An Algebraic Approach to Voting Theory

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Overview

Asking for votes

Full Rankings

Ex. 3 candidates

A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Partial Rankings

Ex. Top 3 from 5 candidates

A	A	A	A	A	A	A	A	A	A	A	A
B	B	B	C	C	C	D	D	D	E	E	E
C	C	E	B	D	E	B	C	E	B	C	D
DE	ED	CD	DE	BE	BD	CE	BE	BC	CD	BD	BC
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Asking for votes

Full Rankings

Ex. 3 candidates

A	*A*	*B*	*B*	*C*	*C*
B	*C*	*A*	*C*	*A*	*B*
C	*B*	*C*	*A*	*B*	*A*

Partial Rankings

Ex. Top 3 from 5 candidates

A	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
B	*B*	*B*	*C*	*C*	*C*	*D*	*D*	*D*	*E*	*E*	*E*
C	*C*	*E*	*B*	*D*	*E*	*B*	*C*	*E*	*B*	*C*	*D*
DE	*ED*	*CD*	*DE*	*BE*	*BD*	*CE*	*BE*	*BC*	*CD*	*BD*	*BC*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Asking for votes

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DE	ED	CD	DE	BE	BD	CE	BE	BC	CD	BD	BC
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Tallying votes

Positional methods

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- Weighting vector

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$$\mathbf{w} = [2, 1, 0]$$

$$\mathbf{w} = [1, 1/2, 0]$$

  - In general

$$\mathbf{w} = [1, t, 0], \qquad 0 \leq t \leq 1$$

Tallying votes

Pairwise (Condorcet) method

A wins over B if $A > B$ more times than $B > A$.

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Comparing methods

Example

A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A
10	25	5	25	5	20

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$$A : 10 + 25 = 35 \qquad B : 5 + 25 = 30 \qquad C : 5 + 20 = 25$$

Comparing methods

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Pairwise Tally

Comparing methods

Example

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$$A : 10 + 25 = 35 \qquad B : 5 + 25 = 30 \qquad C : 5 + 20 = 25$$

Pairwise Tally

$$B > A \quad 50 : 40 \qquad C > A \quad 50 : 40 \qquad C > B \quad 50 : 40$$

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Pos: A > B > C vs. Pairws: C > B > A

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$$(\ 10,\quad 25,\quad 5,\quad 25,\quad 5,\quad 20\ )$$

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  - **Non-dictatorship:** outcome dictated by more than one vote.
- There is no 'ideal' system.

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- Breaking down the profile space
- Borda Count minimizes conflicts

Tally Methods as Linear Maps

Positional Method

$$\mathbf{w} = [1, t, 0], \qquad 0 \leq t \leq 1$$

$$T_{\mathbf{w}} = \begin{pmatrix} 1 & 1 & t & 0 & t & 0 \\ t & 0 & 1 & 1 & 0 & t \\ 0 & t & 0 & t & 1 & 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

Pairwise Method

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} A > B \\ A > C \\ B > A \\ B > C \\ C > A \\ C > B \end{matrix}$$

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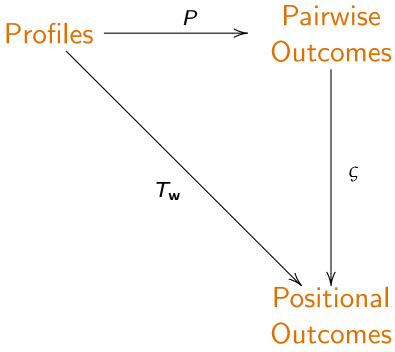
Relating Tally Maps



Breaking down the spaces

Theorem

All pairwise and positional maps are $\mathbb{Q}S_n$-module homomorphisms.

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$$U \cong W_1 \oplus W_2 \oplus W_3 \oplus Y_1 \oplus Z_1$$
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$$y_1 \longmapsto y_2$$

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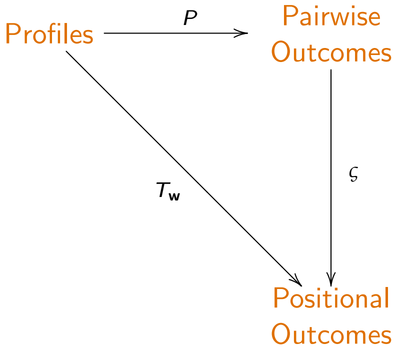
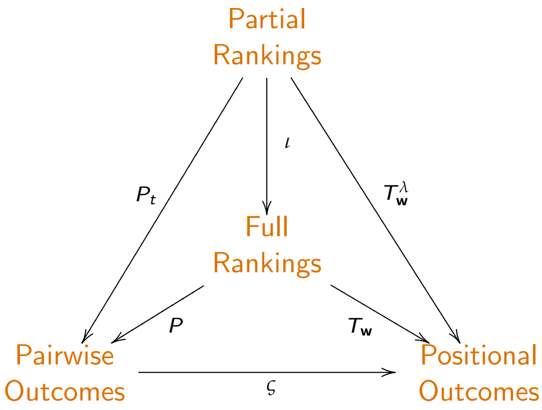


More Comparisons



Results

Full Rankings:

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- Borda Count uniquely allows our maps to commute

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Full Rankings:

- Recovery of Saari's results
- Borda Count uniquely allows our maps to commute
- Tools for a new perspective

Results

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$$\begin{array}{c} A \\ B \\ C \\ \hline D \\ E \end{array}$$

Results

For rankings of top k from n candidates, two cases:

- If pairwise map gives tying candidates each $1/2$ point, a particular scaled linear modification of the Borda Count works uniquely.

A	1
B	$3/4$
C	$2/4$
D	$1/4$
E	0

Results

For rankings of top k from n candidates, two cases:

- If pairwise map gives tying candidates each 1/2 point, a particular scaled linear modification of the Borda Count works uniquely.

A	1	1
B	3/4	3/4
C	2/4	2/4
D	1/4	1/8
E	0	1/8

Results

For rankings of top k from n candidates, two cases:

- If pairwise map gives tying candidates each 1/2 point, a particular scaled linear modification of the Borda Count works uniquely.

A	1	1	7/8
B	3/4	3/4	5/8
C	2/4	2/4	3/8
D	1/4	1/8	0
E	0	1/8	0

Results

For rankings of top k from n candidates, two cases:

- If pairwise map gives tying candidates each 1/2 point, a particular scaled linear modification of the Borda Count works uniquely.

A	1	1	7/8	1
B	3/4	3/4	5/8	5/7
C	2/4	2/4	3/8	3/7
D	1/4	1/8	0	0
E	0	1/8	0	0

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D	1/4	1/8	0	0
E	0	1/8	0	0

- If pairwise map gives tying candidates each something besides 1/2 point, there may be more freedom in our choice of positional method.

Thanks to...

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Reed Institute for Decision Science, Claremont McKenna

Professor Michael Orrison

Fellow thesis students

for more information:
http://www.math.hmc.edu/~zajj/thesis/