_{i+1}]_{s_{i+1}T,T}\big)\\
&= [t_i]_{T,T}[t_{i+1}]_{T,T}([t_i]_{T,T} - [t_{i+1}]_{T,T}) + ([t_i]_{T,s_iT}[t_i]_{s_iT,T})[t_{i+1}]_{s_iT,s_iT}\\
&\qquad - ([t_{i+1}]_{T,s_{i+1}T}[t_{i+1}]_{s_{i+1}T,T})[t_i]_{s_{i+1}T,s_{i+1}T}\\
&= \left(\tfrac{1}{B-A}\right)\left(\tfrac{1}{C-B}\right)\left(\tfrac{1}{B-A} - \tfrac{1}{C-B}\right)\\
&\quad + \left(1 - \tfrac{1}{(B-A)^2}\right)\left(\tfrac{1}{C-A}\right) - \left(1 - \tfrac{1}{(C-B)^2}\right)\left(\tfrac{1}{C-A}\right)\\
&= \left(\tfrac{1}{(C-B)(C-A)(B-A)} + \tfrac{1}{C-A}\right)\\
&\quad - \left(\tfrac{1}{(C-B)(C-A)(B-A)} + \tfrac{1}{C-A}\right)\\
&= 0
\end{aligned}$$

$S = s_iT$:

$$\begin{aligned}
&[t_i]_{T,T}[t_{i+1}]_{T,T}[t_i]_{T,s_iT} + [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_iT}[t_i]_{s_iT,s_iT} - [t_{i+1}]_{T,T}[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_iT}\\
&= [t_i]_{T,s_iT}\left([t_i]_{T,T}[t_{i+1}]_{T,T} + [t_{i+1}]_{s_iT,s_iT}[t_i]_{s_iT,s_iT} - [t_{i+1}]_{T,T}[t_{i+1}]_{s_iT,s_iT}\right)\\
&= [t_i]_{T,s_iT}\left(\tfrac{1}{B-A}\tfrac{1}{C-B} + \tfrac{1}{C-A}\tfrac{1}{B-A} - \tfrac{1}{C-B}\tfrac{1}{C-A}\right)\\
&= 0
\end{aligned}$$

$S = s_{i+1}T$:

$$\begin{aligned}
&[t_i]_{T,T}[t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_{i+1}T}\\
&\quad - \big([t_{i+1}]_{T,T}[t_i]_{T,T}[t_{i+1}]_{T,s_{i+1}T} + [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_{i+1}T}[t_{i+1}]_{s_{i+1}T,s_{i+1}T}\big)\\
&= [t_{i+1}]_{T,s_{i+1}T}\big([t_i]_{T,T}[t_i]_{s_{i+1}T,s_{i+1}T} - [t_{i+1}]_{T,T}[t_i]_{T,T}\\
&\qquad - [t_i]_{s_{i+1}T,s_{i+1}T}[t_{i+1}]_{s_{i+1}T,s_{i+1}T}\big)\\
&= [t_{i+1}]_{T,s_{i+1}T}\left(\left(\tfrac{1}{B-A}\right)\left(\tfrac{1}{C-A}\right) - \left(\tfrac{1}{C-B}\right)\left(\tfrac{1}{B-A}\right) - \left(\tfrac{1}{C-A}\right)\left(\tfrac{1}{B-C}\right)\right)\\
&= 0
\end{aligned}$$

$S = s_i s_{i+1} T$:

$$\begin{aligned}
&[t_i]_{T,T}[t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T} \\
&\qquad -[t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}T} \\
&\qquad\qquad = [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}\left([t_i]_{T,T} - [t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}T}\right) \\
&\qquad\qquad = [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}\left(\left(\tfrac{1}{B-A}\right) - \left(\tfrac{1}{B-A}\right)\right) \\
&\qquad\qquad = 0
\end{aligned}$$

$S = s_{i+1} s_i T$:

$$\begin{aligned}
&[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_{i+1}s_iT} \\
&\qquad -[t_{i+1}]_{T,T}[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT} \\
&\qquad\qquad = [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}\left([t_i]_{s_{i+1}s_iT,s_{i+1}s_iT} - [t_{i+1}]_{T,T}\right) \\
&\qquad\qquad = [t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}\left(\left(\tfrac{1}{C-B}\right) - \left(\tfrac{1}{C-B}\right)\right) \\
&\qquad\qquad = 0
\end{aligned}$$

$S = s_i s_{i+1} s_i T$: The expression

$$\begin{aligned}
&[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT} \\
&\qquad -[t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT}
\end{aligned}$$

cannot be reduced using the determined values.

Therefore, if

$$\begin{aligned}
&[t_i]_{T,s_iT}[t_{i+1}]_{s_iT,s_{i+1}s_iT}[t_i]_{s_{i+1}s_iT,s_is_{i+1}s_iT} \\
&= [t_{i+1}]_{T,s_{i+1}T}[t_i]_{s_{i+1}T,s_is_{i+1}T}[t_{i+1}]_{s_is_{i+1}T,s_is_{i+1}s_iT}
\end{aligned}$$

then the braid relation

$$t_{s_i} t_{s_{i+1}} t_{s_i} v_T = t_{s_{i+1}} t_{s_i} t_{s_{i+1}} v_T$$

is satisfied, concluding the proof of Claim 4. □

Claim 5: If

$$\begin{aligned}
&[x_1]_{s_1T,s_0s_1T}[x_1]_{s_1s_0s_1T,s_0s_1s_0s_1T}[t_1]_{T,s_1T}[t_1]_{s_0s_1T,s_1s_0s_1T} \\
&\qquad = [x_1]_{T,s_0T}[x_1]_{s_1s_0T,s_0s_1s_0T}[t_1]_{s_0T,s_1s_0T}[t_1]_{s_0s_1s_0T,s_1s_0s_1s_0T}
\end{aligned}$$

then the braid relation (4.4)

$$(x_1 t_{s_1} x_1 t_{s_1} + x_1 t_{s_1}) v_T = (t_{s_1} x_1 t_{s_1} x_1 + t_{s_1} x_1) v_T$$

is satisfied.

Proof: Let $a_T = [x_1]_{T,T}$, $b_T = [x_1]_{T,s_0T}$, $d_T = [t_1]_{T,T}$, $e_T = [t_1]_{T,s_1T}$.

$$\begin{aligned} x_1 t_{s_1} v_T &= x_1(d_T v_T + e_T v_{s_1T}) \\ &= a_T d_T v_T + b_T d_T v_{s_0T} + a_{s_1T} e_T v_{s_1T} + b_{s_1T} e_T v_{s_0 s_1 T} \end{aligned}$$

So

$$\begin{aligned} & x_1 t_{s_1} x_1 t_{s_1} v_T \\ &\quad = a_T d_T x_1 t_{s_1} v_T + b_T d_T x_1 t_{s_1} v_{s_0T} \\ &\qquad\quad + a_{s_1T} e_T x_1 t_{s_1} v_{s_1T} + b_{s_1T} e_T x_1 t_{s_1} v_{s_0 s_1 T} \\ &\quad = a_T d_T (a_T d_T v_T + b_T d_T v_{s_0T} + a_{s_1T} e_T v_{s_1T} + b_{s_1T} e_T v_{s_0 s_1 T}) \\ &\qquad + b_T d_T (a_{s_0T} d_{s_0T} v_{s_0T} + b_{s_0T} d_{s_0T} v_T + a_{s_1 s_0 T} e_{s_0T} v_{s_1 s_0 T} + b_{s_1 s_0 T} e_{s_0T} v_{s_0 s_1 s_0 T}) \\ &\qquad + a_{s_1T} e_T (a_{s_1T} d_{s_1T} v_{s_1T} + b_{s_1T} d_{s_1T} v_{s_0 s_1 T} + a_T e_{s_1T} v_T + b_T e_{s_1T} v_{s_0T}) \\ &\qquad + b_{s_1T} e_T (a_{s_0 s_1 T} d_{s_0 s_1 T} v_{s_0 s_1 T} + b_{s_0 s_1 T} d_{s_0 s_1 T} v_{s_1T} \\ &\qquad\qquad + a_{s_1 s_0 s_1 T} e_{s_0 s_1 T} v_{s_1 s_0 s_1 T} + b_{s_1 s_0 s_1 T} e_{s_0 s_1 T} v_{s_0 s_1 s_0 s_1 T}) \\ &\quad = (a_T^2 d_T^2 + b_T b_{s_0T} d_T d_{s_0T} + a_T a_{s_1T} e_T e_{s_1T}) v_T \\ &\qquad + (a_T b_T d_T^2 + a_{s_0T} b_T d_T d_{s_0T} + a_{s_1T} b_T e_T e_{s_1T}) v_{s_0T} \\ &\qquad + (a_T a_{s_1T} d_T e_T + a_{s_1T}^2 d_{s_1T} e_T + b_{s_1T} b_{s_0 s_1 T} d_{s_0 s_1 T} e_T) v_{s_1T} \\ &\qquad + (a_T b_{s_1T} d_T e_T + a_{s_1T} b_{s_1T} d_{s_1T} e_T + a_{s_0 s_1 T} b_{s_1T} e_T d_{s_0 s_1 T}) v_{s_0 s_1 T} \\ &\qquad + (a_{s_1 s_0 T} b_T d_T e_{s_0T}) v_{s_1 s_0 T} \\ &\qquad + (b_{s_1 s_0 T} b_T d_T e_{s_0T}) v_{s_0 s_1 s_0 T} \\ &\qquad + (a_{s_1 s_0 s_1 T} b_{s_1T} e_T e_{s_0 s_1 T}) v_{s_1 s_0 s_1 T} \\ &\qquad + (b_{s_1T} b_{s_1 s_0 s_1 T} e_T e_{s_0 s_1 T}) v_{s_0 s_1 s_0 s_1 T} \end{aligned}$$

and

$$
\begin{aligned}
&(x_1 t_{s_1} x_1 t_{s_1} + x_1 t_{s_1}) v_T \\
&\qquad = (a_T^2 d_T^2 + b_T b_{s_0T} d_T d_{s_0T} + a_T a_{s_1T} e_T e_{s_1T} + a_T d_T) v_T \\
&\qquad\quad + (a_T b_T d_T^2 + a_{s_0T} b_T d_T d_{s_0T} + a_{s_1T} b_T e_T e_{s_1T} + b_T d_T) v_{s_0T} \\
&\qquad\quad + (a_T a_{s_1T} d_T e_T + a_{s_1T}^2 d_{s_1T} e_T + b_{s_1T} b_{s_0s_1T} d_{s_0s_1T} e_T + a_{s_1T} e_T) v_{s_1T} \\
&\qquad\quad + (a_T b_{s_1T} d_T e_T + a_{s_1T} b_{s_1T} d_{s_1T} e_T + a_{s_0s_1T} b_{s_1T} e_T d_{s_0s_1T} + b_{s_1T} e_T) v_{s_0s_1T} \\
&\qquad\quad + (a_{s_1s_0T} b_T d_T e_{s_0T}) v_{s_1s_0T} \\
&\qquad\quad + (b_{s_1s_0T} b_T d_T e_{s_0T}) v_{s_0s_1s_0T} \\
&\qquad\quad + (a_{s_1s_0s_1T} b_{s_1T} e_T e_{s_0s_1T}) v_{s_1s_0s_1T} \\
&\qquad\quad + (b_{s_1T} b_{s_1s_0s_1T} e_T e_{s_0s_1T}) v_{s_0s_1s_0s_1T}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
t_{s_1} x_1 v_T &= t_{s_1}(a_T v_T + b_T v_{s_0T}) \\
&= a_T d_T v_T + b_T d_{s_0T} v_{s_0T} + a_T e_T v_{s_1T} + b_T e_{s_0T} v_{s_1s_0T}.
\end{aligned}
$$

So, since $s_0 s_1 s_0 s_1 T = s_1 s_0 s_1 s_0 T$,

$$
\begin{aligned}
&t_{s_1} x_1 t_{s_1} x_1 v_T \\
&\quad = a_T d_T t_{s_1} x_1 v_T + b_T d_{s_0T} t_{s_1} x_1 v_{s_0T} + a_T e_T t_{s_1} x_1 v_{s_1T} + b_T e_{s_0T} t_{s_1} x_1 v_{s_1s_0T} \\
&\quad = a_T d_T (a_T d_T v_T + b_T d_{s_0T} v_{s_0T} + a_T e_T v_{s_1T} + b_T e_{s_0T} v_{s_1s_0T}) \\
&\qquad + b_T d_{s_0T} (a_{s_0T} d_{s_0T} v_{s_0T} + b_{s_0T} d_T v_T + a_{s_0T} e_{s_0T} v_{s_1s_0T} + b_{s_0T} e_T v_{s_1T}) \\
&\qquad + a_T e_T (a_{s_1T} d_{s_1T} v_{s_1T} + b_{s_1T} d_{s_0s_1T} v_{s_0s_1T} + a_{s_1T} e_{s_1T} v_T + b_{s_1T} e_{s_0s_1T} v_{s_1s_0s_1T}) \\
&\qquad + b_T e_{s_0T} (a_{s_1s_0T} d_{s_1s_0T} v_{s_1s_0T} + b_{s_1s_0T} d_{s_0s_1s_0T} v_{s_0s_1s_0T} \\
&\qquad\qquad\qquad + a_{s_1s_0T} e_{s_1s_0T} v_{s_0T} + b_{s_1s_0T} e_{s_0s_1s_0T} v_{s_1s_0s_1s_0T}) \\
&\quad = (a_T^2 d_T^2 + b_T b_{s_0T} d_T d_{s_0T} + a_T a_{s_1T} e_T e_{s_1T}) v_T \\
&\qquad + (a_T b_T d_T d_{s_0T} + a_{s_0T} b_T d_{s_0T}^2 + a_{s_1s_0T} b_T e_{s_0T} e_{s_1s_0T}) v_{s_0T} \\
&\qquad + (a_T^2 d_T e_T + b_T b_{s_0T} d_{s_0T} e_T + a_T a_{s_1T} d_{s_1T} e_T) v_{s_1T} \\
&\qquad + (a_T b_{s_1T} d_{s_0s_1T} e_T) v_{s_0s_1T} \\
&\qquad + (a_T b_T d_T e_{s_0T} + a_{s_0T} b_T d_{s_0T} e_{s_0T} + a_{s_1s_0T} b_T d_{s_1s_0T} e_{s_0T}) v_{s_1s_0T} \\
&\qquad + (b_T b_{s_1s_0T} d_{s_0s_1s_0T} e_{s_0T}) v_{s_0s_1s_0T} \\
&\qquad + (a_T b_{s_1T} e_T e_{s_0s_1T}) v_{s_1s_0s_1T} \\
&\qquad + (b_T b_{s_1s_0T} e_{s_0T} e_{s_0s_1s_0T}) v_{s_0s_1s_0s_1T}
\end{aligned}
$$

and

$$
\begin{aligned}
&(t_{s_1}x_1t_{s_1}x_1 + t_{s_1}x_1)v_T \\
&\quad = (a_T^2 d_T^2 + b_T b_{s_0T} d_T d_{s_0T} + a_T a_{s_1T} e_T e_{s_1T} + a_T d_T)v_T \\
&\qquad + (a_T b_T d_T d_{s_0T} + a_{s_0T} b_T d_{s_0T}^2 + a_{s_1s_0T} b_T e_{s_0T} e_{s_1s_0T} + b_T d_{s_0T})v_{s_0T} \\
&\qquad + (a_T^2 d_T e_T + b_T b_{s_0T} d_{s_0T} e_T + a_T a_{s_1T} d_{s_1T} e_T + a_T e_T)v_{s_1T} \\
&\qquad + (a_T b_{s_1T} d_{s_0s_1T} e_T)v_{s_0s_1T} \\
&\qquad + (a_T b_T d_T e_{s_0T} + a_{s_0T} b_T d_{s_0T} e_{s_0T} + a_{s_1s_0T} b_T d_{s_1s_0T} e_{s_0T} + b_T e_{s_0T})v_{s_1s_0T} \\
&\qquad + (b_T b_{s_1s_0T} d_{s_0s_1s_0T} e_{s_0T})v_{s_0s_1s_0T} \\
&\qquad + (a_T b_{s_1T} e_T e_{s_0s_1T})v_{s_1s_0s_1T} \\
&\qquad + (b_T b_{s_1s_0T} e_{s_0T} e_{s_0s_1s_0T})v_{s_0s_1s_0s_1T}
\end{aligned}
$$

Let $A = c_T(1)$, $B = c_T(2)$. By definition, for whichever of these S exist, their shifted contents are given by the following table.

$S \rightarrow$	T	s_0T	s_1T	s_0s_1T	s_1s_0T	$s_0s_1s_0T$	$s_1s_0s_1T$	$s_0s_1s_0s_1T$
$c_S(1)$	A	$-A$	B	$-B$	B	$-B$	A	$-A$
$c_S(2)$	B	B	A	A	$-A$	$-A$	$-B$	$-B$

Thus the values of a_S and d_S are given by

$S \rightarrow$	T	s_0T	s_1T	s_0s_1T
a_S	a_T	$-a_T + (a-p)$	a_{s_1T}	$-a_{s_1T} + (a-p)$
d_S	$\frac{1}{B-A}$	$\frac{1}{B+A}$	$-d_T$	d_{s_0T}

$S \rightarrow$	s_1s_0T	$s_0s_1s_0T$	$s_1s_0s_1T$	$s_0s_1s_0s_1T$
a_S	a_{s_1T}	$-a_{s_1T} + (a-p)$	a_T	$-a_T + (a-p)$
d_S	$-d_{s_0T}$	d_T	$-d_{s_0T}$	$-d_T$

Furthermore recall that $b_T b_{s_0T} = -a_T^2 + (a-p)a_T + ap$ and $e_{s_1T} e_T = 1 - d_T^2$. So the coefficients on v_S in $x_1x_2 - x_2x_1$ are determined by the following eight calculations.

$S = T$:

$$
\begin{aligned}
&a_T^2 d_T^2 + b_T b_{s_0T} d_T d_{s_0T} + a_T a_{s_1T} e_T e_{s_1T} + a_T d_T \\
&\quad -(a_T^2 d_T^2 + b_T b_{s_0T} d_T d_{s_0T} + a_T a_{s_1T} e_T e_{s_1T} + a_T d_T) \\
&\qquad = 0
\end{aligned}
$$

$S = s_0 T:$

$$
\begin{aligned}
&a_T b_T d_T^2 + a_{s_0T} b_T d_T d_{s_0T} + a_{s_1T} b_T e_T e_{s_1T} + b_T d_T \\
&\quad -(a_T b_T d_T d_{s_0T} + a_{s_0T} b_T d_{s_0T}^2 + a_{s_1s_0T} b_T e_{s_0T} e_{s_1s_0T} + b_T d_{s_0T}) \\
&\qquad = b_T\big((a_T d_T + a_{s_0T} d_{s_0T} + 1)(d_T - d_{s_0T}) \\
&\qquad\qquad + a_{s_1T} e_T e_{s_1T} - a_{s_1s_0T} e_{s_0T} e_{s_1s_0T}\big) \\
&\qquad = b_T\big((a_T d_T + (-a_T + (a-p))d_{s_0T} + 1)(d_T - d_{s_0T}) \\
&\qquad\qquad + a_{s_1T}(1 - d_T^2) - a_{s_1T}(1 - d_{s_0T}^2)\big) \\
&\qquad = b_T(d_T - d_{s_0T})\big(a_T(d_T - d_{s_0T}) + (a-p)d_{s_0T} + 1 - a_{s_1T}(d_T + d_{s_0T})\big) \\
&\qquad = b_T\left(\frac{2A}{(B-A)(B+A)}\right) \\
&\qquad\qquad \left(\left(\frac{(a-p)A + A^2 + K}{2A}\right)\left(\frac{2A}{(B-A)(B+A)}\right)\right. \\
&\qquad\qquad\quad + (a-p)\left(\frac{1}{B+A}\right) + 1 \\
&\qquad\qquad\quad \left. - \left(\frac{(a-p)B + B^2 + K}{2B}\right)\left(\frac{2B}{(B-A)(B+A)}\right)\right) \\
&\qquad = b_T\left(\frac{2A}{(B-A)(B+A)^2}\right) \\
&\qquad\qquad \left(\left(\frac{(a-p)A + A^2 + K}{(B-A)}\right) + (a-p)\right. \\
&\qquad\qquad\quad \left. + B + A - \left(\frac{(a-p)B + B^2 + K}{(B-A)}\right)\right) \\
&\qquad = 0
\end{aligned}
$$

$S = s_1T$:

$$
\begin{aligned}
a_T a_{s_1T} d_T e_T &+ a_{s_1T}^2 d_{s_1T} e_T + b_{s_1T} b_{s_0 s_1 T} d_{s_0 s_1 T} e_T + a_{s_1T} e_T \\
&-(a_T^2 d_T e_T + b_T b_{s_0T} d_{s_0T} e_T + a_T a_{s_1T} d_{s_1T} e_T + a_T e_T) \\
&= e_T\big((a_T d_T + a_{s_1T} d_{s_1T} + 1)(a_{s_1T} - a_T) \\
&\qquad + (-a_{s_1T}^2 + (a-p)a_{s_1T} + ap)d_{s_0 s_1 T} \\
&\qquad - (-a_T^2 + (a-p)a_T + ap)d_{s_0T}\big) \\
&= e_T\big(((a_T - a_{s_1T})d_T + 1)(a_{s_1T} - a_T) \\
&\qquad + (-a_{s_1T}^2 + (a-p)a_{s_1T} + ap + a_T^2 - (a-p)a_T - ap)d_{s_0T}\big) \\
&= e_T(a_{s_1T} - a_T) \\
&\qquad \big((a_T - a_{s_1T})d_T + 1 - (a_{s_1T} + a_T)d_{s_0T} + (a-p)d_{s_0T}\big) \\
&= e_T(a_{s_1T} - a_T) \\
&\qquad \Bigg(\left(\frac{(a-p)A + A^2 + K}{2A}\right)\left(\frac{2A}{(B-A)(B+A)}\right) \\
&\qquad - \left(\frac{(a-p)B + B^2 + K}{2B}\right)\left(\frac{2B}{(B-A)(B+A)}\right) \\
&\qquad + 1 + (a-p)\frac{1}{B+A}\Bigg) \\
&= \frac{e_T(a_{s_1T} - a_T)}{B+A} \\
&\qquad \Bigg(\left(\frac{(a-p)A + A^2 + K}{(B-A)}\right) \\
&\qquad - \left(\frac{(a-p)B + B^2 + K}{(B-A)}\right) \\
&\qquad + B + A + (a-p)\Bigg) \\
&= 0
\end{aligned}
$$

$S = s_0 s_1 T$:

$$a_T b_{s_1T} d_T e_T + a_{s_1T} b_{s_1T} d_{s_1T} e_T + a_{s_0s_1T} b_{s_1T} e_T d_{s_0s_1T} + b_{s_1T} e_T - a_T b_{s_1T} d_{s_0s_1T} e_T$$
$$= b_{s_1T} e_T \big(a_T d_T - a_{s_1T} d_T + (-a_{s_1T} + (a-p)) d_{s_0T} + 1 - a_T d_{s_0T}\big)$$
$$= b_{s_1T} e_T \left(a_T(d_T - d_{s_0T}) - a_{s_1T}(d_T + d_{s_0T}) + (a-p) d_{s_0T} + 1\right)$$
$$= 0$$

(same as above)

$S = s_1 s_0 T$:

$$(a_{s_1s_0T} b_T d_T e_{s_0T}) - (a_T b_T d_T e_{s_0T} + a_{s_0T} b_T d_{s_0T} e_{s_0T} + a_{s_1s_0T} b_T d_{s_1s_0T} e_{s_0T} + b_T e_{s_0T})$$
$$= b_T e_{s_0T} (a_{s_1s_0T} d_T - a_T d_T - a_{s_0T} d_{s_0T} - a_{s_1s_0T} d_{s_1s_0T} - 1)$$
$$= b_T e_{s_0T} (a_{s_1T} d_T - a_T d_T - (-a_T + (a-p)) d_{s_0T} + a_{s_1T} d_{s_0T} - 1)$$
$$= b_T e_{s_0T} (a_{s_1T}(d_T + d_{s_0T}) - a_T(d_T - d_{s_0T}) - (a-p) d_{s_0T} - 1)$$
$$= 0.$$

(same as above)

$S = s_0 s_1 s_0 T$:

$$b_T b_{s_1s_0T} d_T e_{s_0T} - b_T b_{s_1s_0T} d_{s_0s_1s_0T} e_{s_0T} = 0$$

because $d_{s_0s_1s_0T} = d_T$.

$S = s_1 s_0 s_1 T$:

$$a_{s_1s_0s_1T} b_{s_1T} e_T e_{s_0s_1T} - a_T b_{s_1T} e_T e_{s_0s_1T} = 0$$

because $a_{s_1s_0s_1T} = a_T$.

$S = s_0 s_1 s_0 s_1 T$: The expression

$$b_{s_1T} b_{s_1s_0s_1T} e_T e_{s_0s_1T} - b_T b_{s_1s_0T} e_{s_0T} e_{s_0s_1s_0T}$$

cannot be simplified from the determined values.

Therefore, if

$$[x_1]_{s_1T, s_0s_1T} [x_1]_{s_1s_0s_1T, s_0s_1s_0s_1T} [t_1]_{T, s_1T} [t_1]_{s_0s_1T, s_1s_0s_1T} = [x_1]_{T, s_0T} [x_1]_{s_1s_0T, s_0s_1s_0T} [t_1]_{s_0T, s_1s_0T} [t_1]_{s_0s_1s_0T, s_1s_0s_1s_0T}$$

then the braid relation

$$(x_1 t_{s_1} x_1 t_{s_1} + x_1 t_{s_1}) v_T = (t_{s_1} x_1 t_{s_1} x_1 + t_{s_1} x_1) v_T$$

is satisfied, thus concluding the proof of Claim 5. □

This concludes Part 1, showing that $\mathcal{L}_\lambda^{(a^p),(b^q)}$ is a $\mathcal{H}_k^{\text{ext}}$-module. □

Part 2: $\mathcal{L}_\lambda^{(a^p),(b^q)}$ is simple.
We will first show that any nontrivial submodule of $\mathcal{L}_\lambda^{(a^p),(b^q)}$ contains some basis element v_T, $T \in \mathcal{T}_\lambda$. We will then prove that any basis element v_T generates $\mathcal{L}_\lambda^{(a^p),(b^q)}$, and therefore conclude that $\mathcal{L}_\lambda^{(a^p),(b^q)}$ contains no nontrivial proper submodules.

Claim 1: If $0 \neq v \in \mathcal{L}_\lambda^{(a^p),(b^q)}$, then $\mathcal{H}_k^{\text{ext}} v$ contains some element of the basis v_T, $T \in \mathcal{T}_\lambda$.
Proof. For any $S \in \mathcal{T}_\lambda$, let

$$W_S = (w_1 - c_S(1))^2 + (w_2 - c_S(2))^2 + \cdots + (w_k - c_S(k))^2.$$

By Lemma 4.14,

$$W_S v_T = \left(\sum_{i=1}^{k} (c_T(i) - c_S(i))^2\right) v_T = 0 \quad \text{if and only if} \quad T = S.$$

Therefore, if

$$\mathrm{Pr}_T = \prod_{\substack{S \in \mathcal{T}_\lambda \\ S \neq T}} \left(\frac{W_S}{\sum_{i=1}^{k} (c_T(i) - c_S(i))^2} \right)$$

then

$$\mathrm{Pr}_T v_S = \delta_{ST} v_T.$$

Write

$$v = \sum_{S \in \mathcal{T}_\lambda} d_S v_S, \qquad d_S \in \mathbb{C}.$$

Since $v \neq 0$, there is some $d_T \neq 0$, and so $v_T = \frac{1}{d_T} \mathrm{Pr}_T v \in \mathcal{H}_k^{\text{ext}} v$, concluding the proof of Claim 1. $\square$

We proceed for the remainder of this proof to show that for any basis vector v_T,

$$\mathcal{H}_k^{\text{ext}} v_T = \mathcal{L}_\lambda^{(a^p),(b^q)}.$$

If $c_T(1) \neq \pm\frac{1}{2}((a+p) \pm (b+q))$, then $[x_1]_{T,s_0T} \neq 0$. Define the operator σ_0 on the basis $\{v_T\}_{T \in \mathcal{T}_\lambda}$ of $\mathcal{L}_\lambda^{(a^p),(b^q)}$ by

$$\sigma_0 v_T = \begin{cases} 0 & \text{if } c_T(1) = \pm\frac{1}{2}((a+p) \pm (b+q)), \\ \frac{1}{[x_1]_{T,s_0T}} \left(x_1 - [x_1]_{T,T}\right) v_T & \text{otherwise,} \end{cases} \tag{4.47}$$

and extend linearly. Though σ_0 is not formally an element of $\mathcal{H}_k^{\text{ext}}$, it defines an action of $\mathcal{H}_k^{\text{ext}}$ on $\mathcal{L}_\lambda^{(a^p),(b^q)}$, i.e. $\sigma_0 v_T \in \mathcal{H}_k^{\text{ext}} v_T$. Therefore if v_{s_0T} exists, then

$$\begin{aligned}\sigma_0 v_T &= \frac{1}{[x_1]_{T,s_0T}}\left(x_1 - [x_1]_{T,T}\right) v_T \\ &= \frac{1}{[x_1]_{T,s_0T}}\left([x_1]_{T,T} v_T + [x_1]_{T,s_0T} v_{s_0T} - [x_1]_{T,T} v_T\right) \\ &= v_{s_0T},\end{aligned} \tag{4.48}$$

and so $v_{s_0T} \in \mathcal{H}_k^{\text{ext}} v_T$.

If $c_T(i+1) \neq c_T(i) \pm 1$, then $[t_i]_{T,s_iT} \neq 0$. Define the operator σ_i, $i = 1, \ldots, k$, on the basis $\{v_T\}_{T \in \mathcal{T}_\lambda}$ of $\mathcal{L}_\lambda^{(a^p),(b^q)}$ by

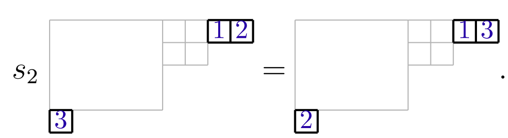
$$\sigma_i v_T = \begin{cases} 0 & \text{if } c_T(i+1) = c_T(i) \pm 1, \\ \frac{1}{[t_i]_{T,s_iT}}\left(t_{s_i} - [t_i]_{T,T}\right) v_T & \text{otherwise} \end{cases} \tag{4.49}$$

and extend linearly. Again, σ_i is not formally an element of $\mathcal{H}_k^{\text{ext}}$, but rather defines an action of $\mathcal{H}_k^{\text{ext}}$ on $\mathcal{L}_\lambda^{(a^p),(b^q)}$. So if v_{s_iT} exists, we have

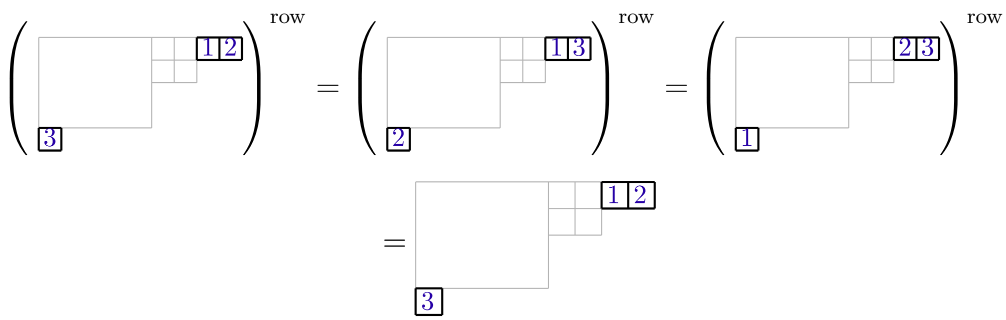
$$\begin{aligned}\sigma_i v_T &= \frac{1}{[t_i]_{T,s_iT}}\left(t_{s_i} - [t_i]_{T,T}\right) v_T \\ &= \frac{1}{[t_i]_{T,s_iT}}\left([t_i]_{T,T} v_T + [t_i]_{T,s_iT} v_{s_iT} - [t_i]_{T,T} v_T\right) \\ &= v_{s_iT},\end{aligned} \tag{4.50}$$

and so $v_{s_iT} \in \mathcal{H}_k^{\text{ext}} v_T$.

Recall from (4.27) that we can view every tableau either as a sequence of partitions, as we have been doing, or as a skew shape filled with integers $1, \ldots, k$ with strictly decreasing rows and columns: the sequence from μ to λ corresponds to the filling of λ/μ where the i^{th} box added in the sequence is filled with i in the skew shape. Viewing T as a standard filling now, consider the placement of labels i and $i+1$. If they are adjacent (in row or column), then $c_T(i+1) = c_T(i) \pm 1$, and so s_iT does not exist. However, if labels i and $i+1$ are nonadjacent, then s_iT is gotten from T by switching i and $i+1$. For example,



Define the tableau T^{row} as the filling of $\lambda/T^{(0)}$ built by placing values left to right, top to bottom, consecutively. For example,



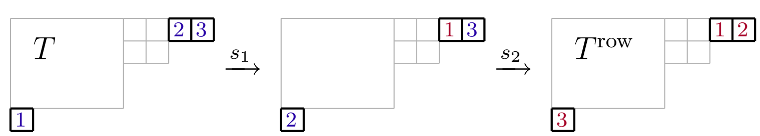
Claim 2: For any tableau $T \in \mathcal{T}_\lambda$ and any submodule $U \subseteq \mathcal{L}_\lambda^{(a^p),(b^q)}$,

$$v_T \in U \quad \text{if and only if} \quad v_{T^{\text{row}}} \in U.$$

Proof. For any T, the following process allows us to construct T^{row} by applying a series of s_i moves, $1 \leq i \leq k$, to T.

1. Reading left to right, top to bottom, find the first box which has a different filling from T^{row}. Let j be the filling in this box and let i be the box immediately before it.
2. Notice $j-1$ is not placed in any boxes north (east or west) or directly west of j, since those boxes are filled with $1, \ldots, i$. Therefore, $j-1$ and j can be switched by applying s_{j-1}.
3. If $s_{j-1}T = T^{\text{row}}$, we are done. Otherwise, begin again at step 1 with $s_{j-1}T$. Since $1 \leq i < j-1 < j \leq k$ for any j, this process terminates in no more than $k!$ steps.

Let $\mathrm{w} = s_{i_\ell} \ldots s_{i_2} s_{i_1}$, $1 \leq i_j \leq k$, be the word generated by this process (where s_{i_1} is the first transposition applied, and so on). In the example above, this process proceeds as follows:



So $\mathrm{w} = s_2 s_1$ and $s_2 s_1 T = T^{\text{row}}$.

If $\mathrm{w}T = s_{i_\ell} \ldots s_{i_2} s_{i_1} T = T^{\text{row}}$, then

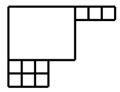
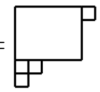
$$\sigma_{i_\ell} \ldots \sigma_{i_2} \sigma_{i_1} v_T = v_{T^{\text{row}}}$$

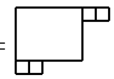
and so $v_{T^{\text{row}}} \in \mathcal{H}_k^{\text{ext}} v_T$. We can apply the same process to find $\mathrm{w}^{-1} T^{\text{row}} = s_{i_1} s_{i_2} \ldots s_{i_\ell} T^{\text{row}} = T$. Thus

$$\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_\ell} v_{T^{\text{row}}} = v_T$$

and so $v_T \in \mathcal{H}_k^{\text{ext}} v_{T^{\text{row}}}$, thus concluding the proof of Claim 2. □

Recall from Section 2.4.1 that if $\mu \in \mathcal{P}((a^p),(b^q))$, then $\mathcal{B}_\mu$ is the set of boxes in μ in rows $p+1$ and below. Let $\mathcal{B}_\mu^\lambda$ be the set of boxes (i,j) in $\mathcal{B}_\mu$ for which box $(a+b+1-i, p+q+1-j)$ is in λ. The criteria in (2.17) imply that these form a skew shape and the shape obtained by moving each of these boxes $(i,j) \in \mathcal{B}_\mu^\lambda$ to their complementary position $(a+b+1-i, p+q+1-j)$ gives another partition in $\mathcal{P}((a^p),(b^q))$. Let $\lambda^{\max}$ be this partition. For example, if

$\lambda =$  and $T^{(0)} =$ 

then $\mathcal{B}_{T^{(0)}}^\lambda = \{(1, p+2)\}$ and $\lambda^{\max} =$  (4.51)

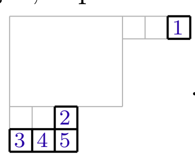
Since each of the boxes in the complementary positions are in λ, the shape obtained by moving all boxes in $\mathcal{B}_\mu^\lambda$ up is also in λ, and therefore there is a tableau

$$S = (\lambda^{\max} = S^{(0)}, \ldots, S^{(k)} = \lambda)$$

from $\lambda^{\max}$ to λ. Let

$$T^\lambda = S^{\text{row}} \tag{4.52}$$

so T^λ is the unique tableau in $\mathcal{T}_\lambda$ with $T^{(0)}$ highest in lexicographical order (see Definition 2.9) and with fillings reading left to right, top to bottom. From the example in (4.51),

$T^\lambda =$  .

Claim 3: For any tableau $T \in \mathcal{T}_\lambda$ and submodule $U \subseteq \mathcal{L}_\lambda^{(a^p),(b^q)}$,

$$v_T \in U \quad \text{if and only if} \quad v_{T^\lambda} \in U.$$

Proof. The following process allows us to construct T^λ from T through a series of s_i moves, $0 \le i \le k$:

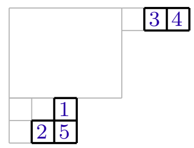
0. Use the process in Claim 2 to move T to T^{row}.
1. Reading left to right, top to bottom, find the last box (i,j) in $\mathcal{B}_{T^{(0)}}^\lambda$.
2. The box in position $(a+b+1-i, p+q+1-j)$ is filled with a 1. Therefore, we can construct a new tableau

$$S = (S^{(0)}, \ldots, S^{(k)} = \lambda) \in \mathcal{T}_\lambda$$

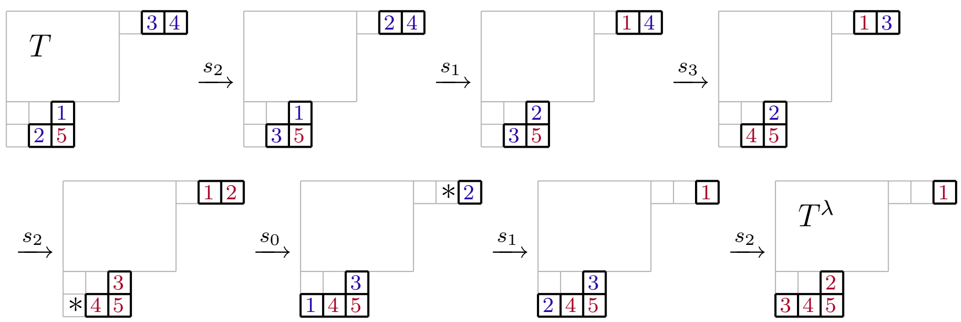
where $S^{(0)}$ is built from $T^{(0)}$ by moving box $(a+b+1-i, p+q+1-j)$ to (i,j), and $S^{(i)} = T^{(i)}$ for $i = 1, \ldots, k$. The resulting filling will have a 1 in box (i,j) and $2, \ldots, k$ identical to T. This new tableau S is equal to s_0T (see the description of (4.39)).

3. Use the process in Claim 2 to move to $(s_0T)^{\text{row}}$.
4. If $(s_0T)^{\text{row}} = T^\lambda$, then we are done. If not, repeat step 1 with $(s_0T)^{\text{row}}$. There are at most $\min(a,b) \times q$ boxes in $\mathcal{B}^\lambda_{T^{(0)}}$, so this process terminates in at most $\min(a,b) \times q$ steps.

Let $\mathrm{w} = s_{i_\ell} \dots s_{i_2} s_{i_1}$, $0 \le i_j \le k$, be the word generated by this process (where s_{i_1} is the first transposition applied, and so on). For example, if

$T =$ 

this process proceeds as follows:



So $\mathrm{w} = s_2 s_1 s_0 s_2 s_3 s_1 s_2$, and $\mathrm{w}T = T^\lambda$.

If $\mathrm{w}T = s_{i_\ell} \dots s_{i_2} s_{i_1} T = T^\lambda$, then

$$\sigma_{i_\ell} \dots \sigma_{i_2} \sigma_{i_1} v_T = v_{T^\lambda}$$

and so $v_{T^\lambda} \in \mathcal{H}^{\text{ext}}_k v_T$. We can apply the same process to find $\mathrm{w}^{-1} T^\lambda = s_{i_1} s_{i_2} \dots s_{i_\ell} T^\lambda = T$. Thus

$$\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_\ell} v_{T^\lambda} = v_T$$

and so $v_T \in \mathcal{H}^{\text{ext}}_k v_{T^\lambda}$, thus concluding the proof of Claim 3. □

By Claim 1, any nonzero submodule $U \subseteq \mathcal{L}^{(a^p),(b^q)}_\lambda$ contains some basis vector v_T. By Claim 3, U therefore contains v_{T^λ}, and consequently contains all basis vectors v_T of $\mathcal{L}^{(a^p),(b^q)}_\lambda$. Thus, $U = \mathcal{L}^{(a^p),(b^q)}_\lambda$ and so $\mathcal{L}^{(a^p),(b^q)}_\lambda$ is simple.

This concludes Part 2, showing that $\mathcal{L}^{(a^p),(b^q)}_\lambda$ is simple, and therefore completes the proof of Theorem 4.15. □

Remark 4.17. *We have shown slightly more than was stated in Theorem 4.15. Namely, if* $\mathcal{L}_\lambda^{(a^p),(b^q)}$ *is a* $\mathcal{H}_k^{\text{ext}}$*-module with basis indexed by* $T \in \mathcal{T}_\lambda$ *and* $w_i \cdot v_T = c_T(i)v_T$ *for* $0 = 1, \ldots, k$*, then*

1. $t_{s_i} \cdot v_T = [t_i]_{T,T}v_T + [t_i]_{T,s_iT}v_{s_iT}$ *and* $x_1 \cdot v_T = [x_1]_{T,T}v_T + [x_1]_{T,s_0T}v_{s_0T}$*, where* $[t_i]_{T,s_iT} = 0$ *if and only if* $c_T(i) = c_T(i+1) \pm 1$*, and* $[x_1]_{T,s_0T} = 0$ *if and only if* $c_T(1) = \frac{1}{2}(\pm(a+p) \pm (b+q))$*,*
2. $[x_1]_{T,S}$ *and* $[t_i]_{T,S}$ *satisfy items (1)-(6) of Theorem 4.15, and*
3. $\mathcal{L}_\lambda^{(a^p),(b^q)}$ *is simple as an* $\mathcal{H}_k^{\text{ext}}$*-module.*

What is more is that the proof that $\mathcal{L}_\lambda^{(a^p),(b^q)}$ *is simple (Part 2) relies only on the action of* $\mathcal{H}_k$*, and so* $\text{Res}_{\mathcal{H}_k}^{\mathcal{H}_k^{\text{ext}}}\left(\mathcal{L}_\lambda^{(a^p),(b^q)}\right)$ *is simple.*

Corollary 4.18. *In the setting of Theorem 4.9,*

$$\text{Res}_{\Phi'(\mathcal{H}_k^{\text{ext}})}^{\text{End}_{\mathfrak{g}}(M\otimes N\otimes V^{\otimes k})}(\mathcal{L}^\mu) \quad \text{and} \quad \text{Res}_{\Phi'(\mathcal{H}_k)}^{\text{End}_{\mathfrak{g}}(M\otimes N\otimes V^{\otimes k})}(\mathcal{L}^\mu)$$

are simple $\mathcal{H}_k^{\text{ext}}$*- and* $\mathcal{H}_k$*-modules, respectively.*

Proof. Any simple $\text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$-module $\mathcal{L}^\mu \subseteq M \otimes N \otimes V^{\otimes k}$ has basis $\{v_T^z \mid T \in \mathcal{T}_\mu\}$, and by Theorem 4.9, w_i acts via Φ' by $w_i \cdot v_T^z = c_T(i)v_T^z$. The restatement of Theorem 4.15 in Remark 4.17 implies $\mathcal{L}^\mu$ is simple as both a $\mathcal{H}_k^{\text{ext}}$-module and a $\mathcal{H}_k$-module. □

Chapter 5

Future Work

5.1 Connections to type C algebras

The degenerate two-boundary Hecke algebra $\mathcal{H}_k$ is strikingly similar to the graded Hecke algebra of type C, denoted $\mathbb{H}$. This observation suggests the possibility of studying representations of type C Hecke algebras using Schur-Weyl duality techniques.

A first step to understanding the similarities between these two algebras is to investigate the center of $\mathcal{H}_k$. The action of central elements is a primary tool in the classification of irreducible representations, and the theory for $\mathbb{H}$ is well developed. A preliminary characterization of the center of $\mathcal{H}_k$ shows it to have similar structure to the center of $\mathbb{H}$—it contains a ring of polynomials symmetric with respect to an action of the type C Weyl group. It remains to be determined whether this subring forms the entire center. We expect that the center will be fully determined through a better understanding of the basis of $\mathcal{H}_k^{\text{ext}}$.

The next step will be to determine the parallels between the representation theory of $\mathbb{H}$ and that of $\mathcal{H}_k$. I have preliminarily compared the Schur-Weyl duality results for $\mathcal{H}_k$ to the previous combinatorial approach via the analysis of intertwining operators in type C (see [Ra]). The calibrated representations (those for which a large abelian subgroup acts semisimply) of the graded Hecke algebra of type C are indexed by skew shapes which can be recognized in the combinatorial structure in Sections 4.2 and 4.3. Furthermore, the dimensions of these representations align accordingly in generic cases. With further study of $\mathbb{H}$, I will be able to describe the correspondence between these combinatorial structures in detail.

As a culmination, my goal is to form a precise statement about how we are able to draw valuable information about $\mathbb{H}$ from $\mathcal{H}_k$. In the quantized version, preliminary work has shown there to be an isomorphism between the affine Hecke algebra of type C and the two-boundary Hecke algebra. In [Lu], Lusztig studies the correspondence between the affine algebras and their graded versions. From this study, we expect an analogous isomorphism between the graded Hecke algebra of type C and the degenerate two-boundary Hecke algebra. The presentation of $\mathcal{H}_k$ in Corollary 4.4 and the following remark nearly grasp such an isomorphism. We can also see that the representation theory is strikingly similar. One contribution I may make through this investigation would be to provide an alternate definition of the graded Hecke algebra of type C which clarifies the isomorphism between it and the affine Hecke algebra of type C as explored in [Lu].

5.2 Centralizers in type B, C, and D

The *graded Birman-Murakami-Wenzl algebra* (otherwise known as the *degenerate affine Wenzl algebra*) $\mathcal{W}_k^{(1)}$ was defined by Nazarov in [Naz] to capture the action of Jucys-Murphy operators on the irreducible representations of Brauer algebras. A consequence of Nazarov's work is that a quotient of $\mathcal{W}_k^{(1)}$ centralizes the action of the orthogonal and symplectic groups on tensor spaces of the form $M \otimes V^{\otimes k}$, where $M = L(\lambda)$ and $V = L(\omega_1)$. In work with Ram and Virk, we have defined the degenerate one-boundary braid group $\mathcal{G}_k^{(1)}$ and shown that $\mathcal{W}_k^{(1)}$ is a quotient of $\mathcal{G}_k^{(1)}$ (paper in progress).

I would like to do an analogous study of the centralizer of the action of $\mathfrak{g}$ on $M \otimes N \otimes V^{\otimes k}$ when $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$ (the Lie algebras associated to the orthogonal and symplectic groups). We expect the centralizer of the action of $\mathfrak{g}$ on $M \otimes N \otimes V^{\otimes k}$ to be a quotient of the degenerate two-boundary braid group $\mathcal{G}_k$ as defined in Section 3. I will define the degenerate two-boundary Birman-Murakami-Wenzl algebra $\mathcal{W}_k$ to be the analogous quotient of $\mathcal{G}_k$ when $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$ to $\mathcal{H}_k^{\text{ext}}$ when $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ or $\mathfrak{sl}_n(\mathbb{C})$. The structure of $\mathcal{G}_k$ is largely based on the structure of the three images of $\mathcal{G}_k^{(1)}$ it contains—one corresponding to M, one corresponding to N, and one corresponding to $M \otimes N$. Similarly, $\mathcal{W}_k$ contains three images of $\mathcal{W}_k^{(1)}$. The structure and representation theory of $\mathcal{W}_k^{(1)}$ is described in [AMR]. One hopes that the construction of $\mathcal{W}_k$ can be done by understanding how the three images of $\mathcal{W}_k^{(1)}$ twist together, and combining that with the known structure of $\mathcal{W}_k^{(1)}$. This approach is completely analogous to that of the analysis of the Hecke algebras done here.

Once the degenerate two-boundary BMW algebra is defined, my next goal will be to explicitly construct the irreducible representations of $\mathcal{W}_k$ using combinatorial tools. The decomposition of $M \otimes N$ when M and N are simple modules corresponding to rectangular partitions is made precise in [Ok], so the decomposition of $M \otimes N \otimes V^{\otimes k}$ is well controlled. We expect the irreducible modules of $\mathcal{W}_k$ to be indexed by a particular class of partitions and to have a basis whose elements are indexed by certain up-down tableaux. Both the type of partitions and of up-down tableaux will be given by the decomposition of $M \otimes N \otimes V^{\otimes k}$. The formulas for the action of $\mathcal{W}_k$ will be in a similar form to the formulas for the action of $\mathcal{H}_k^{\text{ext}}$ given Section 4.3 and the formulas for the action of $\mathcal{W}_k^{(1)}$ in [AMR]. This approach also parallels the analysis of the quantized version of $\mathcal{W}_k^{(1)}$ done in [OR].

Next, I will study of the center of $\mathcal{W}_k$. In work with Ram and Virk, we thoroughly study the center of the graded BMW algebra $\mathcal{W}_k^{(1)}$, and its quantized version, the affine BMW algebra. Each of these algebras is defined with a choice of an infinite family of parameters awkwardly subject to admissibility conditions. By studying the centers of $\mathcal{W}_k^{(1)}$ and the affine BMW algebra, we define two algebras which do not depend on a choice of parameters, but which specialize to $\mathcal{W}_k^{(1)}$ and the affine BMW algebra, respectively. A priori, $\mathcal{W}_k$ should depend on a *pair* of infinite families of parameters subject to admissibility conditions. This leads to the question: Is there an algebra which does not depend on a choice of parameters, but which specializes to $\mathcal{W}_k$ for any admissible choice of parameters? A proper study of the center of $\mathcal{W}_k$ should lead to the definition of such an algebra.

5.3 Non-calibrated representations

In a paper of Orellana and Ram [OR], they construct a functor from quantum group $\mathcal{U}_q\mathfrak{g}$ modules to affine braid group modules. This functor takes finite dimensional $\mathcal{U}_q\mathfrak{g}$-modules to "calibrated" modules, Verma modules to "standard" modules, and irreducible modules to irreducible modules (under appropriate conditions). The affine and cyclotomic Hecke and Birman-Murakami-Wenzl modules are then described as quotients of the affine braid group. They recover representations of these algebras by considering the cases when $\mathfrak{g}$ is type $\mathfrak{sl}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$. Herein, I construct many calibrated modules for $\mathcal{H}_k^{\text{ext}}$ (those for which the ring $\mathbb{C}[w_0, w_1, \ldots, w_k]$ acts semisimply) by mapping $\mathcal{H}_k^{\text{ext}}$ into $\text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$, where M, N, and V are finite dimensional. I hope to construct a similar functor from the set of pairs of $\mathfrak{g}$-modules M and N to irreducible $\mathcal{G}_k$-modules. This will allow me to better understand the representation theory of $\mathcal{G}_k$, $\mathcal{H}_k^{\text{ext}}$, and $\mathcal{W}_k$. I should then be able to describe the combinatorial structure of not only calibrated modules for these three algebras, but standard modules as well. The results should parallel the formulas given in [OR].

5.4 Full centralizers

For small examples, I have found that the image of $\mathcal{H}_k^{\text{ext}}$ in $\text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ recovers the entire centralizer. The fact that this happens as often as it does is remarkable. The beauty of the representation Φ' is that when it surjects, it provides a construction of the full centralizer that relies on the use of very simple operators. When Φ' is not surjective, we can see through an analysis of the decomposition of $M \otimes N \otimes V^{\otimes k}$ that the image of $\mathcal{H}_k^{\text{ext}}$ differs from the full centralizer only by a commutative subalgebra—a portion of image of the center of $\mathfrak{g}$ acting on $M \otimes N$ not contained in the image of $\mathcal{H}_k^{\text{ext}}$.

In my treatment of the decomposition of $M \otimes N$, I describe a partial order on $\mathcal{P}((a^p), (b^q))$. This partial order captures an inductive process for constructing each partition in $\mathcal{P}((a^p), (b^q))$ by moving successive boxes. I hope to use this inductive process to construct and control projections onto each component of $M \otimes N$, thus calculating the full centralizer of the $\mathfrak{g}$-action on $M \otimes N \otimes V^{\otimes k}$.

In the study of the degenerate two-boundary BMW algebra, we expect similar phenomena. I hope to harness similar tools to extend quotients of $\mathcal{W}_k$ to full centralizers in non-generic cases.

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